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ITERATIVE SCHEME FOR MIXED EQUILIBRIUM PROBLEMS, FIXED POINT PROBLEMS AND VARIATIONAL INEQUALITY PROBLEMS OF A COUNTABLE FAMILY OF *k*-STRICT PSEUDO-CONTRACTIONS

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Abstract

In this paper, we introduce an iterative scheme for finding a common solution of mixed equilibrium problems, fixed point problems and variational inequality problems of a countable family of *k*-strict pseudo-contractions in the framework Hilbert spaces. We prove a strong convergence theorem of the proposed scheme. The results presented in this paper improve and extend the corresponding results announced by many others.

1. Introduction

Throughout of this paper, we always assume that K is a closed convex subset of a real Hilbert space H with inner product and norm which denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, $\mathbb{R}$ is the set of real numbers, and $\mathbb{N}$ is the set

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of positive integers. Let $G : K \times K \to \mathbb{R}$ be to find $z \in K$ such that

$$G(z, y) \geq 0, \forall y \in K. \tag{1.1}$$

The set of solution of (1.1) is denoted by $EP(G)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). The mixed equilibrium for two bifuctions of $G_1, G_2 : K \times K \to \mathbb{R}$ is to find $z \in K$ such that

$$G_1(z, y) + G_2(z, y) + \langle Bz, y - z \rangle \geq 0, \forall y \in K. \tag{1.2}$$

In the sequel, we will indicate by $MEP(G_1, G_2, B)$ the set of solution of our mixed equilibrium problem. If $B = 0,$ then we denote $MEP(G_1, G_2, 0),$ with $MEP(G_1, G_2).$ We notice that for $G_2 = 0$ and $B = 0,$ the problem is the well-known equilibrium problem (see [4]).

Let $B : K \to H$ be a mapping. The classical variational inequality, denoted by $VI(B, K),$ is to find $s \in K$ such that $\langle Bs, y - s \rangle \geq 0$ for all $y \in K.$

Let $T : K \to K$ be a self-mapping of K. Recall T is said to be *k-strict pseudo-contraction* if there exists a constant $k \in [0, 1)$ such that

$$\| Tz - Ty \|^2 \leq \| z - y \|^2 + k \| (I - T)z - (I - T)y \|^2 \tag{1.3}$$

for all $z, y \in K.$ The set of fixed points of T is denoted by $Fix(T)$ (i.e., $Fix(T) = \{z \in K : Tz = z\}).$ Note that the class k-strict pseudo-contractions includes the class of nonexpansive mappings which are mappings T on K such that

$$\| Tz - Ty \| \leq \| z - y \| \tag{1.4}$$

for all $z, y \in K$ (see [14]). That is, T is nonexpansive if and only if T is 0-strict pseudo-contractions.

In the recent years, many papers concern the convergence of iterative schemes for nonexpansive mapping and k-strict pseudo-contractions have been extensively studied by many authors [1, 4-7, 10, 12, 14] and references therein.

In this paper, motivated and inspired by these facts, we introduce the new iterative scheme of a countable family of k-strict pseudo-contractions which include [10], [6], and [12] as some special cases.

2. Preliminaries

For every point $z \in H$, there exists a unique nearest point in K, denoted by $P_K z$ such that

$$\| z - P_K z \| \leq \| z - y \| \quad \text{for all } y \in K. \tag{2.1}$$

P_K is called the *metric projection* from H into K. It well known that P_K is a nonexpansive mapping of H into K and satisfies

$$\langle z - y, P_K z - P_K y \rangle \geq \| P_K z - P_K y \|^2. \tag{2.2}$$

Recall that a mapping $B : K \to H$ is called β-*inverse-strongly monotone*, if there exists a positive number β such that $\langle Bz - By, z - y \rangle \geq \beta \| Bz - By \|^2$, $\forall z, y \in K$. Let I be the identity mapping on K. It is well known that if $B : K \to H$ is β-inverse-strongly monotone, then B is $\frac{1}{\beta}$-Lipschitz continuous and monotone mapping. Moreover, if $0 < \lambda < 2\beta$, then $1 - \lambda B$ is a nonexpansive mapping (see [1, 2]).

The following lemmas will be useful for proving in our main results.

Lemma 2.1 (See [3]). *For all* $z, y \in H$, *there holds the inequality*

$$\| z + y \|^2 \leq \| z \|^2 + 2\langle y, z + y \rangle.$$

Lemma 2.2 (See [7]). *Let H be a Hilbert space, K be a nonempty closed subset of H,* $f : H \to H$ *be a contraction with coefficient* $0 < \alpha < 1$, *and A be a strongly positive linear bounded operator with coefficient* $\bar{\gamma} > 0$. *Then,*

(1) *if* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, *then* $\langle z - y, (A - \gamma f)z - (A - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\alpha)\| z - y \|^2$, $z, y \in H$;

(2) *if* $0 < \rho < \| A \|^{-1}$, *then* $\| I - \rho A \| \leq 1 - \rho\bar{\gamma}$.

For solving the mixed equilibrium problem for a bifunction $G : K \times K \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, let us assume that G satisfies the following conditions:

(A1) $G(z, z) = 0$ for all $z \in K$;

(A2) G is monotone, that is, $G(z, y) + G(y, z) \leq 0$ for all $z, y \in K$;

(A3) for each $x, z, y \in K$, $\lim_{t \to 0} G(tx + (1 - t)z, y) \leq G(z, y)$;

(A4) for each $z \in K$, $y \mapsto G(z, y)$ is convex and lower semicontinuous.

Lemma 2.3 (See [4])**.** *Let K be a convex closed subset of a Hilbert space H. Let* $G_1 : K \times K \to \mathbb{R}$, *where* $\mathbb{R}$ *is the set of real numbers, be a bifunction such that*

(l1) $G_1(z, z) = 0$ *for all* $z \in K$;

(l2) G_1 *is monotone and upper hemicontinuous in the first variable*;

(l3) G_1 *is lower semicontinuous and convex in the second variable*.

Let $G_2 : K \times K \to \mathbb{R}$ *be a bifunction such that*

(h1) $G_2(z, z) = 0$ *for all* $z \in K$;

(h2) G_2 *is monotone and weakly upper semicontinuous in the first variable*;

(h3) G_2 *is convex in the second variable*.

Moreover, let us suppose that

(H) *for fixed* $\lambda > 0$ *and* $z \in K$, *there exists a bounded set* $D \subset K$ *and* $a \in D$ *such that for all*

$$y \in K \setminus D, \quad -G_1(a, y) + G_2(y, a) + \frac{1}{\lambda}\langle a - y, y - z\rangle < 0.$$

For $\lambda > 0$ *and* $z \in H$, *let* $F_\lambda : H \to K$ *be a mapping defined by*

$$F_\lambda(z) = \left\{y \in K : G_1(x, y) + G_2(x, y) + \frac{1}{\lambda}\langle y - x, x - z\rangle \geq 0, \forall x \in K\right\}$$

called resolvent of G_1 *and* G_2. *Then*

(1) $F_\lambda(z) \neq \emptyset$;

(2) F_λ *is a single value*;

(3) F_λ *is firmly nonexpansive*;

(4) $MEP(G_1, G_2) = Fix(F_\lambda(z))$ *and it is closed and convex.*

Lemma 2.4 (See [11]). *Let* $\{x_n\}$ *and* $\{v_n\}$ *be bounded sequences in a Banach space X and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = \beta_n x_n + (1 - \beta_n)v_n$ *for all integers* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\| v_{n+1} - v_n \| - \| x_{n+1} - x_n \|) \leq 0.$$

Then $\lim_{n\to\infty}\| v_n - x_n \| = 0.$

Lemma 2.5 (See [13]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \ n \geq 0,$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty}\alpha_n = \infty$

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty}| \alpha_n | < \infty.$

Then $\lim_{n\to\infty} a_n = 0.$

Lemma 2.6 (See [1]). *Let K be a nonempty closed convex subset of a Banach space and* $\{T_n\}$ *be a sequence mapping of K into itself. Suppose that* $\sum_{n=1}^{\infty} \sup\{\| T_{n+1}z - T_n z : z \in K \|\} < \infty$. *Then, for each* $y \in K$, $\{T_n y\}$ *converges strongly to some point of K. Moreover, let T be a mapping of K into itself defined by* $Ty = \lim_{n\to\infty} T_n y$ *for all* $y \in K$. *Then* $\lim_{n\to\infty} \sup\{\| Tz - T_n z \| : z \in K\} = 0$.

Lemma 2.7 (See [2]). *Let K be a nonempty closed convex subset of a Hilbert space H. Let* $S : K \to H$ *be a k-strict pseudo-contraction. Define* $T : K \to H$ *by* $Tx = \mu x + (1 - \mu)Sx$ *for each* $x \in K$. *Then, as* $\mu \in [k, 1)$, T *is a nonexpansive mapping such that* $Fix(T) = Fix(S)$.

3. Main Results

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H, let* G_1 *and* G_2 *be bifunctions from* $K \times K \to \mathbb{R}$, *where* $\mathbb{R}$ *is the set of real numbers, satisfying* (A1)-(A4), *let* $B : K \to H$ *be a* β*-inverse-strongly monotone mapping, and let* $\{T_n\}$ *be a sequence of k-strictly pseudo-contraction of K into itself with fixed point for all* $n \in \mathbb{N}$ *and* $k \in [0, 1)$. *Define* $S_n^k x = kx + (1 - k)T_n x$. *Let f be a contraction of K into itself with the coefficient* $\alpha \in (0, 1)$. *Let A be a strongly positive linear bounded operator on K with coefficient* $\bar{\gamma} > 0$. *Assume that* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ *and* $\Omega := \bigcap_{n=1}^{\infty} Fix(T_n) \cap MEP(G_1, G_2) \cap VI(K, A) \neq \emptyset$. *Let* x_n, y_n *and* u_n *be sequences generated by* $x_1 \in K$ *and*

$$G_1(u_n, y) + G_2(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in K,$$

$$y_n = P_K(u_n - \varphi_n B u_n),$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + \mu_n x_n + ((1 - \mu_n)I - \alpha_n A)S_n^k y_n \tag{3.1}$$

for all $n \in N$, *where* $\varphi_n \in (0, 2\beta)$, *and* α_n, μ_n *are two sequences in* $[0, 1]$ *and* $\lambda_n \subset (0, \infty)$ *satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(ii) $\liminf_{n\to\infty} \lambda_n > 0$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(iii) $0 < a \le \mu_n < b < 1$ *for all* $n \ge 1$, $\lim_{n\to\infty} \mu_n = 0$;

(iv) $\lim_{n\to\infty}(\varphi_{n+1} - \varphi_n) = 0$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\| S_{n+1}^k z - S_n^k z \| : z \in D\} < \infty$ *for any bounded subset D of K. Let S be a mapping of K into itself defined by* $Su = \lim_{n\to\infty} S_n^k u$ *for all* $u \in K$ *and suppose* $Fix(S) = \bigcap_{n=1}^{\infty} Fix(S_n^k)$. *Then* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *converge strongly to* ω, *where* $\omega = P_\Omega(I - A + \gamma f)(\omega)$ *is a unique solution of the variational inequality*

$$\langle (A - \gamma f)\omega, \omega - x \rangle \le 0, \ \forall x \in \Omega. \tag{3.2}$$

Proof. Note that from the conditions (i) and (iii), we will assume that $\alpha_n \le (1 - \mu_n)\| A \|^{-1}$ for all $n \ge 1$. Since A is a strongly positive linear operator, we have $\| A \| = \sup\{|\langle Ax, x\rangle| : x \in K, \| x \| = 1\}$. By Lemma 2.2, we have

$$\| (1 - \mu_n)I - \alpha_n A \| \le (1 - \mu_n) - \alpha_n \bar{\gamma}. \tag{3.3}$$

From the definition of S_n^k, we have S_n^k is nonexpansive by Lemma 2.6. We know that P_K is nonexpansive. We now show that $\{x_n\}$ is bounded. Let $v \in \Omega$. By using Lemma 2.3, we have

$$\begin{aligned} \| y_n - v \| &= \| P_K(u_n - \varphi_n Bu_n) - P_K(v - \varphi_n Bv) \| \\ &\le \| (u_n - \varphi_n Bu_n) - (v - \varphi_n Bv) \| \\ &\le \| u_n - v \| \end{aligned}$$

$$\leq \| F_{\lambda_n} x_n - F_{\lambda_n} v \|$$

$$\leq \| x_n - v \| \tag{3.4}$$

for all $n \geq 1$. Then, we have

$$\begin{aligned}\| x_{n+1} - v \| &= \| ((1-\mu_n)I - \alpha_n A)(S_n^k y_n - S_n^k v) \\ &\quad + \alpha_n \gamma(f(x_n) - f(v)) + \alpha_n(\gamma f(v) - Av) + \mu_n(x_n - v) \| \\ &\leq (1 - \alpha_n(\bar{\gamma} - \alpha\gamma)) \| x_n - v \| + \alpha_n(\bar{\gamma} - \alpha\gamma) \frac{\| \gamma f(v) - Av \|}{\bar{\gamma} - \alpha\gamma}. \end{aligned} \tag{3.5}$$

It follows from (3.5) and induction that

$$\| x_n - v \| \leq \max\left\{ \| x_n - v \|, \frac{\| \gamma f(v) - Av \|}{\bar{\gamma} - \alpha\gamma} \right\}, \ \forall n \geq 1. \tag{3.6}$$

This implies that $\{x_n\}$ is bounded and hence the sets of $\{y_n\}$, $\{u_n\}$, $\{S_n^k y_n\}$ and $\{Bu_n\}$ are also bounded. Next, we show that $\| x_{n+1} - x_n \| \to 0$. Define $x_{n+1} = \beta_n x_n + (1 - \beta_n) e_n$ for all $n \geq 0$. We see that

$$\begin{aligned} &\| e_{n+1} - e_n \| - \| x_{n+1} - x_n \| \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \| \gamma f(x_{n+1}) \| + \| A S_{n+1}^k y_{n+1} \| + \frac{\alpha_n}{1 - \beta_n} \| \gamma f(x_n) \| + \| A S_n^k y_n \| \\ &\quad + \| S_{n+1}^k y_n - S_n^k y_n \| + \| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|. \end{aligned} \tag{3.7}$$

On the other hand, we see that

$$\begin{aligned} \| y_{n+1} - y_n \| &\leq \| P_K(u_{n+1} - \varphi_n B u_{n+1}) - P_K(u_n - \varphi_n B u_n) \| \\ &\leq \| (I - \varphi_n B) u_{n+1} - (I - \varphi_n B) u_n \| \\ &\leq \| u_{n+1} - u_n \|. \end{aligned} \tag{3.8}$$

Supposing $\sum_{n=1}^{\infty} \sup\{\| S_{n+1}^k z - S_n^k z \| : z \in D\} < \infty$, we obtain

$$\lim_{n \to \infty} \| S_{n+1}^k y_n - S_n^k y_n \| = 0. \tag{3.9}$$

On the other hand, we note that

$$G_1(u_n, y) + G_2(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in K \qquad (3.10)$$

and

$$G_1(u_{n+1}, y) + G_2(u_{n+1}, y) + \frac{1}{\lambda_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \ \forall y \in K. \quad (3.11)$$

By the same argument as that in the proof of [4, Lemma 3.7], we have

$$\| u_n - u_{n+1} \|^2 \leq \| u_n - u_{n+1} \| \left(\| x_n - x_{n+1} \| + \left| 1 - \frac{\lambda_{n+1}}{\lambda_n} \right| \| u_n - x_n \| \right). \quad (3.12)$$

Since $\liminf_{n\to\infty} \lambda_n > 0$, we assume that $\lambda_n > d > 0$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \| u_n - u_{n+1} \| &\leq \| x_n - x_{n+1} \| + \left| 1 - \frac{\lambda_{n+1}}{\lambda_n} \right| \| u_n - x_n \| \\ &\leq \| x_n - x_{n+1} \| + \frac{L}{d} | \lambda_n - \lambda_{n+1} |, \end{aligned} \qquad (3.13)$$

where $L = \sup\{\| u_n - x_n \| : n \in \mathbb{N}\}$.

Combining (3.7), (3.8) and (3.13) yields that

$$\begin{aligned} &\| e_{n+1} - e_n \| - \| x_{n+1} - x_n \| \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \| \gamma f(x_{n+1}) - AS_{n+1}^k y_{n+1} \| + \frac{\alpha_n}{1 - \beta_n} \| \gamma f(x_n) - AS_n^k y_n \| \\ &\quad + \left(\| x_n - x_{n+1} \| + \frac{L}{d} | \lambda_n - \lambda_{n+1} | \right) + | \varphi_n - \varphi_{n+1} | \| Bu_n \| \\ &\quad + \| S_{n+1}^k y_n - S_n^k y_n \| - \| x_{n+1} - x_n \|. \end{aligned} \qquad (3.14)$$

It follows from (3.9) and the conditions (i), (ii) and (iv) that

$$\lim_{n\to\infty} \| e_{n+1} - e_n \| - \| x_{n+1} - x_n \| = 0. \qquad (3.15)$$

Hence, Lemma 2.4, we obtain that $\lim_{n\to\infty} \| e_n - x_n \| = 0$.

Consequently, it follows that

$$\lim_{n\to\infty}\| x_{n+1} - x_n \| = \lim_{n\to\infty}(1 - \beta_n)\| e_n - x_n \| = 0. \tag{3.16}$$

Moreover, from (3.8), (3.13), (3.16), and the condition (ii), we also imply that

$$\lim_{n\to\infty}\| y_{n+1} - y_n \| = 0. \tag{3.17}$$

Next, we will prove that $\lim_{n\to\infty}\| u_n - x_n \| = 0.$

Since $x_n = \alpha_{n-1}\gamma f(x_{n-1}) + \mu_{n-1}x_{n-1} + ((1 - \mu_{n-1})I - \alpha_{n-1}A)S_{n-1}^k y_{n-1}$, we have that

$$\begin{aligned}
&\| x_n - S_n^k y_n \| \\
&\le \| x_n - S_{n-1}^k y_{n-1} \| + \| S_{n-1}^k y_{n-1} - S_{n-1}^k y_n \| + \| S_{n-1}^k y_n - S_n^k y_n \| \\
&\le \alpha_{n-1}\| \gamma f(x_{n-1}) - AS_{n-1}^k y_{n-1} \| + \mu_{n-1}\| x_{n-1} - S_{n-1}^k y_{n-1} \| \\
&\quad + \| (S_{n-1}^k y_{n-1}) - S_{n-1}^k y_n \| + \| y_{n-1} - y_n \| \\
&\quad + \sup\{\| S_{n+1}^k z - S_n^k z \| : z \in \{y_n\}\}.
\end{aligned} \tag{3.18}$$

It follows by (3.17), the conditions (i), (ii), and $\sup\{\| S_{n+1}^k z - S_n^k z \| : z \in \{y_n\}\} \to 0,$ (as $n \to \infty$), we have

$$\lim_{n\to\infty}\| x_n - S_n^k y_n \| = 0. \tag{3.19}$$

For $s \in \Omega,$ since $u_n = F_{\lambda_n} x_n,$ it follows from Lemma 2.3 that

$$\begin{aligned}
\| u_n - s \|^2 &\le \langle F_{\lambda_n} x_n - F_{\lambda_n} s,\ x_n - s\rangle = \langle u_n - s,\ x_n - s\rangle \\
&\le \frac{1}{2}(\| u_n - s \|^2 + \| x_n - s \|^2 - \| u_n - x_n \|^2),
\end{aligned}$$

and hence $\| u_n - s \|^2 \le \| x_n - s \|^2 - \| u_n - x_n \|^2.$

We note that

$$\| y_n - s \|^2 \le \langle (u_n - \lambda_n B u_n) - (s - \lambda_n Bs),\ y_n - s \rangle$$

$$\le \frac{1}{2}\{\| (u_n - \lambda_n B u_n) - (s - \lambda_n Bs) \|^2 + \| y_n - s \|^2$$

$$- \| (u_n - \lambda_n B u_n) - (s - \lambda_n Bs) - (y_n - s) \|^2\}$$

$$\le \frac{1}{2}\{\| u_n - s \|^2 + \| y_n - s \|^2 - \| u_n - y_n \|^2$$

$$+ 2\lambda_n \langle u_n - y_n,\ Bu_n - Bs \rangle - \lambda_n^2 \| Bu_n - Bs \|^2\}$$

so, we have

$$\| y_n - s \|^2 \le \| u_n - s \|^2 - \| u_n - y_n \|^2$$

$$+ 2\lambda_n \langle u_n - y_n,\ Bu_n - Bs \rangle - \lambda_n^2 \| Bu_n - Bs \|^2. \qquad (3.20)$$

Set $M_n = \gamma f(x_n) - AS_n^k y_n$, and let $\xi > 0$ be a constant such that

$$\xi > \sup_{n,\, t \ge 1}\{\| M_n \|, \| x_t - s \|\}. \qquad (3.21)$$

We have

$$\| x_n - s \|^2 \le \| (1 - \mu_n)(S_n^k y_n - s) + \mu_n (x_n - s) + \alpha_n M_n \|^2$$

$$\le (1 - \mu_n) \| (y_n - s) \|^2 + \mu_n \| (x_n - s) \|^2 + 2\xi^2 \alpha_n$$

$$\le (1 - \mu_n)(\| x_n - s \|^2 - \| u_n - x_n \|^2) - (1 - \mu_n) \| u_n - y_n \|^2$$

$$+ 2\lambda_n (1 - \mu_n) \| (u_n - y_n) \| \| Bu_n - Bs \|$$

$$- 2\lambda_n^2 \| Bu_n - Bs \|^2 + \mu_n \| x_n - s \|^2$$

$$+ 2\xi^2 \alpha_n - \lambda_n (1 - \mu_n)(2\beta - \lambda_n) \| Bu_n - Bs \|^2 + 2\xi^2 \alpha_n.$$

It follows that

$$\lambda_n (1 - \mu_n)(2\beta - \lambda_n) \| Bu_n - Bs \|^2$$

$$\le \| x_n - x_{n+1} \| \{\| x_n - s \| + \| x_{n+1} - s \|\} + 2\xi^2 \alpha_n.$$

Therefore, $\| Bu_n - Bs \| \to 0$ as $n \to \infty$. We also have that

$$(1-\mu_n)\| u_n - x_n \|^2 \le \| x_n - x_{n+1} \| \{\| x_n - s \| + \| x_{n+1} - s \|\}$$
$$- \lambda_n (1-\mu_n)(2\beta - \lambda_n)\| Bu_n - Bs \| + 2\xi^2 \alpha_n$$

and

$$(1-\mu_n)\| u_n - y_n \|^2 \le \| x_n - x_{n+1} \| \{\| x_n - s \| + \| x_{n+1} - s \|\}$$
$$- \lambda_n (1-\mu_n)(2\beta - \lambda_n)\| Bu_n - Bs \| + 2\xi^2 \alpha_n,$$

by using the conditions (i), (ii) and (3.16), $\| Bu_n - Bs \| \to 0$ imply that $\| u_n - x_n \| \to 0$ and $\| u_n - y_n \| \to 0,$ respectively. In addition, according to $\| x_n - y_n \| \le \| x_n - u_n \| + \| u_n - y_n \|,$ we obtain that

$$\lim_{n\to\infty} \| x_n - y_n \| = 0. \tag{3.22}$$

By using (3.22), (3.19) and $\| y_n - S_n^k y_n \| \le \| y_n - x_n \| + \| x_n - S_n^k y_n \|,$ we have

$$\lim_{n\to\infty} \| y_n - S_n^k y_n \| = 0. \tag{3.23}$$

Since

$$\| Sy_n - y_n \| \le \| Sy_n - S_n^k y_n \| + \| S_n^k y_n - y_n \|$$
$$\le \sup\{\| Sz - S_n^k z \| : z \in \{y_n\}\} + \| S_n^k y_n - y_n \|,$$

by (3.23), $\alpha_n \to 0$ and Lemma 2.6, we have $\| Sy_n - y_n \| \to 0,$ as $n \to \infty$. From $S_n^k x = \mu x + (1-\mu) T_n x,$ we know by Lemma 2.7 that S_n^k is nonexpansive with $Fix(S_n^k) = Fix(T_n)$. We now show that $z \in \Omega$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z (denoted by $x_{n_i} \xrightarrow{w} z$). From $\| u_n - x_n \| \to 0,$ we obtain $u_{n_i} \xrightarrow{w} z$. We show $z \in MEP(G_1, G_2)$. From $\| u_n - y_n \| \to 0,$ it follows that $y_{n_i} \xrightarrow{w} z$. From (3.1) and (A2), we obtain

$$\frac{1}{\lambda_n}\langle y - u_n,\, u_n - x_n\rangle \geq G_1(y,\, u_n) + G_2(y,\, u_n), \tag{3.24}$$

and hence

$$\frac{1}{\lambda_n}\left\langle y - u_{n_i},\, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \geq G_1(y,\, u_n) + G_2(y,\, u_n). \tag{3.25}$$

Since $\frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}} \to 0$ and $u_{n_i} \xrightarrow{w} z$, it follows from (A4) that $0 \geq G_1(y,\, z) + G_2(y,\, z)$ for all $y \in K$. Put $q_t = ty + (1-t)z$ for all $t \in (0, 1]$ and $y \in K$. Then, we have $q_t \in K$ and hence $0 \geq G_1(q_t,\, z) + G_2(q_t,\, z)$. So, from (A1) and (A4), we have

$$\begin{aligned} 0 &= G_1(q_t,\, q_t) + G_2(q_t,\, q_t) \\ &= tG_1(q_t,\, y) + (1-t)G_1(q_t,\, z) + G_2(q_t,\, y) + (1-t)G_2(q_t,\, z) \\ &\leq G_1(q_t,\, y) + G_2(q_t,\, y) \end{aligned}$$

and hence $0 \leq G_1(q_t,\, y) + G_2(q_t,\, y)$. From (A3), we have $0 \leq G_1(z,\, y) + G_2(z,\, y)$ for all $y \in K$. Therefore, $z \in MEP(G_1,\, G_2)$. We show that $z \in (\bigcap_{n=1}^{\infty} Fix(T_n))$. Assume $z \notin (\bigcap_{n=1}^{\infty} Fix(S_n^k))$. Then we have $z \neq S_n^k z,\ \forall n \in \mathbb{N}$. It follows by the Opial's condition (see [4]) and $\lim_{n\to\infty}\| Sy_n - y_n \| = 0$ that

$$\begin{aligned} \liminf_{n\to\infty}\| y_n - z \| &< \liminf_{n\to\infty}\| y_n - Sz \| \\ &\leq \liminf_{n\to\infty}\{\| y_n - Sy_n \| + \| Sy_n - Sz \|\} \\ &\leq \liminf_{n\to\infty}\| y_n - z \|. \end{aligned}$$

This is a contradiction. So, we get $z \in (\bigcap_{n=1}^{\infty} Fix(S_n^k))$ and hence $z \in (\bigcap_{n=1}^{\infty} Fix(T_n))$.

Finally, by the same argument as that in the proof of [8, Theorem 3.1, pp. 197-198], we can show that $z \in VI(C,\, A)$. Hence $z \in \Omega$. Next, we show that $\limsup_{n\to\infty}\langle (A-\gamma f)\omega, \omega - x_n\rangle \le 0,$ where $\omega = P_\Omega(I - A + \gamma f)(\omega)$ is a unique solution of the variational inequality (3.2). We choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (A-\gamma f)\omega, \omega - x_n\rangle = \limsup_{n\to\infty}\langle (A-\gamma f)\omega, \omega - x_{n_i}\rangle$$

$$= \langle (A-\gamma f)\omega, \omega - z\rangle \le 0. \tag{3.26}$$

Therefore,

$$\| x_{n+1} - \omega \|^2$$

$$= \| ((1-\mu_n)I - \alpha_n A)(S_n^k - \omega) + \mu_n(x_n - \omega) + \alpha_n(\gamma f(x_n) - A\omega) \|^2$$

$$\le \| ((1-\mu_n)I - \alpha_n A)(S_n^k - \omega) + \mu_n(x_n - \omega) \|^2 + 2\alpha_n\langle \gamma f(x_n) - A\omega, x_{n+1} - \omega\rangle$$

$$\le ((1-\mu_n) - \alpha_n\bar{\gamma})\| (S_n^k y_n - \omega) \|^2 + \mu_n \| (x_n - \omega) \|^2$$

$$+ 2\alpha_n\gamma\alpha \| x_n - \omega \| \| x_{n+1} - \omega \| + 2\alpha_n\langle \gamma f(\omega) - A\omega, x_{n+1} - \omega\rangle$$

$$\le ((1-\mu_n) - \alpha_n\bar{\gamma})\| (x_n - \omega) \|^2 + \mu_n \| (x_n - \omega) \|^2$$

$$+ \alpha_n\gamma\alpha(\| x_n - \omega \|^2 + \| x_{n+1} - \omega \|^2) + 2\alpha_n\langle \gamma f(\omega) - A\omega, x_{n+1} - \omega\rangle$$

$$\le (1 - \mu_n(\bar{\gamma} - \gamma\alpha))\| (x_n - \omega) \|^2 + \mu_n \| (x_n - \omega) \|^2$$

$$+ \alpha_n\gamma\alpha \| (x_{n+1} - \omega) \|^2 + 2\alpha_n\langle \gamma f(\omega) - A\omega, x_{n+1} - \omega\rangle$$

which implies that

$$\| x_{n+1} - \omega \|^2 \le \left(1 - \frac{(\bar{\gamma} - \alpha\gamma)\alpha_n}{1 - \alpha\gamma\alpha_n}\right) \| x_n - \omega \|^2$$

$$+ \frac{2\alpha_n}{1 - \alpha\gamma\alpha_n}\langle \gamma f(\omega) - A\omega, x_{n+1} - \omega\rangle.$$

It is easily verified from the condition (i), (3.29) and Lemma 2.5, we get that $\{x_n\}$ converges strongly to ω. This completes the proof.

Remarks. In our Theorem 3.1,

(1) If setting $S_n^k \equiv S$, $k = 0$, $\mu_n = 0$, $y_n = u_n$ for all $n \in \mathbb{N}$, $B = I$, and $G_2 = 0$ for all $x, y \in K$, then our Theorem 3.1 reduces to theorem of Plubtieng and Punpaeng [10].

(2) If setting $S_n^k \equiv S_n$, $k = 0$, $\mu_n = 0$, $y_n = u_n$ for all $n \in \mathbb{N}$, $B = I$, and $G_2 = 0$ for all $x, y \in K$, then our Theorem 3.1 reduces to theorem of Khongtham and Plubtieng [6].

(3) If setting $S_n^k \equiv S$, $k = 0$, $\mu_n = 0$, $f(x_n) = x_n$, $\lambda_n = 1$, $u_n = x_n$ for all $n \in \mathbb{N}$, $\gamma = 1$, $A = I$ and $G_1 = 0$, $G_2 = 0$ for all $x, y \in K$, then our Theorem 3.1 reduces to theorem of Takahashi and Toyoda [12].

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