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OSCILLATION THEOREMS OF THE THIRD-ORDER NONLINEAR DELAY DYNAMIC EQUATIONS ON TIME SCALE

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Abstract

In this paper, a class of third-order nonlinear delay dynamic equations on time scales is studied. By using the generalized Riccati transformation and the inequality technique, two new sufficient conditions which ensure that every solution is oscillatory or converges to zero are established. The results obtained essentially generalize and improve earlier ones. Examples are given to illustrate our main results.

1. Introduction

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to the studies by Bohner and Saker [1] and Erbe et al. [2, 3], and there are some results dealing with oscillatory behavior of second-order delay dynamic equations on time scales [4-9]. However,

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there are few results dealing with the oscillation of the solutions of third-order delay dynamic equations on time scales, we refer the reader to the papers [10-12].

In this paper, we consider oscillatory behavior of all solutions of the third-order nonlinear delay dynamic equation

$$(r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\Delta} + q(t)f(x[g(t)]) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0, \tag{1.1}$$

where $\alpha > 0$ is the ratio of two positive odd integers.

Throughout this paper, we will assume the following hypotheses:

(H_1) $\mathbb{T}$ is a time scale (i.e., a nonempty closed subset of the real numbers $\mathbb{R}$) which is unbounded above, and $t_0 \in \mathbb{T}$ with $t_0 > 0$, we define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$.

(H_2) $r_1(t)$, $r_2(t)$, $q(t)$ are positive, real-valued rd-continuous functions defined on $\mathbb{T}$, and $r_1(t)$, $r_2(t)$ satisfy

$$\int_{t_0}^{\infty} \frac{1}{r_1(s)} \Delta s = \infty, \qquad \int_{t_0}^{\infty} \left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha}} \Delta s = \infty.$$

(H_3) $g(t) \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $g(t) \geq t$, and $g(\mathbb{T}) = \mathbb{T}$.

(H_4) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\frac{f(x)}{x^{\alpha}} \geq K > 0$, for $x \neq 0$.

By a solution of (1.1), we mean a nontrivial function $x(t)$ satisfying (1.1) which has the properties $x(t) \in C_{rd}^1([T_0, \infty)_{\mathbb{T}}, \mathbb{R})$ for $T_x \geq t_0$, and $r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha} \in C_{rd}^1([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$. Our attention is restricted to those solutions of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution x of equation (1.1) is said to be *oscillatory* on $[T_x, \infty)_{\mathbb{T}}$ if it is neither eventually positive nor eventually negative. Otherwise it is called *nonoscillatory*. The equation itself is called *oscillatory* if all its solutions are oscillatory.

If $\alpha = 1$, $g(t) = t$, then (1.1) simplifies to the third-order nonlinear dynamic equation

$$(r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}])^{\Delta} + q(t)f(x(t)) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \tag{1.2}$$

If, furthermore, $r_1(t) = r_2(t) = 1$, $f(x) = x$, $g(t) = t$, then (1.1) reduces to the third-order linear dynamic equation

$$x^{\Delta\Delta\Delta}(t) + q(t)x(t) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \tag{1.3}$$

If, in addition, $\alpha = 1$, then (1.1) reduces to the nonlinear delay dynamic equation

$$(r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}])^{\Delta} + q(t)f(x[g(t)]) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0. \tag{1.4}$$

In 2005, Erbe et al. [10] considered the general third-order nonlinear dynamic (1.2). By employing the generalized Riccati transformation techniques, they established some sufficient conditions which ensure that every solution of (1.2) is oscillatory or converges to zero. In 2007, Erbe et al. [11] studied the third-order linear dynamic (1.3), and they obtained Hille and Nehari type oscillation criteria for the (1.3). In 2011, Han et al. [12] extended and improved the results of [10, 11], meanwhile obtained some oscillatory criteria for the (1.4). On this basis, we discuss the oscillation of solutions of (1.1). By using the generalized Riccati transformation and the inequality technique, we obtain some sufficient conditions which guarantee that every solution of (1.1) is oscillatory or converges to zero.

The paper is organized as follows: In Section 1, we present some basic definitions and useful results from the theory of calculus on time scale. In Section 2, we give several lemmas. In Section 3, we use the generalized Riccati transformation and the inequality technique to obtain some sufficient conditions which guarantee that every solution of (1.1) is either oscillatory or converges to zero.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g of two differentiable functions f and g:

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)), \quad (1.5)$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}, \text{ if } gg^{\sigma} \neq 0. \quad (1.6)$$

For $b, c \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_b^c f^{\Delta}(t)\Delta(t) = f(c) - f(b).$$

The integration by parts formula reads

$$\int_b^c f^{\Delta}(t)g(t)\Delta(t) = f(c)g(c) - f(b)g(b) - \int_b^c f^{\sigma}(t)g^{\Delta}(t)\Delta t,$$

and infinite integrals are defined by

$$\int_b^{\infty} f(s)\Delta s = \lim_{t\to\infty}\int_b^t f(s)\Delta s.$$

For more details, see [17, 18].

2. Several Lemmas

Throughout this paper, for sufficiently large $T_0 \in \mathbb{T}$, we denote

$$R_1(t, T_0) := \int_{T_0}^t \left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha}}\Delta s, \quad R_2(t, T_0) := \int_{T_0}^t \frac{R_1(s, T_0)}{r_1(s)}\Delta s,$$

$$Q(t) := \left(K\int_t^{\infty} q(s)\Delta s\right)^{\frac{1}{\alpha}}.$$

In order to the definition of $Q(t)$ meaningful, we denote

$$\int_t^{\infty} q(s)\Delta s < \infty. \tag{2.1}$$

In this section, we present several lemmas that will be needed in the proofs of our results in Section 3.

Lemma 2.1. *Assume that* $x(t)$ *is an eventually positive solution of* (1.1). *Then there exists* $T \in [t_0, \infty)_{\mathbb{T}}$ *such that either*

(I) $x(t) > 0,\ x^{\Delta}(t) > 0,\ (r_1(t)x^{\Delta}(t))^{\Delta} > 0,\ (r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\Delta} < 0,$ $t \in [T, \infty)_{\mathbb{T}};$

or

(II) $x(t) > 0,\ x^{\Delta}(t) < 0,\ (r_1(t)x^{\Delta}(t))^{\Delta} > 0,\ (r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\Delta} < 0,$ $t \in [T, \infty)_{\mathbb{T}}.$

Proof. Assume that $x(t)$ is an eventually positive solution of (1.1), then there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(g(t)) > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. From (1.1), we have

$$(r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\Delta} = -q(t)f(x(g(t))) \le -Kq(t)x^{\alpha}[g(t)] < 0. \tag{2.2}$$

Hence, $r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha}$ is decreasing and therefore eventually of one sign, so $(r_1(t)x^{\Delta}(t))^{\Delta}$ is either eventually positive or eventually negative. We assert that $(r_1(t)x^{\Delta}(t))^{\Delta} > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$.

Assume that $(r_1(t)x^{\Delta}(t))^{\Delta} < 0$ eventually, then there exists $t_1 \in [T, \infty)_{\mathbb{T}}$ such that $(r_1(t_1)x^{\Delta}(t_1))^{\Delta} < 0$. We get

$$r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha} \le r_2(t_1)[(r_1(t_1)x^{\Delta}(t_1))^{\Delta}]^{\alpha} < 0,\ t \in [t_1, \infty)_{\mathbb{T}}.$$

Let $M = -r_2(t_1)[(r_1(t_1)x^{\Delta}(t_1))^{\Delta}]^{\alpha} > 0.$ Then

$$r_1(t)x^{\Delta}(t) \le -M^{\frac{1}{\alpha}} \frac{1}{(r_2(t))^{\frac{1}{\alpha}}}. \tag{2.3}$$

Integrating (2.3) from t_1 to $t\,(t \in [t_1, \infty)_{\mathbb{T}})$ provides

$$r_1(t)x^{\Delta}(t) \le r_1(t_1)x^{\Delta}(t_1) - M^{\frac{1}{\alpha}} \int_{t_1}^{t} \frac{1}{(r_2(s))^{\frac{1}{\alpha}}} \Delta s \to -\infty,\ t \to +\infty.$$

Then there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $r_1(t)x^{\Delta}(t) \le r_1(t_2)x^{\Delta}(t_2) < 0.$ Similarly, we obtain

$$x(t) \le x(t_2) + r_1(t_2)x^{\Delta}(t_2) \int_{t_2}^{t} \frac{1}{(r_1(s))^{\frac{1}{\alpha}}} \Delta s \to -\infty, \quad t \to +\infty,$$

which contradicts with $x(t) > 0.$ So $(r_1(t)x^{\Delta}(t))^{\Delta} > 0,$ this implies that $x^{\Delta}(t) > 0$ or $x^{\Delta}(t) < 0.$ This completes the proof.

Lemma 2.2 [16]. *Assume that* $x(t)$ *is a positive solution of equation* (1.1) *which satisfies case* (I) *in Lemma* 2.1. *Then*

$$x^{\Delta}(t) > \frac{R_1(t, T_0)}{r_1(t)} r_2^{\frac{1}{\alpha}}(t)(r_1(t)x^{\Delta}(t))^{\Delta}, \tag{2.4}$$

$$x(t) > R_2(t, T_0) r_2^{\frac{1}{\alpha}}(t)(r_1(t)x^{\Delta}(t))^{\Delta}. \tag{2.5}$$

If conditions (2.1) *and* (H_3) *hold, then*

$$x(t) < \frac{1}{Q(t)} r_2^{\frac{1}{\alpha}}(t)(r_1(t)x^{\Delta}(t))^{\Delta}. \tag{2.6}$$

Lemma 2.3. *Assume that* $x(t)$ *is a positive solution of equation* (1.1) *which satisfies case* (I) *in Lemma* 2.1, *and* (H_3) *holds. Then*

$$\frac{x^{\Delta}(t)}{x(t)} \geq \frac{R(t, T_0)}{r_1(t)} r_2(\sigma(t)) \left(\frac{(r_1(\sigma(t)) x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))} \right)^{\alpha}, \tag{2.7}$$

where

$$R(t, T_0) = \begin{cases} R_1(t, T_0) Q^{1-\alpha}(\sigma(t)), & 0 < \alpha \leq 1, \\ R_1(t, T_0) R_2^{1-\alpha}(\sigma(t), T_0), & \alpha > 1. \end{cases} \tag{2.8}$$

Proof. Because the $x(t)$ is a positive solution of (1.1) and which satisfies case (I) in Lemma 2.1, and $\sigma(t) \geq t$, hence

$$x(\sigma(t)) \geq x(t). \tag{2.9}$$

From (2.2), we have

$$r_2(t)[(r_1(t) x^{\Delta}(t))^{\Delta}]^{\alpha} \geq r_2(\sigma(t))[(r_1(\sigma(t)) x^{\Delta}(\sigma(t)))^{\Delta}]^{\alpha}. \tag{2.10}$$

Hence,

$$\begin{aligned} \frac{x^{\Delta}(t)}{x(t)} &\overset{(2.9)}{\geq} \frac{x^{\Delta}(t)}{x(\sigma(t))} \overset{(2.4)}{\geq} \frac{R_1(t, T_0)}{r_1(t)} \frac{r_2^{\frac{1}{\alpha}}(t)(r_1(t) x^{\Delta}(t))^{\Delta}}{x(\sigma(t))} \\ &\overset{(2.10)}{\geq} \frac{R_1(t, T_0)}{r_1(t)} \frac{r_2(\sigma(t))[(r_1(\sigma(t)) x^{\Delta}(\sigma(t)))^{\Delta}]^{\alpha}}{x(\sigma(t))}. \end{aligned} \tag{2.11}$$

Thus, when $0 < \alpha \leq 1$,

$$\begin{aligned} \frac{x^{\Delta}(t)}{x(t)} \geq \frac{R_1(t, T_0)}{r_1(t)} r_2(\sigma(t)) \left[\frac{(r_1(\sigma(t)) x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))} \right]^{\alpha} \\ \cdot \left[\frac{r_2^{\frac{1}{\alpha}}(\sigma(t))(r_1(\sigma(t)) x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))} \right]^{1-\alpha} \end{aligned}$$

$$\overset{(2.6)}{\geq} \frac{R_1(t, T_0)Q^{1-\alpha}(\sigma(t))}{r_1(t)} r_2(\sigma(t))\left[\frac{(r_1(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))}\right]^{\alpha}$$

$$\overset{(2.8)}{=} \frac{R(t, T_0)}{r_1(t)} r_2(\sigma(t))\left[\frac{(r_1(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))}\right]^{\alpha};$$

when $\alpha > 1$,

$$\frac{x^{\Delta}(t)}{x(t)} \geq \frac{R_1(t, T_0)}{r_1(t)} r_2(\sigma(t))\left[\frac{(r_1(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))}\right]^{\alpha}$$

$$\cdot\left[\frac{x(\sigma(t))}{r_2^{\frac{1}{\alpha}}(\sigma(t))(r_1(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}\right]^{\alpha-1}$$

$$\overset{(2.5)}{\geq} \frac{R_1(t, T_0)R_2^{\alpha-1}(\sigma(t)), T_0}{r_1(t)} r_2(\sigma(t))\left[\frac{(r_1(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))}\right]^{\alpha}$$

$$\overset{(2.8)}{=} \frac{R(t, T_0)}{r_1(t)} r_2(\sigma(t))\left[\frac{(r_1(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))}\right]^{\alpha}.$$

This completes the proof.

Lemma 2.4. *Assume that* $x(t)$ *is an eventually positive solution of* (1.1) *which satisfies case* (II) *in Lemma* 2.1 *such that*

$$\int_{t_0}^{\infty}\frac{1}{r_1(t)}\int_{t}^{\infty}\left[\frac{1}{r_2(s)}\int_{s}^{\infty} q(u)\Delta u\right]^{\frac{1}{\alpha}}\Delta s\Delta t = \infty. \tag{2.12}$$

Then $\lim_{t\to\infty} x(t) = 0.$

Proof. Assume that $x(t)$ is an eventually positive solution of (1.1) which satisfies case (II) in Lemma 2.1. Then $x(t)$ is decreasing and $\lim_{t\to\infty} x(t) = l \geq 0.$

If $l > 0,$ then it is easy to see that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x[g(t)] \geq x(t) \geq l > 0$ for all $t_1 \in [t_1, \infty)_{\mathbb{T}}.$ From (2.2), we have

$$(r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\Delta} \leq -Kq(t)x^{\alpha}[g(t)] \leq -Kl^{\alpha}q(t). \tag{2.13}$$

If (2.1) does not hold, then integrating (2.12) from t_1 to $t(t \in [t_1, \infty)_{\mathbb{T}}),$ we get

$$r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha} \leq r_2(t_1)[(r_1(t_1)x^{\Delta}(t_1))^{\Delta}]^{\alpha} - Kl^{\alpha}\int_{t_1}^{t} q(s)\Delta s \to -\infty$$

$$(t \to +\infty).$$

This is contrary to $(r_1(t)x^{\Delta}(t))^{\Delta} > 0.$

If (2.1) holds, then integrating (1.1) from t to ∞, we get

$$-r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha} \leq -K\int_{t}^{\infty} q(s)x^{\alpha}[g(s)]\Delta s \leq -Kgl^{\alpha}\int_{t}^{\infty} q(s)\Delta s,$$

$$t \in [t_1, \infty)_{\mathbb{T}}.$$

Hence, we have

$$-(r_1(t)x^{\Delta}(t))^{\Delta} \leq -l\left[\frac{1}{r_2(t)}\int_{t}^{\infty} Kq(s)\Delta s\right]^{\frac{1}{\alpha}}.$$

Integrating the above inequality from t to ∞, we obtain

$$r_1(t)x^{\Delta}(t) \leq -lK^{\frac{1}{\alpha}}\int_{t}^{\infty}\left[\frac{1}{r_2(s)}\int_{s}^{\infty} q(u)\Delta u\right]^{\frac{1}{\alpha}}\Delta s.$$

Integrating the last inequality again from T to t, we have

$$x(t) \leq x(T) - lK^{\frac{1}{\alpha}} \int_T^t \frac{1}{r_1(s)} \int_s^\infty \left[\frac{1}{r_2(u)} \int_u^\infty q(v)\Delta v\right]^{\frac{1}{\alpha}} \Delta u \Delta s.$$

Since condition (2.12) holds, we obtain $\lim_{t\to\infty} x(t) = -\infty$, which contradicts $x(t) > 0.$ Hence $l = 0.$ This completes the proof.

3. Main Results

In this section, we state and prove our main results. Write

$$\mathbb{D}_{\mathbb{T}} \equiv \{(t, s) : t \geq s \geq t_0,\ t, s \in [t_0, \infty)_{\mathbb{T}}\}.$$

Called function $H(t, s)$ has the property P (denoted as $H \in P$), if for sufficiently large $t \geq t_0$ satisfy

(i) $H(t, t) = 0,\ t \geq t_0;\ H(t, s) > 0,\ t > s \geq t_0,\ t, s \in [t_0, \infty)_{\mathbb{T}};$

(ii) $H(t, s)$ has a nonpositive continuous Δ-partial derivative $H^{\Delta_s}(t, s)$ with respect to the second variable.

Theorem 3.1. *Assume that* (H_1)-(H_4) *and* (2.12) *hold*. *Assume that there exist positive functions* $\delta, \varphi \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, $H(t, s) \in P$, $h(t, s) \in C_{rd}(\mathbb{D}_{\mathbb{T}}, \mathbb{R})$ *such that*

$$-H^{\Delta_s}(t, s) - \frac{R^*(s, T_0)}{\delta(\sigma(s))} H(t, s) = \frac{h(t, s)}{\delta(\sigma(s))} H^{\frac{1}{2}}(t, s), \tag{3.1}$$

and for all sufficiently large $T_1 \geq T_0$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t, T_1)} \int_{T_1}^t \left\{H(t, s)Q(s, T_0) - \frac{h_-^2(t, s) r_1(s)}{4\alpha\delta(s)R(s, T_0)}\right\} \Delta s = \infty, \tag{3.2}$$

where

$$R^*(t, T_0) = \delta^\Delta(t) + 2\alpha\delta(t)\frac{R(t, T_0)}{r_1(t)} r_2(\sigma(t))\varphi(\sigma(t)), \tag{3.3}$$

$$Q(t, T_0) = \delta(t)\left\{Kq(t) - (r_2(t)\varphi(t))^\Delta + \frac{\alpha R(t, T_0)}{r_1(t)}(r_2(\sigma(t))\varphi(\sigma(t)))^2\right\}, \tag{3.4}$$

$$h_-(t, s) = \max\{0, -h(t, s)\},\ h_+(t, s) = \max\{0, h(t, s)\}. \tag{3.5}$$

Then (1.1) *is either oscillatory or converges to zero.*

Proof. Assume that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists sufficiently large $T \geq t_0$ such that $x(t) > 0$ and $x(g(t)) > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. By Lemma 2.1, we see that $x(t)$ satisfies either then only case (I) or case (II) may occur.

If case (I) holds, then $x^\Delta(t) > 0,\ t \in [T, \infty)_{\mathbb{T}}$. Define the function $W(t)$ by

$$W(t) = \delta(t)r_2(t)\left(\left[\frac{(r_1(t)x^\Delta(t))^\Delta}{x(t)}\right]^\alpha + \varphi(t)\right), \quad t \in [T, \infty)_{\mathbb{T}}. \tag{3.6}$$

Then $W(t) > 0$ and

$$\begin{aligned}
W^\Delta(t) &= \frac{\delta(t)}{x^\alpha(t)}(r_2(t)[(r_1(t)x^\Delta(t))^\Delta]^\alpha)^\Delta \\
&\quad + \left(\frac{\delta(t)}{x^\alpha(t)}\right)^\Delta (r_2(t)[(r_1(t)x^\Delta(t))^\Delta]^\alpha)^\sigma + (\delta(t)r_2(t)\varphi(t))^\Delta \\
&\overset{(1.1)}{=} \frac{\delta(t)}{x^\alpha(t)}(-q(t)f(x[g(t)])) \\
&\quad + \left(\frac{\delta(t)}{x^\alpha(t)}\right)^\Delta (r_2(t)[(r_1(t)x^\Delta(t))^\Delta]^\alpha)^\sigma + (\delta(t)r_2(t)\varphi(t))^\Delta
\end{aligned}$$

$$\overset{(H_3)}{\leq} \frac{\delta(t)}{x^{\alpha}(t)}(-q(t)Kx^{\alpha}[g(t)])$$

$$+\left(\frac{\delta(t)}{x^{\alpha}(t)}\right)^{\Delta}(r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\sigma}+(\delta(t)r_2(t)\varphi(t))^{\Delta}. \quad (3.7)$$

From $x^{\Delta}(t) > 0,\ g(t) > t,\ \alpha > 0,$ we obtain

$$\frac{x^{\alpha}(g(t))}{x^{\alpha}(t)} > 1. \quad (3.8)$$

Using (3.7) and (3.8), we obtain

$$W^{\Delta}(t) = -K\delta(t)q(t) + \left(\frac{\delta(t)}{x^{\alpha}(t)}\right)^{\Delta}(r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\sigma} + (\delta(t)r_2(t)\varphi(t))^{\Delta}. \quad (3.9)$$

From

$$(x^{\alpha}(t))^{\Delta} = \alpha x^{\Delta}(t)\int_0^1 (hx(\sigma(t)) + (1-h)x(t))^{\alpha-1}dh$$

$$\geq \alpha x^{\Delta}(t)\int_0^1 (hx(t) + (1-h)x(t))^{\alpha-1}dh = \alpha x^{\Delta}(t)x^{\alpha-1}(t),$$

we obtain

$$\left(\frac{\delta(t)}{x^{\alpha}(t)}\right)^{\Delta} = \frac{\delta^{\Delta}(t)x^{\alpha}(t) - \delta(t)(x^{\alpha}(t))^{\Delta}}{x^{\alpha}(t)x^{\alpha}(\sigma(t))} \leq \left(\delta^{\Delta}(t) - \alpha\delta(t)\frac{x^{\Delta}(t)}{x(t)}\right)\frac{1}{x^{\alpha}(\sigma(t))}. \quad (3.10)$$

Hence, we get

$$\left(\frac{\delta(t)}{x^{\alpha}(t)}\right)^{\Delta}(r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\sigma}$$

$$\leq\left(\delta^{\Delta}(t)-\alpha\delta(t)\frac{x^{\Delta}(t)}{x(t)}\right)\frac{1}{x^{\alpha}(\sigma(t))}(r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\sigma}$$

$$=\left(\delta^{\Delta}(t)-\alpha\delta(t)\frac{x^{\Delta}(t)}{x(t)}\right)\left(r_2(t)\left[\frac{(r_1(t)x^{\Delta}(t))^{\Delta}}{x(t)}\right]^{\alpha}\right)^{\sigma}$$

$$\overset{(2.7)}{\leq}\left(\delta^{\Delta}(t)-\alpha\delta(t)\frac{R(t,T_0)}{r_1(t)}r_2(\sigma(t))\left(\frac{(r_1(\sigma(t))x^{\Delta}(\sigma(t)))^{\Delta}}{x(\sigma(t))}\right)^{\alpha}\right)$$

$$\cdot\left(r_2(t)\left[\frac{(r_1(t)x^{\Delta}(t))^{\Delta}}{x(t)}\right]^{\alpha}\right)^{\sigma}$$

$$=\delta^{\Delta}(t)\left(r_2(t)\left[\frac{(r_1(t)x^{\Delta}(t))^{\Delta}}{x(t)}\right]^{\alpha}\right)^{\sigma}$$

$$-\alpha\delta(t)\frac{R(t,T_0)}{r_1(t)}\left(\left(r_2(t)\left[\frac{(r_1(t)x^{\Delta}(t))^{\Delta}}{x(t)}\right]^{\alpha}\right)^{\sigma}\right)^{2}$$

$$\overset{(3.6)}{=}\delta^{\Delta}(t)\left(\frac{W(\sigma(t))}{\delta(\sigma(t))}-r_2(\sigma(t))\varphi(\sigma(t))\right)$$

$$-\alpha\delta(t)\frac{R(t,T_0)}{r_1(t)}\left(\frac{W(\sigma(t))}{\delta(\sigma(t))}-r_2(\sigma(t))\varphi(\sigma(t))\right)^{2}$$

$$=-\delta^{\Delta}(t)r_2(\sigma(t))\varphi(\sigma(t))-\alpha\delta(t)\frac{R(t,T_0)}{r_1(t)}(r_2(\sigma(t))\varphi(\sigma(t)))^{2}$$

$$+\left(\delta^{\Delta}(t)+2\alpha\delta(t)\frac{R(t,\,T_0)}{r_1(t)}r_2(\sigma(t))\varphi(\sigma(t))\right)\frac{W(\sigma(t))}{\delta(\sigma(t))}$$

$$-\,\alpha\delta(t)\frac{R(t,\,T_0)}{r_1(t)}\left(\frac{W(\sigma(t))}{\delta(\sigma(t))}\right)^2. \tag{3.11}$$

From (3.9), (3.11) and (3.3), (3.4), we obtain

$$\begin{aligned}
W^{\Delta}(t) &\le -K\delta(t)q(t)-\delta^{\Delta}(t)r_2(\sigma(t))\varphi(\sigma(t))\\
&\quad-\alpha\delta(t)\frac{R(t,\,T_0)}{r_1(t)}(r_2(\sigma(t))\varphi(\sigma(t)))^2+(\delta(t)r_2(t)\varphi(t))^{\Delta}\\
&\quad+\left(\delta^{\Delta}(t)+2\alpha\delta(t)\frac{R(t,\,T_0)}{r_1(t)}r_2(\sigma(t))\varphi(\sigma(t))\right)\frac{W(\sigma(t))}{\delta(\sigma(t))}\\
&\quad-\alpha\delta(t)\frac{R(t,\,T_0)}{r_1(t)}\left(\frac{W(\sigma(t))}{\delta(\sigma(t))}\right)^2\\
&\le -\delta(t)\left(Kq(t)-(r_2(t)\varphi(t))^{\Delta}+\frac{\alpha R(t,\,T_0)}{r_1(t)}(r_2(\sigma(t))\varphi(\sigma(t)))^2\right)\\
&\quad+\left(\delta^{\Delta}(t)+2\alpha\delta(t)\frac{R(t,\,T_0)}{r_1(t)}r_2(\sigma(t))\varphi(\sigma(t))\right)\frac{W(\sigma(t))}{\delta(\sigma(t))}\\
&\quad-\alpha\delta(t)\frac{R(t,\,T_0)}{r_1(t)}\left(\frac{W(\sigma(t))}{\delta(\sigma(t))}\right)^2\\
&\le -Q(t,\,T_0)+R^*(t,\,T_0)\frac{W(\sigma(t))}{\delta(\sigma(t))}-\frac{\alpha\delta(t)R(t,\,T_0)}{r_1(t)}\left(\frac{W(\sigma(t))}{\delta(\sigma(t))}\right)^2.
\end{aligned} \tag{3.12}$$

From (3.12), we get

$$Q(t,\,T_0)\le -W^{\Delta}(t)+R^*(t,\,T_0)\frac{W(\sigma(t))}{\delta(\sigma(t))}-\frac{\alpha\delta(t)R(t,\,T_0)}{r_1(t)}\left(\frac{W(\sigma(t))}{\delta(\sigma(t))}\right)^2. \tag{3.13}$$

Replace t with s, multiply both sides by $H(t, s)$, integrating from T_1 to $t > T_1$, we obtain

$$\int_{T_1}^{t} H(t, s)Q(s, T_0)\Delta s \leq -\int_{T_1}^{t} H(t, s)W^{\Delta}(s)\Delta s$$

$$+\int_{T_1}^{t} H(t, s)R^{*}(s, T_0)\frac{W(\sigma(s))}{\delta(\sigma(s))}\Delta s$$

$$-\int_{T_1}^{t} H(t, s)\frac{\alpha\delta(s)R(s, T_0)}{r_1(s)}\left(\frac{W(\sigma(s))}{\delta(\sigma(s))}\right)^2\Delta s. \quad (3.14)$$

From $H \in P$ and (3.1), we get

$$-\int_{T_1}^{t} H(t, s)W^{\Delta}(s)\Delta s = H(t, T_1)W(T_1) + \int_{T_1}^{t} H^{\Delta_s}(t, s)W(\sigma(s))\Delta s$$

$$= H(t, T_1)W(T_1) - \int_{T_1}^{t} \frac{R^{*}(s, T_0)}{\delta(\sigma(s))}H(t, s)W(\sigma(s))\Delta s$$

$$+\int_{T_1}^{t} \frac{-h(t, s)}{\delta(\sigma(s))}H^{\frac{1}{2}}(t, s)W(\sigma(s))\Delta s. \quad (3.15)$$

From (3.14), (3.15) and (3.5), we obtain

$$\int_{T_1}^{t} H(t, s)Q(s, T_0)\Delta s$$

$$\leq H(t, T_1)W(T_1) + \int_{T_1}^{t} \frac{-h(t, s)}{\delta(\sigma(s))}H^{\frac{1}{2}}(t, s)W(\sigma(s))\Delta s$$

$$-\int_{T_1}^{t} H(t, s)\frac{\alpha\delta(t)R(s, T_0)}{r_1(s)}\left(\frac{W(\sigma(s))}{\delta(\sigma(s))}\right)^2\Delta s$$

$$\leq H(t, T_1)W(T_1) + \int_{T_1}^{t} \frac{h_-(t, s)}{\delta(\sigma(s))} H^{\frac{1}{2}}(t, s)W(\sigma(s))\Delta s$$

$$- \int_{T_1}^{t} H(t, s)\frac{\alpha\delta(t)R(s, T_0)}{r_1(s)}\left(\frac{W(\sigma(s))}{\delta(\sigma(s))}\right)^2 \Delta s$$

$$= H(t, T_1)W(T_1)$$

$$+ \int_{T_1}^{t}\left(\frac{h_-(t, s)}{\delta(\sigma(s))} H^{\frac{1}{2}}(t, s)W(\sigma(s)) - H(t, s)\frac{\alpha\delta(t)R(s, T_0)}{r_1(s)}\left(\frac{W(\sigma(s))}{\delta(\sigma(s))}\right)^2\right)\Delta s \tag{3.16}$$

such that

$$\frac{h_-(t, s)}{\delta(\sigma(s))} H^{\frac{1}{2}}(t, s)W(\sigma(s)) - H(t, s)\frac{\alpha\delta(t)R(s, T_0)}{r_1(s)}\left(\frac{W(\sigma(s))}{\delta(\sigma(s))}\right)^2$$

$$\leq \frac{h_-^2(t, s)r_1(s)}{4\alpha\delta(\sigma(s))R(s, T_0)}. \tag{3.17}$$

From (3.16), (3.17), we obtain

$$\frac{1}{H(t, T_1)}\int_{T_1}^{t}\left(H(t, s)Q(s, T_0) - \frac{h_-^2(t, s)r_1(s)}{4\alpha\delta(\sigma(s))R(s, T_0)}\right)\Delta s \leq W(T_1).$$

This is contrary to (3.2).

If case (II) holds, from (2.12), by Lemma 2.4, $\lim_{t\to\infty} x(t) = 0.$ This completes the proof.

Theorem 3.2. *Assume that* (H_1)-(H_4), (2.12) *hold. Assume that there exist positive functions* $\delta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, $H(t, s) \in P$, $h(t, s) \in C_{rd}(\mathbb{D}_{\mathbb{T}}, \mathbb{R})$ *such that*

$$H^{\Delta_s}(t, s) + \frac{\delta^{\Delta}(s)}{\delta(\sigma(s))}H(t, s) = -\frac{h(t, s)}{\delta(\sigma(s))}H^{\frac{\alpha}{1+\alpha}}(t, s), \tag{3.18}$$

and for all sufficiently large $T_1 \geq T_0$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t, T_1)} \int_{T_1}^{t} \left\{ KH(t, s)\delta(s)q(s) - \frac{h_-^{\alpha+1}(t, s) r_1^{\alpha}(s)}{(1+\alpha)^{(1+\alpha)}[\delta(s) R_1(s, T_0)]^{\alpha}} \right\} \Delta s = \infty. \tag{3.19}$$

Then (1.1) *is either oscillatory or converges to zero.*

Proof. Assume that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists sufficiently large $T \geq t_0$ such that $x(t) > 0$ and $x(g(t)) > 0$ for all $t \in [T, \infty)_{\mathbb{T}}$. By Lemma 2.1, we see that $x(t)$ satisfies either then only case (I) or case (II) may occur.

If case (I) holds, then $x^{\Delta}(t) > 0,\ t \in [T, \infty)_{\mathbb{T}}$. Define the function $W(t)$ by

$$W(t) = \delta(t) r_2(t) \left(\frac{(r_1(t) x^{\Delta}(t))^{\Delta}}{x(t)} \right)^{\alpha}, \quad t \in [T, \infty)_{\mathbb{T}}. \tag{3.20}$$

Then $W(t) > 0$ and

$$W^{\Delta}(t)$$

$$= \frac{\delta(t)}{x^{\alpha}(t)} (r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\Delta} + \left(\frac{\delta(t)}{x^{\alpha}(t)} \right)^{\Delta} (r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\sigma}$$

$$\overset{(1.1)}{=} \frac{\delta(t)}{x^{\alpha}(t)} (-q(t) f(x[g(t)])) + \left(\frac{\delta(t)}{x^{\alpha}(t)} \right)^{\Delta} (r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\sigma}$$

$$\overset{(H_3)}{\leq} \frac{\delta(t)}{x^{\alpha}(t)} (-q(t) K x^{\alpha}[g(t)]) + \left(\frac{\delta(t)}{x^{\alpha}(t)} \right)^{\Delta} (r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\sigma}$$

$$\overset{(3.8)}{\leq} -K\delta(t)q(t) + \left(\frac{\delta(t)}{x^{\alpha}(t)} \right)^{\Delta} (r_2(t)[(r_1(t)x^{\Delta}(t))^{\Delta}]^{\alpha})^{\sigma}$$

$$\overset{(3.10)}{\leq} -K\delta(t)q(t) + \left(\delta^{\Delta}(t) - \alpha\delta(t)\frac{x^{\Delta}(t)}{x(t)}\right)\left(r_2(t)\left[\frac{(r_1(t)x^{\Delta}(t))^{\Delta}}{x(t)}\right]^{\alpha}\right)^{\sigma}$$

$$\overset{(2.11)}{\leq} -K\delta(t)q(t) + \delta^{\Delta}(t)\left(r_2(t)\left[\frac{(r_1(t)x^{\Delta}(t))^{\Delta}}{x(t)}\right]^{\alpha}\right)^{\sigma}$$

$$-\alpha\delta(t)\frac{R_1(t, T_0)}{r_1(t)}\frac{r_2^{\frac{1}{\alpha}}(\sigma(t))[(r_1(\sigma(t))x^{\Delta}(\sigma(t))]^{\Delta}}{x(\sigma(t))}\left(r_2(t)\left[\frac{(r_1(t)x^{\Delta}(t))^{\Delta}}{x(t)}\right]^{\alpha}\right)^{\sigma}$$

$$\overset{(3.20)}{=} -K\delta(t)q(t) + \delta^{\Delta}(t)\frac{W(\sigma(t))}{\delta(\sigma(t))} - \frac{\alpha\delta(t)R_1(t, T_0)}{r_1(t)}\left(\frac{W(\sigma(t))}{\delta(\sigma(t))}\right)^{1+\frac{1}{\alpha}}.$$

Thus,

$$K\delta(t)q(t) \leq -W^{\Delta}(t) + \delta^{\Delta}(t)\frac{W(\sigma(t))}{\delta(\sigma(t))} - \frac{\alpha\delta(t)R_1(t, T_0)}{r_1(t)}\left(\frac{W(\sigma(t))}{\delta(\sigma(t))}\right)^{1+\frac{1}{\alpha}}. \tag{3.21}$$

Replace t with s, multiply both sides by $H(t, s)$, integrating from T_1 to $t > T_1$, we obtain

$$\int_{T_1}^{t} H(t, s)K\delta(s)q(s)\Delta s$$

$$\leq -\int_{T_1}^{t} H(t, s)W^{\Delta}(s)\Delta s + \int_{T_1}^{t} H(t, s)\delta^{\Delta}(s)\frac{W(\sigma(s))}{\delta(\sigma(s))}\Delta s$$

$$-\int_{T_1}^{t} H(t, s)\frac{\alpha\delta(s)R_1(s, T_0)}{r_1(s)}\left(\frac{W(\sigma(s))}{\delta(\sigma(s))}\right)^{1+\frac{1}{\alpha}}\Delta s. \tag{3.22}$$

From $H \in P$ and (3.18), we obtain

$$-\int_{T_1}^{t} H(t, s)W^{\Delta}(s)\Delta s = H(t, T_1)W(T_1) + \int_{T_1}^{t} H^{\Delta_s}(t, s)W(\sigma(s))\Delta s$$

$$= H(t, T_1)W(T_1) - \int_{T_1}^{t} H(t, s)\delta^{\Delta}(s)\frac{W(\sigma(s))}{\delta(\sigma(s))}\Delta s$$

$$+ \int_{T_1}^{t} -h(t, s)H^{\frac{\alpha}{1+\alpha}}(t, s)\frac{W(\sigma(s))}{\delta(\sigma(s))}\Delta s$$

$$\leq H(t, T_1)W(T_1) - \int_{T_1}^{t} H(t, s)\delta^{\Delta}(s)\frac{W(\sigma(s))}{\delta(\sigma(s))}\Delta s$$

$$+ \int_{T_1}^{t} h_{-}(t, s)H^{\frac{\alpha}{1+\alpha}}(t, s)\frac{W(\sigma(s))}{\delta(\sigma(s))}\Delta s. \tag{3.23}$$

From (3.22), (3.23), we obtain

$$\int_{T_1}^{t} H(t, s)K\delta(s)q(s)\Delta s \leq H(t, T_1)W(T_1)$$

$$+\int_{T_1}^{t}\left(h_{-}(t, s)H^{\frac{\alpha}{1+\alpha}}(t, s)\frac{W(\sigma(s))}{\delta(\sigma(s))} - H(t, s)\frac{\alpha\delta(t)R_1(s, T_0)}{r_1(s)}\left(\frac{W(\sigma(s))}{\delta(\sigma(s))}\right)^{1+\frac{1}{\alpha}}\right)\Delta s. \tag{3.24}$$

Now set

$$X^{\lambda} = H(t, s)\frac{\alpha\delta(s)R_1(s, T_0)}{r_1(s)}\left(\frac{W(\sigma(s))}{\delta(\sigma(s))}\right)^{\lambda},$$

$$Y^{\lambda-1} = \frac{h_{-}(t, s)r_1^{\frac{1}{\lambda}}(s)}{\lambda[\alpha\delta(s)R_1(s, T_0)]^{\frac{1}{\lambda}}},$$

where $\lambda = \dfrac{\alpha + 1}{\alpha} > 1,$ $X \geq 0$ and $Y \geq 0.$ Using the equality

$$\lambda XY^{\lambda-1} - X^{\lambda} \leq (\lambda - 1)Y^{\lambda},$$

which yields

$$\frac{h_{-}(t, s)H^{\frac{\alpha}{1+\alpha}}(t, s)W(\sigma(s))}{\delta(\sigma(s))} - \frac{H(t, s)\sigma\delta(t)R_1(s, T_0)}{r_1(s)}\left(\frac{W(\sigma(s))}{\delta(\sigma(s))}\right)^{1+\frac{1}{\alpha}}$$

$$\leq \frac{h_{-}^{1+\alpha}(t, s)r_1^{\alpha}(s)}{(1+\alpha)^{1+\alpha}[\delta(s)R_1(s, T_0)]^{\alpha}}. \tag{3.25}$$

From (3.24) and (3.25), we have

$$\frac{1}{H(t, T_1)}\int_{T_1}^{t}\left(H(t, s)K\delta(s)q(s) - \frac{h_{-}^{1+\alpha}(t, s)r_1^{\alpha}(s)}{(1+\alpha)^{1+\alpha}[\delta(s)R_1(s, T_0)]^{\alpha}}\right)\Delta s \leq W(T_1).$$

This is contrary to (3.19).

If case (II) holds, from (2.12), by Lemma 2.4, $\lim_{t\to\infty} x(t) = 0.$ This completes the proof.

Example 3.1. Consider the third-order nonlinear delay dynamic equation

$$\left(t^{\frac{2}{3}}\left(\left(\frac{1}{t}x^{\Delta}(t)\right)^{\Delta}\right)^{\frac{5}{3}}\right)^{\Delta} + \frac{1}{t^2}(x(2t))^{\frac{5}{3}}(1 + \ln(1 + x^2(2t))) = 0,$$

$$t \in \overline{2^{\mathbf{Z}}}, \quad t \geq t_0 := 2. \tag{3.26}$$

Here $\alpha = \frac{5}{3},$ $r_1(t) = \frac{1}{t},$ $r_2(t) = t^{\frac{2}{3}},$ $q(t) = \frac{1}{t^2},$ $f(x) = x^{\frac{5}{3}}(1 + \ln(1 + x^2))$ and $g(t) = 2t > t.$

Conditions (H_1)-(H_3) are clearly satisfied, (H_4) holds with $K = 1$, and

$$\int_{t_0}^{\infty} \frac{1}{r_1(t)} \int_t^{\infty} \left[\frac{1}{r_2(s)} \int_s^{\infty} q(u) \Delta u \right]^{\frac{1}{\alpha}} \Delta s \Delta t = \int_2^{\infty} t \int_t^{\infty} \left[\frac{1}{s^{\frac{2}{3}}} \int_s^{\infty} \frac{1}{u^2} \Delta u \right]^{\frac{3}{5}} \Delta s \Delta t$$

$$= \infty,$$

so (1.2) holds. Noting that

$$R_1(t, T_0) = \int_{T_0}^{t} \left(\frac{1}{r_2(s)} \right)^{\frac{1}{\alpha}} \Delta s = \int_{T_0}^{t} s^{-\frac{2}{5}} \Delta s = \frac{t^{\frac{3}{5}} - T_0^{\frac{3}{5}}}{2^{\frac{3}{5}} - 1},$$

$$R_2(t_0, T_0) = \int_{T_0}^{t} \frac{R_1(s, T_0)}{r_1(s)} \Delta s = \int_{T_0}^{t} \frac{s^{\frac{8}{5}} - T_0^{\frac{3}{5}} s}{2^{\frac{3}{5}} - 1} \Delta s$$

$$= \frac{t^{\frac{13}{5}} - T_0^{\frac{13}{5}}}{\left(2^{\frac{3}{5}} - 1\right)\left(2^{\frac{13}{5}} - 1\right)} - \frac{T_0^{\frac{3}{5}}(t^2 - T_0^2)}{3\left(2^{\frac{3}{5}} - 1\right)},$$

$$R(t, T_0) = R_1(t, T_0) R_2^{\alpha - 1}(t, T_0)$$

$$= \frac{t^{\frac{3}{5}} - T_0^{\frac{3}{5}}}{2^{\frac{3}{5}} - 1} \left[\frac{t^{\frac{13}{5}} - T_0^{\frac{13}{5}}}{\left(2^{\frac{3}{5}} - 1\right)\left(2^{\frac{13}{5}} - 1\right)} - \frac{T_0^{\frac{3}{5}}(t^2 - T_0^2)}{3\left(2^{\frac{3}{5}} - 1\right)} \right]^{\frac{2}{3}} > 1.$$

Let $\delta(t) = t$, $\varphi(t) = t^{-\frac{5}{3}}$, since $\sigma(t) = 2t$, we obtain

$$R^*(t, T_0) = \delta^{\Delta}(t) + 2\alpha\delta(t) \frac{R_1(t, T_0)}{r_1(t)} r_2(\sigma(t))\varphi(\sigma(t)) = 1 + \frac{5}{3} \cdot t \cdot R(t, T_0),$$

$$Q(t,\ T_0) = \delta(t)\left\{Kq(t) - (r_2(t)\varphi(t))^{\Delta} + \frac{\alpha R(t,\ T_0)}{r_1(t)}(r_2(\sigma(t))\varphi(\sigma(t)))^2\right\}$$

$$= t\left[\frac{1}{t^2} - \left(\frac{1}{t}\right)^{\Delta} + \frac{5}{3}\cdot t\cdot\left(\frac{1}{2t}\right)^2\cdot R(t,\ T_0)\right] = \frac{3}{2t} + \frac{5}{12}\cdot R(t,\ T_0).$$

Let $H(t,\ s) = (t-s)^2,$ that there exists a function

$$h(t,\ s) = -\frac{R^*(s,\ T_0)\cdot H(t,\ s) - 2s(2t-3s)}{H^{\frac{1}{2}}(t,\ s)} \geq R^*(s,\ T_0)\cdot H^{\frac{1}{2}}(t,\ s)$$

such that

$$-H^{\Delta_s}(t,\ s) - \frac{R^*(s,\ T_0)}{\delta(\sigma(s))}H(t,\ s) = \frac{h(t,\ s)}{\delta(\sigma(s))}H^{\frac{1}{2}}(t,\ s)$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,\ T_1)}\int_{T_1}^{t}\left\{H(t,\ s)\cdot Q(s,\ T_0) - \frac{h_-^2(t,\ s)r_1(s)}{4\alpha\delta(s)R(s,\ T_0)}\right\}\Delta s$$

$$\geq \limsup_{t\to\infty}\frac{1}{H(t,T_1)}\int_{T_1}^{t}\left\{H(t,s)\left[\frac{3}{2s}+\frac{5}{12}\cdot R(s,T_0)\right] - \frac{\left[R^*(s,T_0)\cdot H^{\frac{1}{2}}(t,s)\right]^2}{\frac{20}{3}\cdot s^2\cdot R(s,T_0)}\right\}\Delta s$$

$$= \limsup_{t\to\infty}\frac{1}{H(t,T_1)}\int_{T_1}^{t}\left\{H(t,s)\left(\left[\frac{3}{2s}+\frac{5}{12}\cdot R(s,T_0)\right] - \frac{[R^*(s,T_0)]^2}{\frac{20}{3}\cdot s^2\cdot R(s,T_0)}\right)\right\}\Delta s$$

$$= \limsup_{t\to\infty}\frac{1}{H(t,T_1)}\int_{T_1}^{t}\left\{H(t,s)\left(\left[\frac{3}{2s}+\frac{5}{12}\cdot R(s,T_0)\right] - \frac{\left[1+\frac{5}{3}\cdot s\cdot R(s,T_0)\right]^2}{\frac{20}{3}\cdot s^2\cdot R(s,T_0)}\right)\right\}\Delta s$$

$$= \limsup_{t\to\infty} \frac{1}{H(t, T_1)} \int_{T_1}^{t} \left\{ H(t, s)\left(\left[\frac{3}{2s} + \frac{5}{12} \cdot R(s, T_0)\right]\right.\right.$$

$$\left.\left.\left. - \left[\frac{3}{20 \cdot s^2 \cdot R(s, T_0)} + \frac{1}{2s} + \frac{5}{12} \cdot R(s, T_0)\right]\right)\right\} \Delta s$$

$$= \limsup_{t\to\infty} \frac{1}{H(t, T_1)} \int_{T_1}^{t} \left\{ H(t, s)\left(\frac{1}{s} - \frac{3}{20 \cdot s^2 \cdot R(s, T_0)}\right)\right\} \Delta s$$

$$\geq \limsup_{t\to\infty} \frac{1}{(t - T_1)^2} \int_{T_1}^{t} \left\{ (t, s)^2 \left(\frac{1}{s} - \frac{3}{20 \cdot s^2}\right)\right\} \Delta s = \infty, \quad T_1 \geq T_0,$$

so (3.1), (3.2) hold. Then, by Theorem 3.1, every solution $x(t)$ of (3.26) is either oscillatory or converges to zero.

Example 3.2. Consider the third-order nonlinear delay dynamic equation

$$\left(t^{-\frac{1}{3}} ((t^{-1} x^{\Delta}(t))^{\Delta})^{\frac{1}{3}} \right)^{\Delta} + t^{-2} \left(x^{\frac{1}{3}}(2t) + x^{\frac{7}{3}}(2t) \right) = 0, \quad t \in \overline{2^{\mathbf{z}}}, \quad t \geq t_0 := 2. \tag{3.27}$$

Here $\alpha = \dfrac{1}{3}$, $r_1(t) = t^{-1}$, $r_2(t) = t^{-\frac{1}{3}}$, $q(t) = t^{-2}$, $f(x) = x^{\frac{1}{3}}(1 + x^2)$ and $g(t) = 2t \geq t$.

Conditions (H_1)-(H_3) are clearly satisfied, (H_4) holds with $K = 1$, and

$$\int_{t_0}^{\infty} \frac{1}{r_1(t)} \int_{t}^{\infty} \left[\frac{1}{r_2(s)} \int_{s}^{\infty} q(u) \Delta u\right]^{\frac{1}{\alpha}} \Delta s \Delta t$$

$$= \int_{2}^{\infty} t \int_{t}^{\infty} \left[s^{\frac{1}{3}} \int_{s}^{\infty} u^{-2} \Delta u \right]^{3} \Delta s \Delta t = 8 \int_{2}^{\infty} t \int_{t}^{\infty} s^{-2} \Delta s \Delta t$$

$$= 16 \int_{2}^{\infty} \Delta t = \infty,$$

so (2.12) holds. Noting that

$$R_1(t, T_0) = \int_{T_0}^{t} \left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha}} \Delta s = \int_{T_0}^{t} s\Delta s = \frac{t^2 - T_0^2}{3}.$$

Let $H(t, s) = (t - s)^2$, $\delta(t) = t$, since $\sigma(t) = 2t$, there exists a function

$$h(t, s) = -\frac{H(t, s) - 2s(2t - 3s)}{H^{\frac{1}{4}}(t, s)} \geq -H^{\frac{3}{4}}(t, s) = -(t - s)^{\frac{3}{2}}$$

such that

$$H^{\Delta_s}(t, s) + \frac{\delta^{\Delta}(s)}{\delta(\sigma(s))} H(t, s) = -\frac{h(t, s)}{\delta(\sigma(s))} H^{\frac{\alpha}{1+\alpha}}(t, s)$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t, T_1)} \int_{T_1}^{t} \left\{ KH(t, s)\delta(s)q(s) - \frac{h_-^{\alpha+1}(t, s) r_1^{\alpha}(s)}{(1+\alpha)^{(1+\alpha)}[\delta(s) R_1(s, T_0)]^{\alpha}} \right\} \Delta s$$

$$= \limsup_{t\to\infty} \frac{1}{(t - T_1)^2} \int_{T_1}^{t} \left\{ 1 \cdot (t - s)^2 \cdot s \cdot \frac{1}{s^2} - \frac{h_-^{\frac{4}{3}}(t, s) \cdot s^{-\frac{1}{3}}}{\left(\frac{4}{3}\right)^{\frac{4}{3}} \cdot \left[s \cdot \frac{s^2 - T_0^2}{3} \right]^{\frac{1}{3}}} \right\} \Delta s$$

$$\geq \limsup_{t\to\infty} \frac{1}{(t - T_1)^2} \int_{T_1}^{t} \left\{ \frac{(t - s)^2}{s} - \frac{3^{\frac{5}{3}}}{4^{\frac{4}{3}}} \cdot \frac{(t - s)^2}{s^{\frac{2}{3}} \cdot [s^2 - T_0^2]^{\frac{1}{3}}} \right\} \Delta s$$

$$\geq \limsup_{t\to\infty} \frac{1}{(t - T_1)^2} \int_{T_1}^{t} \left\{ \left[1 - \frac{3^{\frac{5}{3}}}{4^{\frac{4}{3}}} \right] \cdot \frac{(t - s)^2}{s} \right\} \Delta s = \infty, \quad T_1 \geq T_0,$$

so (3.18), (3.19) hold. Then, by Theorem 3.2, every solution $x(t)$ of (3.27) is either oscillatory or converges to zero.

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