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A REPHRASED FORM OF THE GOVERNING EQUATIONS OF NATURAL CONVECTION BOUNDARY LAYER FLOWS

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Abstract

In this paper, the author derives a rephrased form of equations of conservation for a two-dimensional incompressible flow field inside a natural convection boundary layer.

The novelty of this work is the obtainment of an approximate analytical solution to the system of equations of mass and momentum conservation for natural convection boundary layer flows with the related boundary conditions, without the contribution of energy equation.

According to the adopted approach, there exists no distinction of the flow field as external or internal; hence this proposed method can be effective for both types of flow.

1. Introduction

Primarily, let us suppose the investigated flow patterns as incompressible

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steady and laminar. The necessary and sufficient condition for the existence of such a flow is the following [6]:

$$\begin{aligned} &\frac{\partial T}{\partial x} > 0 \\ &\wedge \\ &\frac{\partial \rho}{\partial x} < 0. \end{aligned} \tag{1.1}$$

Moreover, we can also assume that the work done by frictional forces during the fluid motion is insignificant, hence the dissipation of mechanical energy is negligible.

In the sequel, by simplifying the initial original problem in a two-dimensional one, we can write out the fundamental equations of conservation as follows [11, 12, 14]:

Mass conservation:

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0. \tag{1.2A}$$

Momentum conservation for axes xx' and yy':

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g_x + \frac{\mu}{\rho}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right), \tag{1.2B}$$

$$V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + g_y + \frac{\mu}{\rho}\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2}\right), \tag{1.2C}$$

where the term $\frac{\partial P}{\partial x}$ depends on the geometry of the obstacle or the duct for external or internal flows, respectively.

Energy conservation:

$$V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} = a \frac{\partial^2 T}{\partial y^2}, \tag{1.3}$$

where a denotes the coefficient of thermal diffusion.

However, according to the particular method that we will develop here, the system of equations (1.2A), (1.2B) and (1.2C) will be solved approximately, without the association of equation (1.3).

2. Towards a Rephrased Form of Momentum Equation

If we concentrate on the two equations of momentum conservation, then we have to clarify primarily that by taking also into account the free convection of fluid matter due to buoyancy effect, then the components of gravitational acceleration on these equations cannot be neglected. Buoyancy is mainly caused by combination of differences in fluid density and also the circumstantial body force must be proportional to density [3, 4, 16]. Body forces can be either gravity, or Coriolis force in atmosphere and oceans. Convection flow is driven by buoyancy in unstable conditions [6, 16]. Next, we can implement the known from literature Prandtl's simplified assumptions, which concern only boundary layer flows and are synopsized as follows [14]:

$$V_x >> V_y,$$

$$\frac{\partial V_x}{\partial y} >> \frac{\partial V_x}{\partial x},$$

$$\frac{\partial^2 V_x}{\partial y^2} >> \frac{\partial^2 V_x}{\partial x^2},$$

$$\frac{\partial T}{\partial y} >> \frac{\partial T}{\partial x},$$

$$\frac{\partial P}{\partial x} >> \frac{\partial P}{\partial y}.$$

Hence, if one takes also into consideration the buoyancy effect in accordance with Prandtl's approximations, then the fundamental equations of conservation are written out as follows [14]:

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0, \tag{2.1}$$

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = g\beta(T - T_\infty) + \frac{\mu}{\rho} \frac{\partial^2 V_x}{\partial y^2}, \tag{2.2A}$$

$$\frac{\partial P}{\partial y} = 0, \tag{2.2B}$$

$$V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} = a \frac{\partial^2 T}{\partial y^2}, \tag{2.3}$$

where β denotes the coefficient of thermal expansion and generally emerges from the following relationship [3, 11, 16]:

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P. \tag{2.4A}$$

For boundary layer incompressible flows, this coefficient can be estimated by the following approximate expression [3, 11, 16]:

$$\beta \approx -\frac{1}{\rho} \frac{\rho - \rho_\infty}{T - T_\infty}. \tag{2.4B}$$

The boundary conditions which complete this system of PDEs are:

$$y = 0 \Rightarrow \begin{aligned} V_x &= 0 \\ V_y &= 0 \\ T &= T_s \end{aligned} \tag{2.5A}$$

and

$$y \to +\infty \Rightarrow \begin{aligned} V_x &= 0 \\ \partial V_y / \partial y &= 0 \\ T &= T_\infty, \end{aligned} \tag{2.5B}$$

where T_s is the temperature of the solid boundary and T_∞ is the temperature outside boundary layer. These terms are considered as a constant. We must also clarify that this boundary condition does not concern the frictionless core for internal flows.

By a combination of equations (2.1) and (2.2A), the following equality arises:

$$V_y \frac{\partial V_x}{\partial y} - V_x \frac{\partial V_y}{\partial y} = g\beta(T - T_\infty) + \frac{\mu}{\rho} \frac{\partial^2 V_x}{\partial y^2}. \tag{2.6}$$

On the other hand, it is also known from multi-valued Calculus that the following identity holds:

$$\frac{\partial}{\partial y}\left(\frac{V_y}{V_x}\right) \equiv \frac{V_x \cdot \dfrac{\partial V_y}{\partial y} - V_y \cdot \dfrac{\partial V_x}{\partial y}}{V_x^2}.$$

Hence, one infers:

$$V_y \cdot \frac{\partial V_x}{\partial y} - V_x \cdot \frac{\partial V_y}{\partial y} \equiv -V_x^2 \frac{\partial}{\partial y}\left(\frac{V_y}{V_x}\right). \tag{2.7}$$

Thus, the equation of momentum conservation for axis xx' which coincides with the governing direction of the flow, results in the following rephrased form:

$$-V_x^2 \frac{\partial}{\partial y}\left(\frac{V_y}{V_x}\right) = g\beta(T - T_\infty) + \frac{\mu}{\rho} \frac{\partial^2 V_x}{\partial y^2} \Leftrightarrow$$

$$V_x^2 \frac{\partial}{\partial y}\left(-\frac{V_y}{V_x}\right) = g\beta(T - T_\infty) + \frac{\mu}{\rho} \frac{\partial^2 V_x}{\partial y^2} \Leftrightarrow$$

$$V_x^2 \frac{\partial}{\partial y}\left(-\frac{1}{\dfrac{V_x}{V_y}}\right) = g\beta(T - T_\infty) + \frac{\mu}{\rho} \frac{\partial^2 V_x}{\partial y^2} \Leftrightarrow$$

$$V_y^2 \frac{\partial}{\partial y}\left(\frac{V_x}{V_y}\right) = g\beta(T - T_\infty) + \frac{\mu}{\rho} \frac{\partial^2 V_x}{\partial y^2} \Leftrightarrow$$

$$\frac{V_x V_y}{\dfrac{V_x}{V_y}} \frac{\partial}{\partial y}\left(\frac{V_x}{V_y}\right) = g\beta(T - T_\infty) + \frac{\mu}{\rho} \frac{\partial^2 V_x}{\partial y^2} \Leftrightarrow$$

$$V_xV_y \frac{\partial}{\partial y}\left(\ln\left|\frac{V_x}{V_y}\right|\right) = g\beta(T - T_\infty) + \frac{\mu}{\rho}\frac{\partial^2 V_x}{\partial y^2} \Leftrightarrow$$

$$e^{\ln V_x + \ln V_y} \frac{\partial}{\partial y}(\ln|V_x| - \ln|V_y|) = g\beta(T - T_\infty) + \frac{\mu}{\rho}\frac{\partial^2 V_x}{\partial y^2}. \tag{2.8}$$

Moreover, it is also known from single-valued Calculus that for any differentiable function $\phi(x)$, the following identity holds [9]:

$$\frac{x}{\phi(x)} \cdot \frac{d\phi(x)}{dx} \equiv \frac{d(\ln \phi(x))}{d \ln x}. \tag{2.9}$$

The above identity guaranties the validity of Prandtl's simplified assumptions, for logarithmic scales as well.

Hence, taking into account that $V_x >> V_y$, we can obtain the following approximate modified form for the equation of momentum conservation along the boundary layer direction:

$$V_x \frac{\partial}{\partial y}(\ln|V_x|) = g\beta(T - T_\infty) + \frac{\mu}{\rho}\frac{\partial^2 V_x}{\partial y^2} \Leftrightarrow$$

$$\frac{\mu}{\rho}\frac{\partial^2 V_x}{\partial y^2} - \frac{\partial V_x}{\partial y} + g\beta(T - T_\infty) = 0 \Leftrightarrow$$

$$\frac{\partial^2 V_x}{\partial y^2} - \frac{\rho}{\mu}\frac{\partial V_x}{\partial y} = -\frac{\rho}{\mu} g\beta(T - T_\infty). \tag{2.10}$$

Letting $y \to +\infty$, equation (2.10) yields:

$$\lim_{y\to+\infty} \frac{\partial^2 V_x}{\partial y^2} = \lim_{y\to+\infty}\left(-\frac{\rho}{\mu} g\beta(T - T_\infty)\right) = -\frac{\rho}{\mu} g\beta(T_\infty - T_\infty) = 0. \tag{2.11}$$

Since Prandtl's assumptions assert us that $V_x >> V_y$, $\frac{\partial V_x}{\partial y} >> \frac{\partial V_x}{\partial x}$, $\frac{\partial^2 V_x}{\partial y^2} >> \frac{\partial^2 V_x}{\partial x^2}$, $\frac{\partial T}{\partial y} >> \frac{\partial T}{\partial x}$, equation (2.10) in accordance with (2.5A) and (2.5B) substantially reduces to the following system:

$$\frac{d^2V_x(y)}{dy^2} - \frac{\rho}{\mu}\frac{dV_x(y)}{dy} = -\frac{\rho}{\mu}g\beta T(y) + \frac{\rho}{\mu}g\beta T_\infty, \tag{2.12}$$

$$y = 0 \Rightarrow V_x(y) = 0, \tag{2.12A}$$

$$y \to +\infty \Rightarrow |V_x(y)| + \left|\frac{dV_x(y)}{dy}\right| + \left|\frac{d^2V_x(y)}{dy^2}\right| = 0. \tag{2.12B}$$

Obviously, the complementary homogeneous equation of equation (2.12) has the form:

$$\frac{d^2V_x(y)}{dy^2} - \frac{\rho}{\mu}\frac{dV_x(y)}{dy} = 0. \tag{2.13}$$

The corresponding auxiliary algebraic equation is:

$$r^2 - \frac{\rho}{\mu}r = 0 \tag{2.13A}$$

with roots $r_1 = 0,\ r_2 = \frac{\rho}{\mu}$.

Hence, the general solution of equation (2.13) is [7, 8, 17]:

$$V_{x0}(y) = C_1 + C_2 e^{\frac{\rho}{\mu}y}, \quad \forall C_1, C_2 \in R^*. \tag{2.14}$$

In continuing, according to the method of variation of parameters [7, 8, 12, 17], let us replace the constants, (or parameters), in equation (2.14) by the arbitrary single-valued functions $u_1(y)$ and $u_2(y)$.

Therefore, we can seek for a particular solution of the inhomogeneous equation (2.12) expressed in the form:

$$V_{xp}(y) = u_1(y) + u_2(y)e^{\frac{\rho}{\mu}y}. \tag{2.15}$$

Next, differentiating equation (2.15) with respect to variable y, we obtain:

$$V'_{xp}(y) = u'_1(y) + u'_2(y)e^{\frac{\rho}{\mu}y} + \frac{\rho}{\mu}u_2(y)e^{\frac{\rho}{\mu}y}. \tag{2.16}$$

Since $u_1(y)$ and $u_2(y)$ are actually arbitrary functions, we can impose two auxiliary conditions on them without violating the generality of our presented mathematical formalism. The first condition is obviously that the function $V_{xp}(y)$ consists in a solution to equation (2.12). Concurrently, we can choose the second condition, such that to simplify our algebraic manipulations.

Therefore, let us formulate the following constraint for these functions, which must hold identically:

$$u_1'(y) + u_2'(y)e^{\frac{\rho}{\mu}y} \equiv 0. \tag{2.17}$$

On the other hand, by differentiating equation (2.16) with respect to variable y and also taking into account the above restriction, we deduce:

$$V_{xp}''(y) = \frac{\rho}{\mu}u_2'(y)e^{\frac{\rho}{\mu}y} + \frac{\rho^2}{\mu^2}u_2(y)e^{\frac{\rho}{\mu}y}. \tag{2.18}$$

Substituting the above data back into equation (2.12), we find

$$\frac{\rho}{\mu}u_2'(y)e^{\frac{\rho}{\mu}y} + \frac{\rho^2}{\mu^2}u_2(y)e^{\frac{\rho}{\mu}y} - \frac{\rho^2}{\mu^2}u_2(y)e^{\frac{\rho}{\mu}y} = -\frac{\rho}{\mu}g\beta T(y) + \frac{\rho}{\mu}g\beta T_\infty \Leftrightarrow$$

$$u_2'(y)e^{\frac{\rho}{\mu}y} = -g\beta T(y) + g\beta T_\infty \overset{\text{equation (2.17)}}{\Leftrightarrow}$$

$$u_1'(y) = -g\beta T(y) + g\beta T_\infty \Leftrightarrow$$

$$u_1(y) = -g\beta\int_0^t T(y)dt + g\beta y T_\infty, \tag{2.19}$$

where the term $\int_0^t T(y)dt$ obviously constitutes in the set of the antiderivatives of the function $T(y)$ with respect to y.

Returning to equation (2.15), we obtain

$$V_{xp}(y) = -g\beta\int_0^t T(y)dt + g\beta y T_\infty + u_2(y)e^{\frac{\rho}{\mu}y}. \tag{2.20}$$

Thus, the complete solution of equation (2.12) reads

$$V_x(y) = C_1 + C_2 e^{\frac{\rho}{\mu} y} - g\beta \int_0^t T(y)dt + g\beta y T_\infty + u_2(y) e^{\frac{\rho}{\mu} y},$$

$$\forall u_2 \in C_{(a,b)}^{(1)}, \quad a, b \in R \cup \{-\infty, +\infty\}. \tag{2.21}$$

Apparently this solution ought to verify equations (2.12A), (2.12B) as well as the equation of energy conservation.

3. Discussion

In this paper, the author obtained an approximate rephrased form of equations of conservation for a two-dimensional, boundary layer incompressible flow field.

The novelty of this paper mainly concentrates on the accomplishment of an approximate analytical solution to the system of equations of mass and momentum conservation for a generic type of natural convection boundary layer flows without the simultaneous contribution of energy equation.

However, this resultant solution concerning velocity profile that is presented here ought to satisfy throughout energy equation. We also emphasize that the method that we performed in this article concerns both external and internal flows.

Evidently, for isothermal flows this expression which describes the sequential velocity profiles along the boundary layer becomes more simplified.

References

[1] G. Arfken, Mathematical Methods for Physicists, 2nd ed., Academic Press, New York, 1970.

[2] A. Bar-Cohen and W. M. Rohsenow, Thermally optimum spacing of vertical natural convection cooled parallel plates, J. Heat Transfer 106 (1984), 116-123.

[3] R. B. Bird, W. E. Stewart and E. N. Lighfoot, Transport Phenomena, 2nd ed., John Wiley & Sons, 2002.

[4] S. W. Churchil and H. S. Chu, Correlating equations for laminar and turbulent free convection from a vertical plate, Int. J. Heat Mass Transfer 18 (1975), 1323-1329.

[5] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. II, John Wiley & Sons, New York, 1962 (Reprint 1989).

[6] F. R. Incropera and D. P. Dewitt, Fundamentals of Heat and Mass Transfer, 5th ed., John Wiley & Sons, 2002.

[7] M. Krasnov, Ordinary Differential Equations, MIR Publ., Moscow, 1983.

[8] M. Krasnov, M. Kiselov and G. Makarenko, A Book of Ordinary Differential Equations, MIR Publ., Moskow, 1981.

[9] D. Logan, Applied Mathematics: A Contemporary Approach, John Wiley & Sons, Inc., 1987.

[10] G. R. Liu, Mesh Free Methods, Moving Beyond the Finite Element Method, CRC Press, 2002.

[11] L. J. Nicolescu and M. I. Stoka, Mathematics for Engineering 1, Tunbridge Wells-Kent, 1974.

[12] N. Pavel, Differential equations, flow invariance and applications, Research Notes in Math., Vol. 113, London 1984.

[13] G. D. Raithby et al., Handbook of Heat Transfer Fundamentals, Chap. 6, McGraw-Hill, New York, 1985.

[14] H. Schlichting, Boundary Layer Theory, McGraw-Hill, New York, 1968 (Reprint 1977).

[15] B. L. Van der Waerden, Modern Algebra, Volumes I and II, Frederick Ungar Publ. Co., New York, 1964.

[16] J. R. Welty et al., Fundamentals of Momentum Heat and Mass Transfer, 4th ed., John Wiley & Sons, 2001.

[17] H. K. Wilson, Ordinary Differential Equations, Addison-Welsey, New York, 1971.