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ON THE Δ-CONVERGENCE OF MODIFIED *S*-ITERATION SCHEMES FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE NON-SELF MAPPINGS IN A CAT(0) SPACE

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Abstract

In this paper, the modified *S*-iteration schemes are defined for approximating a fixed point of total asymptotically nonexpansive non-self mappings in CAT(0) spaces. Some Δ-convergence theorems are proved under suitable conditions in CAT(0) spaces.

1. Introduction

In [1], Agarwal et al. introduced the *S*-iteration process and modified *S*-iteration process in a Banach space

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - a_n)Tx_n + a_nTy_n, \quad n \in N, \\ y_n = (1 - b_n)x_n + b_nTx_n, \end{cases} \tag{1.1}$$

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$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - a_n)T^n x_n + a_n T^n y_n, \quad n \in N, \\ y_n = (1 - b_n)x_n + b_n T^n x_n, \end{cases} \tag{1.2}$$

where the sequences $\{a_n\}$ and $\{b_n\}$ are in $(0, 1)$.

These iterations have evoked many authors' great interest, see [2-5] for details.

Fixed point theory in a CAT(0) space has been first studied by Kirk [7]. He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory in a CAT(0) space has rapidly developed and many papers have appeared (see, e.g. [9-22]). In [15], Dhompongsa and Panyanak obtained Δ-convergence theorems for the Mann and Ishikawa iterations for nonexpansive single-valued mappings in CAT(0) spaces. Laowang and Panyanak [17] extended results of [15] for nonexpansive non-self mappings in CAT(0) spaces.

The purpose of this paper is to construct a modified *S*-iteration process for approximating a fixed point of total asymptotically nonexpansive non-self mappings. Some Δ-convergence theorems are proved under suitable conditions in CAT(0) spaces. Our results improve and extend the corresponding recent results in [2, 5, 17, 21, 22].

2. Preliminaries

A geodesic space is said to be a CAT(0) if all geodesic triangles of appropriate sizes satisfy the following comparison axiom.

Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle for Δ.

A geodesic space X is a CAT(0) space. Then geodesic triangle Δ is said to satisfy the CAT(0) inequality if or all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta} \subset E^2$ (Euclidean space) such that $d(x, y) \le d(\bar{x}, \bar{y})$.

In this paper, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = td(x, y), \quad d(z, y) = (1-t)d(x; y). \tag{2.1}$$

A subset C of a CAT(0) space is convex if $[x, y] \subset C$ for all $x, y \in C$.

The following lemma plays an important role in this paper.

Lemma 2.1 [15]. *A geodesic space X is a CAT*(0) *space if and only if the following inequality holds*:

$$d^2(z, (1-t)x \oplus ty) \leq (1-t)d^2(z, x) + td^2(z, y) - t(1-t)d^2(x, y) \tag{2.2}$$

for all $x, y, z \in X$ *and all* $t \in [0, 1]$. *In particular*, *if* $x, y, z \in X$ *and* $t \in [0, 1]$, *then*

$$d(z, (1-t)x \oplus ty) \leq (1-t)d(z, x) + td(z, y). \tag{2.3}$$

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n). \tag{2.4}$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}. \tag{2.5}$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}. \tag{2.6}$$

Proposition 2.2 [14]. *Let X be a complete CAT*(0) *space*, $\{x_n\}$ *be a bounded sequence in X and C be a closed convex subset of X. Then*

(1) *there exists a unique point* $u \in C$ *such that* $r(u, \{x_n\}) = \inf_{x\in C} r(x, \{x_n\})$;

(2) $A(\{x_n\})$ *and* $A_C(\{x_n\})$ *both are singleton.*

Definition 2.3 [6, 9]**.** Let X be a CAT(0) space. A sequence $\{x_n\}$ in X is said to Δ-*converge* to $q \in X$, if q is the unique asymptotic center of $\{u_n\}$ for each subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = q$ and call q the Δ-*limit* of $\{x_n\}$.

Lemma 2.4 [15]**.** *Let X be a complete CAT*(0) *space and $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{q\}$; $\{u_n\}$ is sequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges. Then $q = u$.*

Lemma 2.5. (1) *Let X be a complete CAT*(0) *space, C be a closed convex subset of X. If $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C* [11].

(2) *Every bounded sequence in a complete CAT*(0) *space always has Δ-convergent subsequence* [9].

Lemma 2.6 [21]**.** *Let X be a complete CAT*(0) *space and $x \in X$. Suppose $\{t_n\}$ is a sequence in $(0, 1)$ and $\{x_n\}, \{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \le r, \qquad \limsup_{n\to\infty} d(y_n, x) \le r$$

and

$$\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$$

for some $r \ge 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 2.7 [21]**.** *Let $\{a_n\}$, $\{\lambda_n\}$ and $\{c_n\}$ be the sequences of nonnegative numbers such that*

$$a_{n+1} \le (1+\lambda_n)a_n + c_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If there exists a subsequence of $\{a_n\}$ which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Let (X, d) be a metric space, and let C be a nonempty and closed subset of X. Recall that C is said to be a *retract* of X if there exists a continuous map $P : X \to C$ such that $Px = x$, $\forall x \in C$. A map $P : X \to C$ is said to be a *retraction* if $P^2 = P$. If P is a retraction, then $Py = y$ for all y in the range of P.

Definition 2.8. A non-self mapping $T : C \to X$ is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \le Ld(x, y), \quad \forall n \ge 1, x, y \in C. \tag{2.7}$$

Definition 2.9. Let K be a nonempty subset of X and $T : K \to X$ is said to be $(\{\mu_n\}, \{\upsilon_n\}, \rho)$ *total asymptotically nonexpansive non-self mapping* if there exist nonnegative sequences $\{\mu_n\}$ and $\{\upsilon_n\}$ with $\mu_n \to 0$, $\upsilon_n \to 0$ and a strictly increasing continuous function $\rho : [0, \infty) \to [0, \infty)$ with $\rho(0) = 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \le d(x, y) + \upsilon_n \rho(d(x, y)) + \mu_n,$$
$$\forall n \ge 1, x, y \in C, \tag{2.8}$$

where P is a nonexpansive retraction of X onto C.

Remark 2.10. It is to know that each nonexpansive non-self mapping is an asymptotically nonexpansive non-self mapping with a sequence $\{k_n = 1\}$, and each asymptotically nonexpansive mapping is a $(\{\mu_n\}, \{\upsilon_n\}, \rho)$ total asymptotically nonexpansive mapping with $\mu_n = 0$, $\upsilon_n = k_n - 1$, $\forall n \ge 1$ and $s(t) = t$, $t \ge 0$.

Recently, Wang et al. [22] proved the demiclosed principle total asymptotically nonexpansive non-self mappings in CAT(0) spaces. We cite as following two lemmas.

Lemma 2.11. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : K \to X$ be a uniformly L-Lipschitzian and*

$(\{\mu_n\}, \{\upsilon_n\}, \rho)$ *total asymptotically nonexpansive non-self mapping. Let* $\{x_n\}$ *be a bounded sequence in K such that* $\{x_n\} \rightharpoonup q$ *and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. *Then* $Tq = q$.

Lemma 2.12. *Let K be a nonempty closed convex subset of a complete CAT*(0) *space X and* $T : K \to X$ *be an asymptotically nonexpansive non-self mapping with a sequence* $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. *Let* $\{x_n\}$ *be a bounded sequence in K such that* $\Delta - \lim_{n\to\infty} x_n = q$ *and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. *Then* $Tq = q$.

3. Main Results

Theorem 3.1. *Let K be a nonempty, closed and convex subset of a complete CAT*(0) *space X. Let* $T_i : K \to X$ *be a uniformly L-Lipschitzian and total asymptotically nonexpansive non-self with sequence* $\{\mu_n^{(i)}\}$ *and* $\{v_n^{(i)}\}$ *satisfying* $\lim_{n\to\infty} \mu_n^{(i)} = 0$ *and* $\lim_{n\to\infty} \upsilon_n^{(i)} = 0$, *and strictly increasing function* $\rho^{(i)} : [0, \infty) \to [0, \infty)$ *with* $\rho^{(i)}(0) = 0$, $i = 1, 2$. *Let* $\{x_n\}$ *be defined as follows*:

$$\begin{cases} x_1 \in K, \\ y_n = P((1 - b_n)x_n \oplus b_n T_2(PT_2)^{n-1}x_n), \\ x_{n+1} = P((1 - a_n)T_1(PT_1)^{n-1}x_n \oplus a_n T_1(PT_1)^{n-1}y_n), \end{cases} \tag{3.1}$$

where $\{\mu_n^{(i)}\}, \{\upsilon_n^{(i)}\}, \rho^{(i)}, \{a_n\}$ *and* $\{b_n\}$ *satisfy the following conditions*:

(1) $\sum \mu_n < \infty, \sum \upsilon_n < \infty$;

(2) *there exist constants* $a, b \in (0, 1)$ *with* $0 < b(1 - a) \le \frac{1}{2}$ *such that* $\{a_n\}, \{b_n\} \subset [a, b]$;

(3) *there exists a constant* $M > 0$ *such that* $\rho(r) \le rM$, $r \ge 0$;

(4) $d(x_n, T_1(PT_1)^{n-1}y_n) \leq d(T_1(PT_1)^{n-1}x_n, T_1(PT_1)^{n-1}y_n)$.

Then the sequence $\{x_n\}$ *defined in* (3.1) Δ-*converges to a fixed point of T*.

Proof. We separate our proof in four steps as follows:

Step 1. Set $\mu_n = \max\{\mu_n^{(1)}, \mu_n^{(2)}\}$ and $\upsilon_n = \max\{\upsilon_n^{(1)}, \upsilon_n^{(2)}\}$, $n = 1, 2, \ldots, \infty$ such that $\sum \mu_n < \infty$ and $\sum \upsilon_n < \infty$.

For any $q \in F(T_1) \cap F(T_2)$, we have

$$\begin{aligned} d(x_{n+1}, q) &= d(P(1-a_n)T_1(PT_1)^{n-1}x_n \oplus a_nT_1(PT_1)^{n-1}y_n, q) \\ &\leq d((1-a_n)T_1(PT_1)^{n-1}x_n \oplus a_nT_1(PT_1)^{n-1}y_n, q) \\ &\leq (1-a_n)d(T_1(PT_1)^{n-1}x_n, q) + a_n(T_1(PT_1)^{n-1}y_n, q) \\ &\leq (1-a_n)(d(x_n, q) + \upsilon_n\rho(d(x_n, q)) + \mu_n) \\ &\quad + a_n(d(y_n, q) + \upsilon_n\rho(d(y_n, q)) + \mu_n) \\ &\leq (1-a_n)(d(x_n, q) + \upsilon_nMd(x_n, q) + \mu_n) \\ &\quad + a_n((1+\upsilon_nM)d(y_n, q) + \mu_n) \\ &= (1-a_n)(1+\upsilon_nM)d(x_n, q) + a_n(1+\upsilon_nM)d(y_n, q) + \mu_n, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} d(y_n, q) &= d(P(1-b_n)x_n \oplus b_nT_2(PT_2)^{n-1}x_n, q) \\ &\leq (1-b_n)d(x_n, q) + b_nd(T_2(PT_2)^{n-1}x_n, q) \\ &\leq (1-b_n)d(x_n, q) + b_n(d(x_n, q) + \upsilon_n\rho(d(x_n, q)) + \mu_n) \\ &\leq (1+b_n\upsilon_nM)d(x_n, q) + b_n\mu_n. \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$\begin{aligned} d(x_{n+1}, q) &\le (1 - a_n)(1 + \upsilon_n M) d(x_n, q) \\ &\quad + a_n(1 + \upsilon_n M)(1 + b_n \upsilon_n M) d(x_n, q) \\ &\quad + a_n(1 + \upsilon_n M) b_n \mu_n + \mu_n \\ &= (1 + (1 + a_n b_n + a_n b_n \upsilon_n M)\upsilon_n M) d(x_n, q) \\ &\quad + (1 + (1 + \upsilon_n M) a_n b_n)\mu_n. \end{aligned} \tag{3.4}$$

Since $\sum \mu_n < \infty$ and $\sum \upsilon_n < \infty$, it follows from Lemma 2.7 that $\lim_{n\to\infty} d(x_n, q)$ exists for each $q \in F(T_1) \cap F(T_2)$.

Step 2. For each $q \in F(T_1) \cap F(T_2)$, we assume that $\lim_{n\to\infty} d(x_n, q) = r$. From (3.3), we have

$$d(y_n, q) \le (1 + b_n \upsilon_n M) d(x_n, q) + b_n \mu_n \tag{3.5}$$

and

$$\lim_{n\to\infty} \inf d(y_n, q) \le \lim_{n\to\infty} \sup d(y_n, q) \le r. \tag{3.6}$$

In addition, since

$$d(T_1(PT_1)^{n-1} y_n, q) \le (1 + \upsilon_n M) d(y_n, q) + \mu_n, \tag{3.7}$$

we have $\lim_{n\to\infty} \sup d(T_1(PT_1)^{n-1} y_n, q) \le r$, similarly, we also can show that

$$\lim_{n\to\infty} \sup d(T_2(PT_2)^{n-1} y_n, q) \le r, \quad \lim_{n\to\infty} \sup d(T_1(PT_1)^{n-1} x_n, q) \le r$$

and

$$\lim_{n\to\infty} \sup d(T_2(PT_2)^{n-1} x_n, q) \le r.$$

Since $\lim_{n\to\infty} d(x_{n+1}, q) = r$, it is easy to prove that

$$\lim_{n\to\infty} d((1-a_n)T_1(PT_1)^{n-1}x_n \oplus a_nT_1(PT_1)^{n-1}y_n, q) = r. \tag{3.8}$$

It follows from Lemma 2.6 that

$$\lim_{n\to\infty} d(T_1(PT_1)^{n-1}x_n, T_1(PT_1)^{n-1}y_n) = 0. \tag{3.9}$$

It follows from condition (4), we have that

$$\begin{aligned} d(x_n, T_1(PT_1)^{n-1}x_n) &\le d(x_n, T_1(PT_1)^{n-1}y_n) \\ &\quad + d(T_1(PT_1)^{n-1}y_n, T_1(PT_1)^{n-1}x_n) \\ &\le 2d(T_1(PT_1)^{n-1}x_n, T_1(PT_1)^{n-1}y_n). \end{aligned} \tag{3.10}$$

According (3.9), we obtain

$$\lim_{n\to\infty} d(x_n, T_1(PT_1)^{n-1}x_n) = 0. \tag{3.11}$$

Since

$$\begin{aligned} d(x_n, T_1x_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_1(PT_1)^n x_{n+1}) \\ &\quad + d(T_1(PT_1)^{n-1}x_{n+1}, T_1(PT_1)^n x_n) + d(T_1(PT_1)^n x_{n+1}, T_1x_n) \\ &\le (1+L)d(x_{n+1}, x_n) + d(x_{n+1}, T_1(PT_1)^n x_{n+1}) \\ &\quad + Ld(T_1(PT_1)^{n-1}x_n, x_n) \\ &\le (1+L)d(x_n, (1-a_n)T_1(PT_1)^{n-1}x_n \oplus a_nT_1(PT_1)^{n-1}y_n) \\ &\quad + d(x_{n+1}, T_1(PT_1)^n x_{n+1}) + Ld(T_1(PT_1)^{n-1}x_n, x_n) \\ &\le (1-a_n + (2-a_n)L)d(x_n, T_1(PT_1)^{n-1}x_n) \\ &\quad + d(x_{n+1}, T_1(PT_1)^n x_{n+1}) + (1+L)a_nd(x_n, T_1(PT_1)^{n-1}y_n) \end{aligned}$$

$$\leq (1+2L)d(x_n, T_1(PT_1)^{n-1}x_n) + d(x_{n+1}, T_1(PT_1)^n x_{n+1})$$

$$+ (1+L)a_n d(T_1(PT_1)^{n-1}x_n, T_1(PT_1)^{n-1}y_n), \tag{3.12}$$

it follows from (3.9) again and (3.11) that $\lim_{n\to\infty} d(x_n, T_1x_n) = 0.$

Similarly, we also can prove that $\lim_{n\to\infty} d(x_n, T_2x_n) = 0.$

In fact, observe that we have

$$r = \lim_{n\to\infty} d(y_n, q) = \lim_{n\to\infty} d(P(1-b_n)x_n \oplus b_nT_2(PT_2)^{n-1}x_n, q)$$

$$\leq \lim_{n\to\infty}\{(1+b_n\upsilon_nM)d(x_n, q) + b_n\mu_n\} \leq r. \tag{3.13}$$

So we have that

$$\lim_{n\to\infty} d((1-b_n)x_n \oplus b_nT_2(PT_2)^{n-1}x_n, q) = r. \tag{3.14}$$

By assuming the same proof in Step 2, we have $\lim_{n\to\infty} d(x_n, q) = r$ and $\lim_{n\to\infty} \sup d(T_2(PT_2)^{n-1}x_n, q) \leq r.$

From Lemma 2.6, we have that

$$\lim_{n\to\infty} d(x_n, T_2(PT_2)^{n-1}x_n) = 0. \tag{3.15}$$

Finally, since

$$d(x_n, T_2x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T_2(PT_2)^n x_{n+1})$$

$$+ d(T_2(PT_2)^n x_{n+1}, T_2(PT_2)^n x_n) + d(T_2(PT_2)^n x_{n+1}, T_2x_n)$$

$$\leq (1+L)d(x_{n+1}, x_n) + d(x_{n+1}, T_2(PT_2)^n x_{n+1})$$

$$+ Ld(T_2(PT_2)^{n-1}x_n, x_n)$$

$$\leq (1+L)d(x_n,\, (1-a_n)T_1(PT_1)^{n-1}x_n \oplus a_nT_1(PT_1)^{n-1}y_n)$$

$$+\, d(x_{n+1},\, T_2(PT_2)^n x_{n+1}) + Ld(T_2(PT_2)^{n-1}x_n,\, x_n)$$

$$\leq (1+L)(1-a_n)d(x_n,\, T_1(PT_1)^{n-1}x_n)$$

$$+\, (1+L)a_nd(T_1(PT_1)^{n-1}x_n,\, T_1(PT_1)^{n-1}y_n)$$

$$+\, d(x_{n+1},\, T_2(PT_2)^n x_{n+1}) + Ld(T_2(PT_2)^{n-1}x_n,\, x_n), \quad (3.16)$$

it follows from (3.9) again and (3.15) that $\lim_{n\to\infty} d(x_n,\, T_2x_n) = 0.$ This is the end of proof of Step 2.

Step 3. Let $W_w(\{x_n\}) := \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\}).$

Let $u \in W_w.$ Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}.$ By Lemma 2.5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in K.$ Since $\lim_{n\to\infty} d(x_n,\, T_1x_n) = \lim_{n\to\infty} d(x_n,\, T_2x_n) = 0,$ it follows from Lemma 2.11 that $v \in F(T_1) \bigcap F(T_2).$ So, $\lim_{n\to\infty} d(x_n,\, q)$ exists. By Lemma 2.4, we have that $q = v \in F(T_1) \bigcap F(T_2).$ This implies that $W_w(\{x_n\}) \subset F(T_1) \bigcap F(T_2).$

Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}.$ Since $u \in W_w(x_n) \subset F(T_1) \bigcap F(T_2),$ according the conclusion of Step 1 we know that $\{d(x_n,\, u)\}$ is convergent.

By Lemma 2.4, $x = u.$ This implies that $W_w(\{x_n\})$ contains only one point.

Step 4. In fact, it follows the conclusion of Step 1 that $\{d(x_n,\, q)\}$ is convergent for each $q \in F(T_1) \bigcap F(T_2).$ According to the conclusion of Step 2, $\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = 0.$ Since $W_w(\{x_n\}) \subset F(T),$ and

$W_w(\{x_n\})$ consists of exactly one point. This shows that $\{x_n\}$ Δ-converges to a point of $F(T)$.

Theorem 3.2. *Let K be a nonempty closed and convex subset of a complete CAT*(0) *space X. Let* $T : K \to X$ *be a uniformly L-Lipschitzian and total asymptotically nonexpansive non-self with sequence* $\{\mu_n\}$ *and* $\{v_n\}$ *satisfying* $\lim_{n\to\infty} \mu_n = 0$ *and* $\lim_{n\to\infty} \upsilon_n = 0$, *and strictly increasing function* $\rho : [0, \infty) \to [0, \infty)$ *with* $\rho(0) = 0$. *Let* $\{x_n\}$ *be defined as follows*:

$$\begin{cases} x_1 \in K, \\ y_n = P((1-b_n)x_n \oplus b_n T(PT)^{n-1} x_n), \\ x_{n+1} = P((1-a_n)T(PT)^{n-1} x_n \oplus a_n T(PT)^{n-1} y_n), \end{cases} \tag{3.17}$$

where $\{\mu_n\}$, $\{\upsilon_n\}$, ρ, $\{a_n\}$ *and* $\{b_n\}$ *satisfy the following conditions*:

(1) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \upsilon_n < \infty$;

(2) *there exist constants* $a, b \in (0, 1)$ *with* $0 < b(1-a) \le \frac{1}{2}$, *such that* $\{a_n\}, \{b_n\} \subset [a, b]$;

(3) *there exists a constant* $M^* r$, $r > 0$ *such that* $\rho(r) \le M^* r$, $r \ge 0$.

Then the sequence $\{x_n\}$ *defined in* (3.17) Δ-*convergences to a fixed point of T*.

Proof. It can put $S = T$ in Theorem 3.1 such that we use similar steps to obtain our conclusion.

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