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INDEPENDENCE SATURATION NUMBER OF SOME CLASSES OF GRAPHS

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Abstract

Let $IS(v, G)$ denote the maximum cardinality among all independent sets of G containing v. Then $IS(G) = \min\{IS(v) : v \in V(G)\}$ is called the independence saturation number of G. In this paper, we compute the independence saturation number of some classes of graphs such as central graph, total graph, line graph of star graph $K_{1,n}$ and double star graph $K_{1,n,n}$, central graph of cycle graph, expansion graphs, corona graphs, Mycielskian graphs and maximal triangle free graphs.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. The *neighbourhood* of a vertex $x \in V(G)$ in the graph G is denoted by $N(x)$ and the *closed neighbourhood* $\{x\} \cup N(x)$ by $N[x]$. If X is a subset of $V(G)$, then $N[X] = \bigcup_{x \in X} N[x]$ and the subgraph induced by

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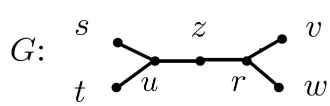
X is denoted by $G[X]$. Terms not defined here are used in the sense of Harary [11].

Acharya [1] initiated a study of domsaturation. Arumugam and Kala [3, 4, 5] obtained further results on domsaturation, connected domsaturation and global domsaturation. A collection of articles in graph saturation parameters have been focussed on [2, 8, 15]. Arumugam and Subramanian [6, 15] introduced the concept of *independence saturation number*. A subset S of V in a graph G is said to be *independent* if no two vertices in S are adjacent. The minimum cardinality of a maximal independent set is called the *independent domination number* of G and is denoted by $i(G)$. The maximum cardinality of an independent set in G is called the *independence number* of G and is denoted by $\beta_0(G)$. Let $IS(v, G)$ denote the maximum cardinality among all independent sets of G containing v. Then $IS(G) = \min\{IS(v) : v \in V(G)\}$ is called the *independence saturation number* of G. A vertex $v \in V$ is called an *IS-vertex* if $IS(v) = IS(G)$. Let $v \in V$ be such that $IS(v) = IS(G)$. Any maximal independent set of cardinality $IS(G)$ containing v is called an *IS-set* of G. Thus *IS*-set is a maximal independent set and hence is a dominating set. Hence $i(G) \leq IS(G) \leq \beta_0(G)$.

In the graph shown below, $i(G) = 2$, $\beta_0(G) = 5$ and $IS(G) = 3$.



For the above graph, u and r are the *IS*-vertices. $\{u, v, w\}$ is an *IS*-set containing u and $\{r, s, t\}$ is an *IS*-set containing r. Hence $IS(u) = IS(r) = IS(G) = 3$. But $IS(v) = IS(w) = IS(s) = IS(t) = IS(z) = 5$. Hence $IS(G) = \min\{3, 5\} = 3$. Basic results are as follows:

Observation 1.1 [6]**.** (i) For the cycle C_n of length n, $IS(v) = \lfloor n/2 \rfloor$ for every vertex v and hence $IS(C_n) = \lfloor n/2 \rfloor$.

(ii) $IS(K_n) = 1$ and $IS(K_{m,n}) = \min\{m, n\}$.

Theorem 1.2 [9]**.** *Let G be a maximal triangle free graph of order* $n \geq 2$ *and minimum degree* $\delta(G)$. *For every vertex* $v \in V(G)$, $N_G(v)$ *is an independent domination set and so* $i(G) \leq \delta(G) \leq \lfloor n/2 \rfloor$.

Proposition 1.3. [10] *For any graph G,*

(a) $i(\exp(G, r)) = r \cdot i(G)$,

(b) $i(cor(G, r)) = r|V(G)| - (r-1)\alpha(G)$.

2. Further Results on Independence Saturation Number

In the following propositions, we investigate the independence saturation number of the central graph of star graph $K_{1,n}$, double star $K_{1,n,n}$ and cycle graph C_n. Also we compute independence saturation number for the total graph, line graph of star graph $K_{1,n}$, and double star graph families $K_{1,n,n}$ as defined in [16, 19].

A study of harmonious, achromatic coloring on middle graph, central graph, total graph, line graph of various classes of graphs can be found in [16, 17, 18, 19]. Double star $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the existing *n* pendant vertices. It has $2n+1$ vertices and 2*n* edges. Let

$$V(K_{1,n,n}) = \{v\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$$

and

$$E(K_{1,n,n}) = \{e_1, e_2, \ldots, e_n\} \cup \{s_1, s_2, \ldots, s_n\}.$$

The central graph $C(G)$ of a graph *G* is formed by adding an extra vertex on each edge of *G* and then joining each pair of vertices of the original graph which were previously non-adjacent. The total graph of *G* has vertex set $V(G) \cup E(G)$, and edges joining all elements of this vertex set which are adjacent or incident in *G*. The line graph of *G* denoted by $L(G)$ is the graph with vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

Proposition 2.1. *For any star graph* $K_{1,n}$, *we have*

(i) $IS(C(K_{1,n})) = 2$,

(ii) $IS(T(K_{1,n})) = 1$,

(iii) $IS(L(K_{1,n})) = 1$.

Proof. (i) By the definition of central graph, each edge vv_i in $K_{1,n}$ is subdivided by the vertex e_i in $C(K_{1,n})$ and the vertices $v_1, v_2, \ldots, v_n$ induce a clique of order n in $C(K_{1,n})$, i.e.,

$$V(C(K_{1,n})) = \{v\} \cup \{v_i : 1 \le i \le n\} \cup \{e_i : 1 \le i \le n\}.$$

Since v_i $(1 \le i \le n)$ induce a clique of order n and v_i is adjacent to e_i, $\{v_i\} \cup \{e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\}$ is a maximum independent set containing v_i $(1 \le i \le n)$. Hence, $IS(v_i) = n$. Also $\{e_1, e_2, \ldots, e_i, \ldots, e_n\}$ is a maximum independent set containing e_i $(1 \le i \le n)$. Hence, $IS(e_i) = n$. Moreover $\{v, v_i\}$ is a maximum independent set of $C(K_{1,n})$ containing v. Hence, $IS(v) = 2$ and so $IS(C(K_{1,n})) = 2$.

(ii) By the definition of total graph, we have $V(T(K_{1,n})) = \{v\} \cup \{v_i : 1 \le i \le n\} \cup \{e_i : 1 \le i \le n\}$, in which the vertices $v, e_1, e_2, \ldots, e_n$ induce a clique of order $n+1$. Since v is adjacent to all the vertices of $T(K_{1,n})$, $IS(v) = 1$ and so $IS(T(K_{1,n})) = 1$.

(iii) Since $L(K_{1,n}) \cong K_n$, $IS(L(K_{1,n})) = 1$. □

Proposition 2.2. *For any double star graph* $K_{1,n,n}$, *we have*

(i) $IS(C(K_{1,n,n})) = n+1$,

(ii) $IS(T(K_{1,n,n})) = n+1$,

(iii) $IS(L(K_{1,n,n})) = n$.

Proof. (i) By the definition of central graph, each edge vv_i and v_iu_i $(1 \le i \le n)$ in $K_{1,n,n}$ are subdivided by the vertices e_i and s_i in $C(K_{1,n,n})$. The vertices $v, u_1, u_2, ..., u_n$ induce a clique of order $n+1$ (say K_{n+1}) and the vertices v_i $(1 \le i \le n)$ induce a clique of order n in $C(K_{1,n,n})$, i.e., $V(C(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n\} \cup \{e_i : 1 \le i \le n\}$ $\cup \{s_i : 1 \le i \le n\}$. Now $\{v\} \cup \{s_i : 1 \le i \le n\}$ is a maximum independent set containing v. Hence, $IS(v) = n+1$. Then $\{e_i : 1 \le i \le n\} \cup \{s_i : 1 \le i \le n\}$ is a maximum independent set containing e_i (or s_i). Hence, $IS(e_i) = 2n$ and $IS(s_i) = 2n$. Also note that

$$\{v_i\} \cup \{u_i\} \cup \{e_1, e_2, ..., e_{i-1}, e_{i+1}, ..., e_n\} \cup \{s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n\}$$

is a maximum independent set containing v_i (or u_i). Hence, $IS(v_i) = 2n$ and $IS(u_i) = 2n$. Hence, $IS(G) = \min\{n+1, 2n\} = n+1$.

(ii) By the definition of total graph, we have

$$V(T(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n\}$$
$$\cup \{e_i : 1 \le i \le n\} \cup \{s_i : 1 \le i \le n\}$$

in which the vertices $v, e_1, e_2, ..., e_n$ induce a clique of order $n+1$. Note that $\{v\} \cup \{s_i : 1 \le i \le n\}$ is a maximum independent set containing v (or s_i). Hence, $IS(v) = n+1$ and $IS(s_i) = n+1$. Then for any $j = 1$ to n, $\{e_j\} \cup$ $\{u_i : 1 \le i \le n\}$ is a maximum independent set containing e_i (or u_i), $i = 1$ to n. Hence, $IS(e_i) = n+1$ and $IS(u_i) = n+1$. Also $\{v_i\} \cup [\cup_{j \ne i} u_j] \cup$ $\{e_j : j \ne i\}$ is a maximum independent set containing v_i. Hence $IS(v_i) =$ $1 + n - 1 + 1 = n+1$. Therefore, $IS(T(K_{1,n,n})) = n+1$.

(iii) By the definition of line graph, each edge of $K_{1,n,n}$ taken to be as vertex in $(L(K_{1,n,n}))$. The vertices $e_1, e_2, \ldots, e_n$ induce a clique of order n and the vertices $s_1, s_2, \ldots, s_n$ are all pendant in $(L(K_{1,n,n}))$, i.e., $V(L(K_{1,n,n})) = \{e_i : 1 \le i \le n\} \cup \{s_i : 1 \le i \le n\}$. We observe that $\{e_i\} \cup [\cup_{j \ne i} s_j]$ is a maximum independent set containing e_i. Hence, $IS(e_i) = n$. Then $\{s_i : 1 \le i \le n\}$ is a maximum independent set containing s_i. Hence, $IS(s_i) = n$ and so $IS(L(K_{1,n,n})) = n$. □

Proposition 2.3. *For any cycle* $C_n = (v_1, v_2, \ldots, v_n)$, *we have* $IS(C(C_n)) = n - 1$

Proof. By the definition of central graph, each edge $v_i v_j$ $(i < j$ and $1 \le i, j \le n)$ in C_n is subdivided by the vertex e_i in $C(C_n)$ and $\deg(v_i) = n - 1$, $\deg(e_i) = 2$. Note that $\{v_i, v_{i+1}, e_{i+2}, e_{i+3}, \ldots, e_{i+n-2}\}$ is a maximum independent set of $C(C_n)$ containing v_i $(1 \le i \le n)$. Hence, $IS(v_i) = n - 2 + 1 = n - 1$. Also $\{e_1, e_2, \ldots, e_n\}$ is a maximum independent set containing e_i. Hence, $IS(e_i) = n$ $(1 \le i \le n)$ and so $IS(G) = n - 1$. □

In the following proposition, we determine the independence saturation number of expansion and corona of graphs. For r a positive integer, the expansion $\exp(G; r)$ of a graph G is the graph obtained from G by replacing each vertex v of G with an independent set I_v of size r and replacing each edge vw by a complete bipartite graph with partite sets I_v and I_w. The corona $cor(G)$ (sometimes denoted $G \circ K_1$) is the graph obtained from G by adding a pendant edge at each vertex of G. In general, the generalized corona $cor(G; r)$ is the graph obtained from G by adding r pendant edges to each vertex of G.

Proposition 2.4. *For any graph G,*

(a) $IS(\exp(G, r)) = r \cdot IS(G)$,

(b) $IS(cor(G, r)) = r|V(G)| - r + 1$.

Proof. (a) Let D be any independent set of $\exp(G, r)$. Each set I_v in D that corresponds a vertex v in G. Note that $\{v : I_v \subseteq D\}$ is an independent set of G. Let z be any IS-vertex of G and S be any IS-set of G containing z in G. Then $IS(G) = |S| = \min\{IS(v)\}$. For every two vertices v, w in S, there corresponds two sets I_v, I_w in $\exp(G, r)$. Since v and w are non-adjacent, all the vertices of $I_v \cup I_w$ are independent. Hence $\cup I_v$, $v \in S$ is a maximum independent set containing $w, w \in I_z$. Hence, $IS(\exp(G, r)) = r.IS(G)$.

(b) Let D be any maximum independent set of $cor(G, r)$. For every vertex v of G, D contains all of r leaves adjacent to v. Let $v \in cor(G, r)$ and I_v be any maximum independent set containing v. If v is not a leaf, then D contains $|V(G)| - 1$ leaves and v. Hence, $IS(v) = (|V(G)| - 1)r + 1 = r|V(G)| - r + 1$. If v is a leaf, then $IS(v) = r|V(G)|$. Therefore,

$$IS(G) = \min\{r|V(G)| - r + 1, r|V(G)|\} = r|V(G)| - r + 1. \qquad \square$$

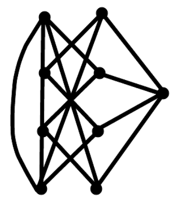
In Theorem 2.5, we investigate the independence saturation number $IS(G)$ of Mycielskian graph G. For a graph $G = (V, E)$, the *Mycielskian* of G is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and is disjoint from V, and edge set $E' = E \cup \{xy', x'y : xy \in E\} \cup \{x'u : x' \in V'\}$. The vertices x and x' are called *twins* of each other and u is called the *root* of $\mu(G)$. Also the graph $\mu(G) - u$ is called the *shadow graph* of G and is denoted by $Sh(G)$. A collection of articles related to Mycielskian graphs can be found in [12, 13, 14].

Theorem 2.5. *For any graph* G, $IS(\mu(G)) = \min\{\beta_0(G) + 1, 2IS(v, G)\}$, $v \in V(G)$.

Proof. Let u be the root of $\mu(G)$. Since u is adjacent to every v', $v \in V(G)$, β_0 set of $G \cup \{u\}$ is a maximum independent set containing u in $\mu(G)$. Hence, $IS(u, \mu(G)) = \beta_0 + 1$. Let $v' \in \mu(G)(G)$, $v \in V(G)$. Let $S = \{v, v_1, v_2, ..., v_n\}$ be any IS-set of G. Then $S' = \{v, v', v_1, v_1', v_2, v_2', ..., v_n, v_n'\}$

is a maximum independent set containing v in $\mu(G)$ and so $IS(v, \mu(G)) = 2IS(v, G)$. We observe that $\{v' : v \in V(G)\}$ is a maximal independent set containing v'. Hence, $IS(v', \mu(G)) = \max\{n(G), 2IS(v, G)\}$ and $IS(v, \mu(G)) \leq IS(v', \mu(G))$. Hence, $IS(\mu(G)) = \min\{\beta_0(G) + 1, 2IS(v, G)\}, v \in V(G)$. □

For example, $IS(\mu(C_4)) = 3$. The graph Mycielskian of C_4 is given below.



In the following theorem, we compute the independence saturation number of maximal triangle free graphs. A graph G is called *maximal triangle-free* (MTF) if G has no triangles but the addition of any edge produces a triangle. For instance, any complete bipartite graph is a maximal triangle-free graph.

Theorem 2.6. *Let G be a maximal triangle free graph of order $n \geq 2$ and minimum degree $\delta(G)$. Then $IS(G)$ is either $\delta(G)$ or $\Delta(G)$.*

Proof. Let $u \in V(G)$ and I_u be any maximum independent set containing u. Choose a vertex $v \in V(G)$ such that $u \in N(v)$. Since G is a maximal triangle free graph, $N(v)$ is an independent set containing u. We show that $N(v)$ is maximal. Suppose there exists $w \in V(G) - N[v]$ such that w is not adjacent to any vertex of $N(v)$. Then $G + uv$ is triangle free, it contradicts the fact that G is maximal triangle free graph. Hence, $N(v)$ is maximal and so $|I_u| \geq \deg(v)$. Since every vertex in G is of degree either $\delta(G)$ or $\Delta(G)$, $|I_u| \geq \delta(G)$ or $|I_u| \geq \Delta(G)$. Let I be any independent set of G containing u. Now we show that $I \subseteq N(v)$ for some v such that $u \in N(v)$. Suppose not, then there exists $z \in I$ such that $z \notin N(v)$ for every v such that $u \in N(v)$. Then $G + vz$ is triangle free, it contradicts the fact that G is maximal triangle free graph. Hence, $|I| \leq \deg(v)$ for some v such that $u \in N(v)$. Since every vertex in G is of degree either $\delta(G)$ or $\Delta(G)$ and $|I_u| \leq \deg(v)$ for some v such that $u \in N(v)$, $|I_u|$ is either $\delta(G)$ or $\Delta(G)$. Note that $IS(G) = \min\{|I_u| : u \in V(G)\}$. Hence, $IS(G)$ is either $\delta(G)$ or $\Delta(G)$. □

In Figure 1.1, we have $IS(G) = \delta(G) = 2$ and in Figure 1.2, we have $IS(G) = \Delta(G) = 3$.

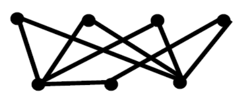


Figure 1.1

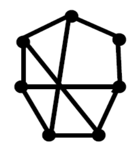


Figure 1.2

Theorem 2.7. *Let G be a maximal triangle free graph of order $n \geq 2$ and minimum degree $\delta(G)$. Then $IS(G) = \delta(G)$ if and only if there exists v such that $\deg(w) = \delta(G)$ for all $w \in N(v)$.*

Proof. Assume that $IS(G) = \delta(G)$. Suppose for every $v \in V(G)$, there exists $w \in N(v)$ such that $\deg(w) > \delta(G)$. Since G is MTF, $N(w)$ is a maximal independent set containing v. Hence, $IS(v) \geq \delta + 1$ and so $IS(G) \geq \delta + 1$. It contradicts the assumption. Conversely, to prove that $IS(v) = IS(G) = \delta(G)$. Since $N(w)$ is a maximal independent set containing v, $IS(v) \geq \delta(G)$. Let I be any independent set containing v. From the Proof of Theorem 2.6, $I \subseteq N(w)$ for some vertex w such that $w \in N(v)$. Hence, $IS(v) = \delta(G)$. Since $IS(G)$ is either $\delta(G)$ or $\Delta(G)$, $IS(v) = IS(G) = \delta(G)$. □

Conclusion and Scope

By the definition of $IS(G)$, we have $i(G) \leq IS(G) \leq \beta_0(G)$. It is clear that $IS(G)$ is equal to $i(G)$ for the graphs G mentioned in Proposition 2.1, Proposition 2.2 and Proposition 2.3. Hence, following is the interesting problem for further investigation.

Problem. Characterize the class of graphs G for which $IS(G) = i(G)$.

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