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UNDERGRADUATE STUDENTS' MISTAKES AND THEIR BEHAVIOR ATTENDING IN EXAMINATION: SPECIALLY IN GROUP THEORY

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Abstract

In this paper, we report on the evaluation of answer scripts of Mathematics (general paper) in the B.Sc. Second Semester Examination under Gauhati University, India which is held in the month of June 2012. We have taken sample of answer scripts from 10 different colleges under this university. In this examination, the question paper was in abstract algebra and matrices which was setting mainly on three topics, those are groups, rings and matrices giving equal weight to each topic. But it is observed that average 16% students give complete correct answers to the questions in groups, 37% in rings and 43% in matrices. So the aim of this study is to investigate how the professional teaching and learning method are effective on the concept of group theory for college students in university courses. And also to explore students' mistakes done in answering the questions (groups only) supplied by the university examination control cell. In this paper, it is also to investigate the students' behavior to attending compulsory and optional questions involving group theory in the examination hall.

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1. Introduction

The group theory is a part of abstract algebra. Abstract algebra has traditionally thought to be the most difficult course because of abstract nature of the course and, particularly, many students have shocked in group theory. The concept of group is an example of a new mental object the construction of which causes fundamental difficulties in the transition from school to university mathematics. As Robert and Schwargenberger [1] pointed out, one root of this difficulty is historical and epistemological. According to Nardi [2], the problems from which these concepts arose an essential manner are not assessable to students who are beginning to study (and expect to understand) the concepts today. There are lots of researchers (Almeida [3] and Hazzan and Zazkis [4]) who identified many problems in the teaching and learning of abstract algebra in their literatures. The researchers have also come to the conclusion that students' mistakes do not arise only from the complexity of the subject but also from weakness in the students' mathematical foundations. The concept such as group theory in an introduction to abstract algebra courses is, particularly, difficult for the students to learn and to maintain positive attitude (Campbell and Zazkis [5]). There has been a large number of researches on the teaching and learning of group theory. Some would like to make modern algebra and group theory more visual (Bardzel and Shannon [6] and Cox et al. [7]). Scientists such as Selden and Selden [11] and Hazzan and Leron [12] described misconceptions in abstract algebra. However, the current trend in pedagogy in abstract algebra has focused on the learning perspective.

In many colleges, abstract algebra is the first course for students in which they must go beyond learning "imitative behavior patterns" for mimicking the solution of a large number of variations on a small number of problems. In such a course, students must come to grips with abstract concepts, work with important mathematical principles, and learn to write proofs. Although there are no formal studies, many students report that this course, they tended to turn off from abstract mathematics. Since a significant percentage of the student audience for abstract algebra consists of future mathematics teachers, it is particularly important that the profession of mathematics education develops effective pedagogical strategies for improving the attitude of high school mathematics teachers towards mathematical abstraction. There is another reason, related to abstraction, for the importance of abstract algebra, in general, and quotient groups, in particular. An individual's knowledge of the concept of group should include an understanding of various mathematical properties and constructions independent of particular examples, indeed including groups consisting of undefined elements and a binary operation satisfying the axioms (Dubinsky et al. [14]). Even if one being with a very concrete group, the transition from the group to one of its quotients changes the nature of the elements and forces a student to deal with elements (e.g., cosets) that are, for her or him, undefined. This relationship between abstract groups and quotient groups has important historical antecedents (Nicholson [15]).

2. Method of Data Collection

For this study, we collect the necessary data from 184 answer scripts of B.Sc. (Bachelor of Science) Second Semester students under Gauhati University, India of the subject Mathematics (general). The Second Semester Examination held in the month of June 2012 and the sample answer scripts are taken from 10 colleges as C_1 (college code 1)-22, $C_2 - 11$, $C_3 - 24$, $C_4 - 19$, $C_5 - 18$, $C_6 - 5$, $C_7 - 26$, $C_8 - 29$, $C_9 - 8$ and $C_{10} - 22$. The data was collected in the duration of answer scripts evaluation process of first week of the month of July 2012, after finishing the Second Semester Examination in this university. Whenever we are going to collect the data it is keeping in mind to micro observing the correctness of the particular answer for each question and the part of the each question where are applicable.

3. Sample of Original Question Paper

The original question paper (sample) is given in the Appendix A (Sample Paper). In this sample paper shown only the questions involve in group theory, because this study mainly focuses on the group theory, so the other parts ring theory and matrix theory are omitted to mention in this case. There are six questions which are given numbers as $1, 2, ..., 6.$ Each question has sub-questions and these are numbering like as $a, b, ...$ The total marks of this question paper is 60 and total time duration is scheduled 2 and half an hour. The marks for each question and each part of a question are mentioned in the right part of each question in the sample paper.

4. Method of Data Analysis

In this study, the method of data analysis is used mainly quantitative analysis as well as percentile analysis method. Recent time, it is observed that the number of students learning mathematics in higher education under Gauhati University as well as other universities in Assam, India gradually decrease. So in this study, quantitative analysis makes it important to analyse the data and also to investigate how much students will be successful to learning mathematics specially here count in group theory by professional teaching and learning method in colleges under this university. Since successful achievement in an examination will create positive attitude to the subject and it encourages to learning in higher education on that subject. As in this study used 184 answer scripts to collect the necessary data for the study whose are taken as sample from 10 different colleges under Gauhati University. So the evaluation process of answer scripts will be completely based on the role and regulation prescribed by Gauhati University, examination control cell. In the evaluation process give full marks for the correct answer against to each question and to each part of question as shown in the sample question paper. Here consider ‘0’ mark for not correct answer and partial marks are given to those answers whose are correct some steps or nearly to correct answer or done only little wrong steps against the question bearing 2 or more than 2 marks. For the 1 mark question, partial marks are not entertained. The details of 184 students’ scores of answering the questions in group theory are given in Table 1. In this table, we display the scores of the students for individual each question (including each part), so there are 18 questions in group theory. On the other hand, since each sub-question of 3 and 4 consists more than two parts, so we are given another Table 2 where the scores are count each sub-question is single question in group theory.

Table 1. Students' scores on group theory (details)

Question no. as original	Question parts	Correct answer		Partial correct answer		Not correct answer		No response		Go to other option	
		Total	%	Total	%	Total	%	Total	%	Total	%
1(a)	No part	58	32%	-	-	81	44%	45	24%	0	0%
1(b)	No part	93	51%	-	-	61	33%	30	16%	0	0%
1(c)	No part	38	21%	-	-	53	29%	93	51%	0	0%
2(a)	No part	34	18%	33	18%	63	34%	54	29%	0	0%
3(a)	1st part	62	34%	-	-	50	27%	24	13%	48	26%
	2nd part	34	18%	27	15%	53	29%	22	12%		
	3rd part	34	18%	25	14%	51	28%	26	14%		
3(b)	1st part	24	13%	-	-	13	8%	10	5%	137	74%
	2nd part	10	5%	10	5%	16	9%	11	6%		
	3rd part	22	12%	-	-	14	8%	11	6%		
	4th part	23	12%	-	-	13	7%	11	6%		
4(a)	1st part	38	21%	21	11%	87	47%	31	17%	7	4%
	2nd part	23	13%	19	10%	98	53%	37	20%		
4(b)	1st part	59	32%	-	-	53	29%	65	35%		
	2nd part	38	21%	26	14%	51	27%	62	34%		
4(c)	1st part	5	3%	-	-	1	0.5%	1	0.5%	177	96%
	2nd part	2	1%	3	2%	1	0.5%	1	0.5%		
4(d)	1st part	0	0%	0	0%	3	2%	4	2%		
	2nd part	0	0%	1	0.5%	1	0.5%	5	3%		

Table 2. Students' scores in questions 3 and 4

Questions	Complete correct		Partial correct		Not correct		Not response		Choose other option	
	Total	%	Total	%	Total	%	Total	%	Total	%
3(a)	10	5%	76	41%	32	17%	18	10%	48	26%
3(b)	6	3%	23	13%	10	5%	8	4%	137	74%
4(a)	15	8%	50	27%	83	45%	29	16%	7	4%
4(b)	33	18%	37	20%	47	26%	60	32%	7	4%
4(c)	2	1%	4	2%	1	0.5%	0	0%	177	96%
4(d)	0	0%	0	0%	1	0.5%	6	3%	177	96%

5. Discussion on Data

From the sample question paper, it is clear that questions 1 and 2 are compulsory*. Here the questions 1(a), (b), (c) and 2(a) involve in group theory which are compulsory. All the sub-questions of question 3 are optional** among the three topics groups, rings and matrices.

N.B. '∗' - compulsory questions which are important to score full marks.
'**' - optional questions which are not important to score full marks but there are given other questions against them.

There are two sub-questions 3(a) and 3(b) involved in group theory, one sub-question 3(c) in ring theory and two sub-questions 3(d) and 3(e) in matrices theory. On the other hand, the all sub-questions in questions 4, 5 and 6 are optional among the same topic. Here question 4 is in group theory. We give the number of students who are trying to give correct answer or partial correct answer or do not give correct answer/give wrong answer and also the students who do not respond as well as the number of students who are going to the other option for individual each question are shown in Table 1. Table 2 shows the number of students who are completely correct/partial correct/not correct/no response/go to other option to each sub-question of questions 3 and 4. Now by analyzing the data which are given in Table 1 and Table 2, we discuss in following sections. Since questions 1 and 2 are compulsory and questions 3 and 4 are optional, so we put the discussion in the two subsections separately.

5.1. Students behavior on the compulsory questions

The number of students gave correct answer in compulsory questions are maximum 93 (51%) for question 1(b) and minimum 34 (18%) for question 2(a). On the other hand, maximum 93 (51%) and minimum 30 (16%) students do not give response for the questions 1(c) and 1(b), respectively. In between the maximum 81 (44%) and minimum 53 (29%) students commit mistake for the compulsory questions.

5.2. Students behavior on the optional questions

As mentioned above, there are two types of optional questions: (i) all sub-questions of 3 are between group and others (ring and matrix) and (ii) all sub-questions 4 are among themselves in group. For choosing optional question for answering in an examination a student much care that he/she can give most correct answer to that question than others.

(i) 48 (26%) students take other option against the question 3(a) and on the other hand a large number 137 (74%) students take other option against the question 3(b) (Tables 1 and 2). The performance of students regard in these two questions is very significant. Only 10 (5%) and 6 (3%) students can give complete correct answer for the questions (including all parts of both) (a) and (b), respectively. A large portion 76 (41%) and 23 (13%) students give partial correct answer for questions (a) and (b) respectively. 32 (17%) students give wrong answer for (a) and 10 (5%) students give wrong answer for (b) (Table 2).

As the question 3(a) consists three parts shown in sample question paper, considering correctness for each part of this question we see that maximum 62 (34%) students give correct definition of group, i.e., 1st part of this question. For the parts 2nd and 3rd the same number 34 (18%) students give the correct answer and relatively same portion 27 (15%) and 25 (14%) students are done little mistake for this two parts, respectively (Table 1). In case the question 3(b) consists four parts, comparatively same portion of students 24 (13%), 22 (12%) and 23 (12.3%) for the parts 1st, 2nd and 4th give correct answer of this question, respectively, and only 10 (5%) are given correct answer and other 10 (5%) are given partial correct answer for the 3rd part.

(ii) In question 4, it is significance that a greater portion 177 (96%) students try to answer the pair of sub-questions a and b and on contrary only a few 7 (4%) students try to give answer the pair of questions c and d. So it is strongly produced that students much feel easy to answer the first pair (a and b) and they feel hard to answer the second pair (c and d). From this study, it is revealed that students' concepts are not clear in subgroup and normal subgroup as well as quotient group. If we observe Table 2, then we see that maximum 33 (18%) students give complete correct answer for question 4(b) and 15 (8%) give complete correct answer for 4(a) in first pair. But not a single student gives complete correct answer for 4(d) and only 2 (1%) students give complete correct answer for 4(c) (Table 2).

Again since the sub-questions a and b of the question 4 consist two parts of each, a maximum 59 (32%) students have capable to give correct answer for the 1st part of the question 4(b) and a maximum 98 (53%) students cannot give correct answer for the 2nd part of the question 4(a) (Table 1). But 5 (3% maximum) students are capable to give correct answer for the 1st part of the question 4(c) and not a single student can give correct answer for any other part of question 4(d).

6. Type of Students Mistakes

In this section, we explore the type of mistake done by the students in the examination hall. Since students' written response or answer (correct/not correct) will sign of his/her knowledge in the specific subject or topic, the mistake or wrong answer will also give some sign of their learning process of a particular subject and it gives their direction of misconception about the knowledge to a specific subject or topic. It makes importance of the role for a teacher or educator on teaching to students for a particular subject or topic. Here we discuss the maximum number of students who are commonly done the same mistake for answering the questions or each part of questions in the following sections.

6.1. Section: on question 1#

This question has three sub-questions in group theory shown in the sample paper. So we discuss about the type of student mistakes briefly in the following three subsections.

6.1.1. On sub-question 1(a)

"Find the order of ω and ω^2 in the multiplicative group $G = \{1, \omega, \omega^2\}$, where ω is cube root of unity".

The correct answer for this is '3 and 3 are the order of ω and ω^2 respectively,' or shortly written as "3 and 3," which is considered for correct answer also. But it is noticeable that only 58 (32%) students can give this correct answer and a greater portion 126 (68%) students give wrong or do not know the correct answer. Most of them will done common mistake by putting the expressions "$\omega^3 = 1$ and $(\omega^2)^2 = \omega$ and so the orders of ω and ω^2 are 3 and 2, respectively." By this mistake it is seen that these students must know that ω is cube root of 1, so they thought $\omega^3 = 1$ which is correct, though they do not know about the order of a element in a group. But the second is $(\omega^2)^2 = \omega$ is also logically correct as $(\omega^2)^2 = \omega^4 = \omega\omega^3 = \omega$ on put $\omega^3 = 1$ but it is not the correct answer for required sense of order. It is observe that some students give any two numbers as he/she likes without knowing the order of the element of a group.

6.1.2. On sub-question 1(b)

"If *H* is a subgroup of a finite group *G*, where $o(G) = 20$ and $o(H) = 4$, then find the number of distinct right (left) coset of *H* in *G*."

The correct answer for this question is "$\frac{o(G)}{o(H)} = \frac{20}{4} = 5$" or shortly '5' is the number of distinct right (left) cosets. In this question, 93 (51%) students put the correct answer but 91 (49%) students give either wrong answer or give no response. It shows that half of total students have no idea about the formula of distinct right (left) coset as given in above, which is the consequence of the Lagrange's theorem. Most of students put the result in reverse way, i.e., $\frac{o(H)}{o(G)} = \frac{4}{20} = \frac{1}{5}$, doing this type of mistake shows that they have some idea about formula but they mislead in their memory in the examination hall. But they do not know that the number of cosets cannot be a fraction number, it is always a positive number.

6.1.3. On sub-question 1(c)

"What is the identity element of the quotient group G/N ?"

The correct answer for this is "*Ne*" (*eN*) or "*N*1" (1*N*) or *N* for multiplicative groups *G* and *N* or "*N* + 0" (0 + *N*) or *N* or '*N* + *e*' (*e* + *N*) for additive groups *G* and *N*. For this question, only 38 (21%) students give the correct answer, 53 (29%) students give wrong answer and a greater portion 93 (51%) students give no response for it. It is seen that almost equal portion (18 and 20) students give the correct answer like *eN* and 1*N*, respectively, but 3 students give the correct answer by put 0 + *N* and no student put *e* + *N* in this study. A large number of students will put the identity of the quotient group G/N as 'e' or '1' or '0' only. Of course, these elements are identities for the different composition groups. This shows that these students have concept about the identity of a group, but they have no concepts on the identity of the quotient group G/N. A few (3) students did mistake by putting answer as "*G*" only. This shows they have confusion with the several answers which one is correct.

6.2. Section: on question 2#

In this question, only the sub-question (a) is involved in group theory, which is 'Define a homomorphism and isomorphism from a group to another group". The correct definitions of the homomorphism and isomorphism are 'If $(G, *)$ and (H, o) are groups, then a function $f : G \to H$ is a homomorphism if $f(x * y) = f(x)\, o\, f(y)$, for all $x, y \in G$. If f is also bijection (one-one and onto), then *f* is called an *isomorphism*.

For this question, equal portion 34 (18.5%) and 33 (18%) students have given correct and partial correct answers, respectively. A large number 63 (34%) students give wrong answers for this question and others give no response it. Since the definition of homomorphism some how it is confusing to give the actual definition, most of the students are not concerning about the different types of binary operation on different groups *G* and *H*. They think the binary operation are same to the groups *G* and *H* and it is multiplication so they write for ‘ $f(x * y) = f(x) * f(y),$ ’ as ‘ $f(xy) = f(x)f(y).$ ’ Though, it is considered correct answer but it is important to care in learning about different types of operation on groups. Most of the students give wrong definition as ‘If *G* and $G^{/}$ are groups, then a function $f : G \to G^{/}$ is a homomorphism if *f* is one-one and if *f* is onto, then it is isomorphism’. Some others give wrong answer as “If *G* and $G^{/}$ are groups, then a function $f : G \to G^{/}$ is a homomorphism if *f* is one-one and ‘ $f(xy) = f(x)f(y)$ ’ then *f* is called *isomorphism*.” This shows that they must be confusion the definition between homomorphism and isomorphism. A few 4 (2%) students give meaningless mathematical sentence for the definition of homomorphism and isomorphism.

6.3. Section: on question 3#

This question is optional and though it has four sub-questions, but students can give answers only three of them. The sub-questions (a) and (b) are in group theory, so we have discussed the students’ mistakes for this in the following two subsections separately.

6.3.1. On sub-question 3(a)

“Define a group. Prove that in a group *G*, identity element is unique and inverse of each $a \in G$ is unique.” The solution strategy of this question is

1st part. A nonempty set *G* with a binary composition denoted by ‘$*$’ and satisfying the following properties is called a *group*, $\forall a, b, c \in G$,

G_1: Associative, i.e., $(a * b) * c = a * (b * c), \forall a, b, c \in G,$

G_2 : Identity, $\forall a \in G, \exists e \in G$ (unique) such that $a * e = e * a = a$,

G_3 : Inverse, $\forall a \in G, \exists b \in G$ (unique) such that $a * b = b * a = e$.

2nd part. Proof. Let $e_0 \in G$ identity, $\therefore e_0 x = x = xe_0$ for all $x \in G$. In particular, setting $x = e$ in $x = xe_o$, then $e = ee_o$ so $e = e_o$ since let e be another identity, or

Proof. Let, if possible, e_1 and e_2 be two identities of a group G with binary composition '·'. Then $e_1 \cdot e_2 = e_1$ $\because e_1$ is identity. Again, $e_1 \cdot e_2 = e_2$ $\because e_2$ is identity. So $e_1 = e_2$.

3rd part. Let, if possible, b and c be two inverse of $a \in G$, where G is group with binary composition '·'. Then $a \cdot b = e$ and $a \cdot c = e$, so $a \cdot b = a \cdot c$ $\therefore b = c$ [by Left cancellation law].

From Table 2, it is seen that only 10 (5%) students can give complete correct answer, on the other hand, 76 (41%) and 32 (17%) students give partial correct and incorrect answers, respectively. In Table 1, (details) it is seen that maximum 62 (34%) students give correct definition of group and same number 34 (18%) students give correct proof of the 2nd and 3rd parts. But a greater portion of students give partial and incorrect answers. Most of the students give incomplete/wrong definition of group as (A) "G be a group if it satisfies: (i) closure property (ii) associative property and (iii) existence property" and (B) "Let G be a set and $a, b, c \in G$ if (i) $a + b = b + a$ (ii) $(a + b) + c = a + (b + c)$ and (iii) $a + e = e + a$, then G is called a *group*". Some students give commutative property as one of group properties, some others give only two properties, like identity and inverse. Even one student wrote that "If G be a set of some elements like identity, inverse and some other elements then G is called *group*".

For the second and third parts, the solution methods are same so the mistakes done by the students are also same. Most of them first take two elements of identities/inverses (respectively, 2nd/3rd) and later some step they put these two numbers are equal without given any reasons, why/how.

Some samples of answers which are incomplete/incorrect, like: (A) Let e and e' be two identities of G. Then $e \cdot e' = e$ and $e' \cdot e = e'$ $(e \cdot e' = e'$ also$)$ so $e = e'$. (B) Let e and e' be two identities of G and a be any element of G. Then $e \cdot a = e$ and $e' \cdot a = e'$, so $e = e'$. Some students left without writing any line in the last part. For the 3rd part, some sample of incomplete/ incorrect answer like: (A) Let b and c be two inverses of a (or in G). Then $ab = e$ and $ac = e$, so $b = c$, and (B) Let b and c be two inverses of a (or in G). Then $a * b = e = b * a$ and $a * c = e' = c * a$, so $b = c$.

6.3.2. On sub-question 3(b)

"Define a cyclic group. Prove that every cyclic group is abelian. Show that the multiplicative group $\{1, -1, i, -i\}$ is cyclic. Write down the generators of this cyclic group".

The answer of this question is

1st part. A group G is called *cyclic* if $\exists a \in G$ such that $G = \langle a \rangle$, where $\langle a \rangle = \{a^n : n \in Z\} = \{\text{all powers of } a\}$ or if a group is generated by a single element a is called a *cyclic group*.

2nd part. Let G be cyclic group generated by a and $x, y \in G$. Then $x = a^n$, $y = a^m$ for some $n, m \in Z$. Now $x \cdot y = a^n \cdot a^m = a^{n+m} = a^{m+n}$ $= a^m \cdot a^n = y \cdot x$.

3rd and 4th parts. Here we consider the multiplicative group $\{1, -1, i, -i\}$ cyclic is as $(i)^1 = i$, $(i)^2 = 1$, $(i)^3 = -i$, $(i)^4 = -1$ and i is generator (3rd part) and $(-i)^1 = -i$, $(-i)^2 = -1$, $(-i)^3 = i$, $(-i)^4 = 1$ and $-i$ is generator (4th part). The wrong answer other than these two explanations is considered as no response.

From Table 2, we see that only 6 (3%) students can give correct answers for all parts of this question and 10 (5%) students give incorrect answer of any parts of it. On the other hand from Table 1, it is seen that comparatively same 24 (13%), 22 (12%), 23 (12.5%) students give correct answers for 1st, 3rd and 4th parts, respectively. Same number 10 (5%) students give correct and partial correct answers for the 2nd part. And in the ranges 12-16 and 10-12 students give incorrect and give no response, respectively. Some common mistakes done by the students are mentioned in the following.

For 1st part. 13 (8%) students give wrong definitions. These are: (a) 3 (1.6%) students write as "If a group G has a cycle, then it is called *cyclic group*." (b) 6 (3%) students write as "If a group G is such that $a \in G$ and $a^n = 1$ $(= e,\ 2$ of them), then G is called *cyclic group*". (c) 4 (2%) students give incomplete definition as: "A group G is called *cyclic* if a^n ...."

For 2nd part. For this part, 10 (5%) and 16 (9%) are given partial and not correct answers, respectively. The mistakes are given: (a) 8 (4%) students relatively same partial correct answer give as "Let G be a cyclic group and $x = a^n$, $y = a^m$ be two elements of G. Now $x \cdot y = a^n \cdot a^m = a^m \cdot a^n = y \cdot x$. (3 of them write this, $x + y = a^n + a^m = a^m + a^n = y + x$). (b) 2 (1%) students give as: "Let G be a cyclic group and $x = a^n$, $y = a^n$ be two elements of G then clearly $xy = yx$. (c) More than 10 (5%) students done wrong as "let G be cyclic. Then $x = a^n$ and $y = b^m$, so $xy = a^n b^m = (ab)^{mn}$...." (d) 4 (2%) students write simply like as "$x \cdot y = a^n \cdot b^n = (ab)^n = (ba)^n = b^n \cdot a^n = yx$.

3rd and 4th parts. It is observed that only 6 (3%) students can give both explanations for 3rd and 4th so they give i and $-i$ are generators, respectively. And others give explanation either 3rd part, 16 students and i is generator or explanation 4th part, 17 students and $-i$ is generator. Most of them (12 students (in 3rd part)) wrongly give explanation as $(i)^1 = i$, $(i)^2 = -1$, $(i)^3 = i$, $(i)^4 = 1$ and some students give as $(-1)^1 = -1$, $(-1)^2 = 1$, $\sqrt{(-1)} = i, \ldots$ so give -1 is generator but this type of wrong explanation is considered in no response category.

6.4. Section: on question 4#

This question has four sub-questions and the students are directed to answer either the pair *a* and *b* or the pair *c* and *d*. But all the sub-questions involve in group theory. We discuss the students' mistakes in the following four subsections in briefly:

6.4.1. On sub-question 4(a)

Prove that the intersection of two subgroups of a group is a subgroup of the group. Give an example to show that the union of two subgroups is not a group.

The solution strategy is "Let H_1 and H_2 be two subgroups of a group G then to show that $H_1 \cap H_2$ is subgroup of G. Suppose

$$a, b \in H_1 \cap H_2$$

$$\Rightarrow a, b \in H_1 \text{ and } a, b \in H_2,$$

$$\Rightarrow ab^{-1} \in H_1 \text{ and } ab^{-1} \in H_2 \because H_1 \text{ and } H_2 \text{ are subgroups of } G$$

$$\Rightarrow ab^{-1} \in H_1 \cap H_2 \text{ so } H_1 \cap H_2 \text{ is a subgroup of } G.$$

The union $H_1 \cup H_2$ may not a subgroup. For example,

$$G = \{\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\},$$

$$H_1 = \{\ldots -4, -2, 0, 2, 4, \ldots\},$$

$$H_2 = \{\ldots -6, -3, 0, 3, 6, \ldots\},$$

$H_1 \cup H_2 = \{\ldots -6, -4, -3, -2, 0, 2, 3, 4, 6, \ldots\}$ is not a subgroup as it is not a group for induced composition. As $3, 4 \in H_1 \cup H_2$, but $3 + 4 = 7 \notin H_1 \cup H_2$, i.e., closure property is not satisfied.

From Table 2, we see that 50 (27%) and 83 (45%) students give partial correct and incorrect answers for this sub-question, respectively. On the other hand, 87 (47%) students give wrong answer in 1st part of this sub-question.

Most of them (50) commit wrong as “Let H_1 and H_2 be two subgroups of a group *G*. Suppose

$a, b \in H_1 \cap H_2$

$\Rightarrow a, b \in H_1$ and $a, b \in H_2$

$\Rightarrow ab^{-1} \in H_1$ and $ab^{-1} \in H_2$ [no reasons put here]

$\Rightarrow ab^{-1} \in H_1 \cap H_2$ so $H_1 \cap H_2$ is a subgroup of *G*” {give partial correct}.

Some students (30) give wrong put as

“$\Rightarrow ab^{-1} \in H_1$ and $ab^{-1} \in H_2,$ where $a, b \in H_1$ and $a, b \in H_2$

$\Rightarrow ab^{-1} \in H_1 \cap H_2,$ so $H_1 \cap H_2$ is a subgroup” directly.

Some students (25) give wrong answer by writing as “Let H_1 and H_2 be two subgroups of a group *G*. Suppose $a, b \in H_1 \cap H_2 \Rightarrow a, b \in H_1$ and $a, b \in H_2.$ Now $a, b \in H_1$ and $a, b \in H_2,$ so $a, b \in G,$ Since *G* is a group $a - b \in G$ and so $a - b \in H_1 \cap H_2.$ Hence $H_1 \cap H_2$ is a subgroup of *G*”. There are observed so many mistakes for individual student that these make no relation in group theory.

In 2nd part, we see that 98 (53%) students done mistake, the greater portion 58 (32%) of them as “The union $H_1 \cup H_2$ may not a subgroup. For example,

$$H_1 = \{..., -4, -2, 0, 2, 4, ...\},$$

$$H_2 = \{..., -5, -3, -1, 0, 1, 3, 5, ...\},$$

$H_1 \cup H_2 = \{..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...\}$ is not a subgroup as $3, 4 \in H_1 \cup H_2,$ but $3 + 4 = 7 \notin H_1 \cup H_2,$ i.e., closure property is not satisfied.” Some others 25 (14%) students take another example for H_1 and $H_2,$ but they failed to prove.

6.4.2. On sub-question 4(b)

"State and prove Lagrange's theorem on order of a subgroup of a finite group". The proof is:

1st part. If H is a subgroup of a finite group G, then $o(H)$ is a divisor of $o(G)$.

2nd part. Let $\{a_1H, a_2H, ..., a_kH\}$ be the family of all the distinct cosets of H in G. Then $G = a_1H \cup a_2H \cup \cdots \cup a_kH$ because each $g \in G$ lies in the coset gH and $gH = a_iH$ for some i. Moreover since the coset of H in G are distinct, so it is disjoint. It follows then $o(G) = o(a_1H) + o(a_2H) + \cdots + o(a_kH)$. But $o(a_iH) = o(H)$ for all i,

$$\therefore \quad o(G) = o(H) + o(H) + \cdots + o(H) \ (k \text{ times}) = ko(H).$$

Hence it follows the required result.

From Table 2, we see that 47 (26%) students give incorrect answer and when we observe as two parts of this, then we have seen comparatively equal number 53 (29%) and 51 (28%) students commit wrong for 1st and 2nd parts, respectively.

In 1st part, a greater portion of students state as "If H is a subgroup of a group G then order of G is divided by order of H." Here they do not mention the finiteness of G. Again some state as "If n is the order of H and m is the order of G, then $m = kn$." Some students put as: "Let H be subgroup of G then $o(G)$ is divisor of $o(H)$".

In 2nd part, a greater portion of students commit wrong/partial wrong doing as "Let $\{a_1H, a_2H, ..., a_kH\}$ be the coset of H in G. Then $G = a_1H \cup a_2H \cup \cdots \cup a_kH$. Then $o(G) = o(a_1H) + o(a_2H) + \cdots + o(a_kH)$. But $o(a_iH) = o(H)$ for all i,

$$\therefore \quad o(G) = o(H) + o(H) + \cdots + o(H) \ (k \text{ times}) = ko(H).$$

Hence it follows the required proof." Some students write 1st step and last step, but in middle steps they do some unnecessary steps/wrong steps and some of them try to prove $o(aH) = o(H)$, and leave it here. Some students write whole pages trying to proof it, with unnecessary and wrong steps.

6.4.3. On sub-question 4(c)

"Define a normal subgroup of a group. Prove that subgroup N of a group G is normal subgroup if and only if $gxg^{-1} \in N, \forall g \in G$ and $\forall x \in N$".

Normal subgroup: A subgroup H of a group G will be a normal subgroup of G if right and left cosets of H coincide, i.e., $Hg = gH, \forall g \in G$.

Let us suppose H is a normal subgroup of G. Then $\forall g \in G$, we have $Hg = gH$, multiplying to both by g^{-1} give $g^{-1}Hg = g^{-1}gH = H \Rightarrow g^{-1}hg \in H$, where $h \in H$ and $\forall g \in G$.

Conversely suppose

$$h \in H \text{ and } \forall g \in G, \text{ we have } g^{-1}hg \in H \Rightarrow g^{-1}Hg \subset H. \tag{i}$$

Again since

$$\begin{aligned} g^{-1}Hg \subset H \Rightarrow g^{-1}Hgg^{-1} \subset Hg^{-1} \Rightarrow gg^{-1}H \subset gHg^{-1} \\ \Rightarrow H \subset gHg^{-1}. \end{aligned} \tag{ii}$$

From (i) and (ii), we have $g^{-1}Hg = H$, $Hg = gH$. Hence H is normal subgroup of G.

From Table 2, we see that only 2 (1%) students give complete correct answer of this question. 5 (3%) students done partial and incorrect answers on this question. On the other hand, 1 (.5%) student gives incorrect answer to both parts of this question and 4 (2%) students give partial correct answer (Table 1). The incorrect answer of the student is given as "A subgroup H of a group G is called *normal* if $H \triangleright G$". The 4 students have given partial correct, 3 of them correct up to first part but the converse part they do not complete/left incomplete. One student completes first part and also forward converse part up to step (i), but cannot show further step.

6.4.4. On sub-question 4(d)

"Define a quotient group. Prove that every quotient group of an abelian group is abelian".

Solution strategy:

Quotient group: The set G/H, all cosets of a normal subgroup H is group w.r.t. multiplication of cosets and this group is called *quotient group*.

Proof. Let H be a subgroup of an abelian group G. Then H is a normal subgroup.

Let $a, b \in G$. Then $Ha, Hb \in \frac{G}{H}$.

Now $(Ha)(Hb) = Hab = Hba = (Hb)(Ha)$

($\therefore$ G is an abelian group, so $ab = ba$)

$\therefore \frac{G}{H}$ is abelian.

Hence every quotient group of an abelian group is abelian.

In this question, only one student tries to attempt but not any correct part steps.

7. Findings

In this study, we have many outcomes in the two dimensions of students' behavior and the students' misconceptions in the group theory in attending the examination hall. We have mentioned the findings of these two dimensions in the following two subsections briefly:

7.1. On students behavior

We observe the following types of students behavior on the examination:

1. On an average, only 16% students can success for correct answer in

this examination and average 30% students give no responses in compulsory questions. There are at least 33% students choose option trying to answer other than group theory.

2. The study revealed that students have felt hard to attempt normal subgroups and quotient group. So students need more importance on teaching normal subgroups and quotient groups in the professional teaching method.

3. We observed that the sequence of answering the questions in most of the scripts (40%) students was same. So these students have done similar type of correct steps or wrong steps whatever their sequence are followed. So it is a sign of some defective control and unfair examination system and need effective measure to control examination system.

4. There are some doubtful performances for one particular student on objective questions answers as the students trying to give answers several students in a sequential order.

7.2. On students' mistakes

Students' mistakes do not arise only from the complexity of the subject but also from weakness in the students' mathematical foundations. The concept such as group theory in an introduction to abstract algebra courses is particularly difficult for the students to learn and to maintain positive attitude. On discussion in Section 6, we observed that students have done variety of their mistakes for a particular question answer. As the students' mistakes on an examination directed their misconceptions on learning a particular topic, so in this study, it is revealed that there are so many misconceptions on group theory, some of them are described in the following section. Some more significance mistakes are observed in this study and these are mentioned shortly as follows:

1. Students are fallen in confusion to give the correct answer.

2. They put any other wrong answer for the objective type questions answer without knowing the correct one.

3. Most of students giving partial or incorrect answers are led to different wrong process or wrong solution strategies.

4. Even some students write the question repeatedly against writing the answer, showing that they cannot give the correct answer or they want to time pass in the examination hall only.

8. Recommended Conclusion

In the above findings, from this study, it is clear that the students' performance in group theory is very poor in the semester examination under Gauhati University. This poor performance of the students will reflect to be the drawback as well as inefficiency of traditional teaching and learning method. So on teaching group theory they must need to add some more effective and audio-visual technology other than professional teaching and learning method to overcome the complexities on learning group theory. So it is important to maintain students attention, motivation and curiosity, many researchers have suggested using the computer applications to teach abstract algebra which can promote meaningful learning of the concepts that can be overcome the difficult to group theory. Otherwise when the subject is abstract it is more difficult to establish meaningful connections with what is already known. Abstract algebra software combined with proper teaching techniques can offer prosperous learning environments where students engage in discovery, control their learning and work collaboratively in groups. There are a number of studies to integrating technology into the abstract algebra and group theory class. For example, Rainbolt [8] used GAP software as a tool in abstract algebra class. Keppelman and Webb [9] used Finite Group Behavior (FGB) program and they discuss the features and philosophy of FGB. Cannon and Playoust [10] used the Magma Computer algebra system in their class. And an important attack on this approach has been made by Ed. Dubinsky and his colleagues. According to this approach, learning is central and the research focuses on how students learn mathematics. In fact, replacing the professional lecture method with constructive, interactive methods involving computer activities and co-operative learning can change radically the amount of meaningful learning achieved by average students (Leron and Dubinsky [13]). In effective teaching and learning process, Serpil [16] argued ISETL (Interactive SET Language) is a power full tool for teaching and learning mathematics, particularly in group theory in abstract algebra.

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Appendix A (Sample Paper)

Part-I (Objective type)

1. Answer the following questions: $1 \times 7 = 7$

(a) Find the order of ω and ω^2 in the multiplicative group $G = \{1, \omega, \omega^2\}$, where ω is cube root of unity.

(b) If H is a subgroup of a finite group G, where $o(G) = 20$ and $o(H) = 4$, then find the number of distinct right (left) coset of H in G.

(c) What is the identity element of the quotient group G/N ?

(d) ---------- (involve in ring).

(e), (f) and (g) ---------- (involve in matrix).

Part-II (Very short answer type)

2. Answer the following questions: $2 \times 4 = 8$

(a) Define a homomorphism and isomorphism from a group to another group.

(b) and (c) ---------- (involve in ring).

(d) ---------- (involve in matrix).

Part-III (Short answer type)

3. Answer any three questions: $5 \times 3 = 15$

(a) Define a group. Prove that in a group G, identity element is unique and inverse of each $a \in G$ is unique. $1 + 2 + 2 = 5$

(b) Define a cyclic group. Prove that every cyclic group is abelian. Show that the multiplicative group $\{1, -1, i, -i\}$ is cyclic. Write down the generators of this cyclic group. $1 + 2 + 1 + 1 = 5$

(c) ---------- (involve in ring).

(d) and (e) ---------- (involve in matrix).

Part-IV

Answer either (a) and (b) or (c) and (d) from each of the following questions:

4(a) Prove that the intersection of two subgroups of a group is a subgroup of the group. Give an example to show that the union of two subgroups is not a group.

$3 + 2 = 5$

(b) State and prove Lagrange's theorem on order of a subgroup of a finite group. $1 + 4 = 5$

(c) Define a normal subgroup of a group. Prove that subgroup N of a group G is normal subgroup if and only if $g \times g^{-1} \in N, \ \forall g \in G$ and $\forall x \in N$. $1 + 5 = 6$

(d) Define a quotient group. Prove that every quotient group of an abelian group is abelian. $2 + 2 = 4$

5. ---------- (all four questions (a), (b), (c) and (d) are involved in ring).

6. ---------- (all four questions (a), (b), (c) and (d) are involved in matrix).