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# A NOTE ON SOLVING PRECONDITIONED LINEAR SYSTEM

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**Abstract**

In this note we point out that there is an erroneous conclusion in Theorem 3b(ii) of [Linear Algebra Appl. 364 (2003), 253-279] due to a sufficient condition considered. By adopting a sufficient condition, already present in [Linear Algebra Appl. 364 (2003), 253-279], we prove that the correctness of the aforementioned conclusion is restored and a stronger result can then be proved.

## 1. Introduction

Consider the following linear system:

$$Ax = b, \tag{1.1}$$

where $A$ is an $n \times n$ square matrix, $x$ and $b$ are $n$-dimensional vectors.

A preconditioned system of (1.1) is

$$PAx = Pb. \tag{1.2}$$

The preconditioner $P$ can be taken as different types for solving linear

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system (1.1). Here we let the preconditioner be $P(\alpha) = I + S(\alpha)$, where

$$S(\alpha) = \begin{bmatrix} 0 & -\alpha_1 a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_2 a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1} a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{1.3}$$

The preconditioned system (1.2) with preconditioner (1.3) has been discussed by many authors.

Let $A = I - L - U$, where $L$ and $U$ are a strictly lower triangular and a strictly upper triangular matrices, respectively. Applying the preconditioner $P(\alpha) = I + S(\alpha)$ on (1.1), we obtain the equivalent linear system

$$\hat{A}(\alpha)x = \hat{b}(\alpha),$$

with $\hat{A}(\alpha) = (I + S(\alpha))A$, $\hat{b}(\alpha) = (I + S(\alpha))b$, where $\hat{A}(\alpha) = (\hat{a}_{ij}(\alpha))$,

$$\hat{a}_{ij}(\alpha) = \begin{cases} a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j} & i = 1, 2, \ldots, n-1, \;\; j \neq i+1, \\ (1 - \alpha_i) a_{i,i+1} & i = 1, 2, \ldots, n-1, \;\; j = i+1, \\ a_{nj} & i = n, \;\; j = 1, 2, \ldots, n, \end{cases}$$

$\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_{n-1}] \in R^{n-1}$, $\alpha_i \geq 0$, $i = 1, 2, \ldots, n-1$. We rewrite

$$\hat{A}(\alpha) = \hat{D}(\alpha) - \hat{L}(\alpha) - \hat{U}(\alpha), \tag{1.4}$$

with $\hat{D}(\alpha)$, $\hat{L}(\alpha)$, $\hat{U}(\alpha)$ diagonal and strictly lower and strictly upper triangular matrices. Defining the matrices $\hat{D}_\alpha = diag(\alpha_1 a_{12} a_{21}, \ldots, \alpha_{n-1} a_{n-1,n} a_{n,n-1}, 0)$ and $S(\alpha)L = \hat{L}_\alpha + \hat{D}_\alpha$, where $\hat{D}_\alpha$, $\hat{L}_\alpha$ are the diagonal and the strictly lower triangular components of $S(\alpha)L$. Then from (1.4), we have

$$\hat{D}(\alpha) = I - \hat{D}_\alpha, \;\; \hat{L}(\alpha) = L + \hat{L}_\alpha, \;\; \hat{U}(\alpha) = (I + S(\alpha))U - S(\alpha).$$

Clearly, $U - S(\alpha)$, $\hat{L}(\alpha)$, $\hat{U}(\alpha)$ are nonnegative, the diagonal elements of $\hat{D}(\alpha)$ are positive. We consider the following splittings:

$$\hat{A}(\alpha) = \begin{cases} M(\alpha) - N(\alpha) = (I + S(\alpha)) - (I + S(\alpha))(L + U), \\ M'(\alpha) - N'(\alpha) = I - [L + \hat{L}_\alpha + \hat{D}_\alpha + (I + S(\alpha))U - S(\alpha)], \\ M''(\alpha) - N''(\alpha) = (I - \hat{D}_\alpha) - [L + \hat{L}_\alpha + (I + S(\alpha))U - S(\alpha)]. \end{cases}$$

The corresponding Jacobi type iteration matrices as well as Gauss-Seidel type ones are as follows, respectively:

$$B(\alpha) = M^{-1}(\alpha)N(\alpha) = L + U,$$

$$\hat{B}'(\alpha) = M'^{-1}(\alpha)N'(\alpha) = L + \hat{L}_\alpha + \hat{D}_\alpha + (I + S(\alpha))U - S(\alpha),$$

$$\hat{B}''(\alpha) = M''^{-1}(\alpha)N''(\alpha) = (I - \hat{D}_\alpha)^{-1}(L + \hat{L}_\alpha + (I + S(\alpha))U - S(\alpha)),$$

and

$$H(\alpha) = (I - L)^{-1}U,$$

$$\hat{H}'(\alpha) = [I - L - \hat{L}_\alpha]^{-1}[\hat{D}_\alpha + (I + S(\alpha))U - S(\alpha)],$$

$$\hat{H}''(\alpha) = (I - L - \hat{L}_\alpha - \hat{D}_\alpha)^{-1}((I + S(\alpha))U - S(\alpha)).$$

In Theorem 3.1 of reference [2] among others, the following inequalities were presented

$$\rho(\hat{H}''(\alpha)) < \rho(\hat{H}'(\alpha)) < 1, \tag{1.5}$$

$$\rho(\hat{B}''(\alpha)) < \rho(\hat{B}'(\alpha)) < 1, \tag{1.6}$$

under the assumption that $A$ is an irreducible nonsingular $M$-matrix and $\alpha_i = 1,\ i \in \{j : a_{j,\,j+1} \neq 0\}$. But their proof of this result is based on the fact that $\hat{A}(\alpha)$ is irreducible. However it is not right. For example, let

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -\frac{1}{3} & 0 & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{pmatrix}.$$

Then $A$ is an irreducible nonsingular $M$-matrix with $a_{12}a_{23} = 1$ and for $\alpha_i = 1, i = 1, 2, 3,$ we have

$$\hat{A}(\alpha) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -\frac{1}{3} & 1 & 0 & -\frac{1}{3} \\ -\frac{4}{9} & -\frac{1}{9} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{pmatrix}.$$

So $\hat{A}(\alpha)$ is an irreducible nonsingular $M$-matrix. Since $\hat{D}_\alpha = 0,$ it is easy to see that $\hat{H}''(\alpha) \equiv \hat{H}'(\alpha)$ and

$$\rho(\hat{H}''(\alpha)) = \rho(\hat{H}'(\alpha)).$$

And so (1.5) is not right. By the following example one can see although $A$ is an irreducible nonsingular $M$-matrix with $a_{12}a_{23} = 1,$

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix},$$

let $\alpha_i = 1, i = 1, 2, 3.$ Then

$$\hat{A}(\alpha) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix},$$

$\hat{A}(\alpha)$ is reducible and it is easy to see that $\rho(\hat{H}''(\alpha)) = \rho(\hat{H}'(\alpha)) = \frac{1}{2}.$ For the Jacobi type methods (1.6), there are also similar errors. In general, we cannot obtain the inequalities (1.5) and (1.6). Here we give an active assumption on $A$ so that the two inequalities are true.

## 2. Preliminaries

We need some notations and definitions in this paper.

For an $n \times n$ matrix $A$, the *directed graph* $\Gamma(A)$ of $A$ is defined to be the pair $(V, E)$, where $V = \{1, \ldots, n\}$ is a set of vertices and $E = \{(i, j) : a_{ij} \neq 0, i, j = 1, \ldots, n\}$ is a set of arcs. A *path from i to j of length k* in $\Gamma(A)$ is a sequence of vertices $\sigma = (i_0, i_1, \ldots, i_k)$, where $i_0 = i$ and $i_k = j$ such that $(i_0, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k)$ are arcs of $\Gamma(A)$. A path $\sigma$ is called a *closed path* if $i = j$. We say a directed graph $\Gamma(A)$ to be *strongly connected* if for any two vertices $i, j$ there is a path from $i$ to $j$ in $\Gamma(A)$. A matrix $A$ is said to be *irreducible* if $\Gamma(A)$ is strongly connected.

A matrix $A = (a_{ij})$ is called a *Z-matrix* if for any $i \neq j$, $a_{ij} \leq 0$ and *M-matrix* if $A = sI - B$, $B \geq 0$ and $s \geq \rho(B)$, where $\rho(B)$ denotes the spectral radius of $B$.

$A = M - N$ is said to be a *splitting* of $A$ if $M$ is nonsingular. A splitting $A = M - N$ is said to be an *M-splitting* if $M$ is a nonsingular $M$-matrix and $N \geq 0$.



## 3. A Modified Result

**Lemma 3.1** [4]. *Let* $A = M_1 - N_1 = M_2 - N_2$ *be both M-splittings of A and*

$$N_1 \geq N_2,\ N_1 \neq N_2,\ N_2 \neq 0.$$

*Then exactly one of the following statements holds*:

(1) $0 \leq \rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_1) < 1.$

(2) $\rho(M_2^{-1}N_2) = \rho(M_1^{-1}N_1) = 1.$

(3) $\rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1) > 1.$

*In the case A irreducible, all inequalities of* (1) *and* (3) *are strict.*

**Theorem 3.2.** *Let* $A = (a_{ij}) \in R^{n\times n}$ $(n \geq 3)$ *with* $a_{i,i+1}a_{i+1,i} > 0$, $i = 1, \ldots, n-1$ *and let* $A$ *be a nonsingular M-matrix. Then for any* $\alpha_i \in (0, 1]$, $i = 1, \ldots, n-1$, $\rho(\hat{H}''(\alpha)) < \rho(\hat{H}'(\alpha)) < 1$.

**Proof.** Clearly, $A$ is irreducible. From $\hat{A}(\alpha) = (\hat{a}_{ij}(\alpha))$, we know

$$\hat{a}_{ij}(\alpha) = \begin{cases} a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j}, & 1 \leq i < n, \\ a_{nj}, & i = n. \end{cases}$$

Hence $\hat{a}_{i+1,i}(\alpha) < 0$, $1 \leq i < n$ and $\hat{a}_{i,i+2}(\alpha) = a_{i,i+2} - \alpha_i a_{i,i+1} a_{i+1,i+2} < 0$, $1 \leq i < n-1$.

This implies that $\Gamma(\hat{A}(\alpha))$ has some paths such as $(n, n-1, \ldots, 3, 2, 1)$, $(1, 3, \ldots, n)$ and $(2, 4, \ldots, n-1)$ when $n$ is odd, and $(n, n-1, \ldots, 3, 2, 1)$, $(1, 3, \ldots, n-1)$ and $(2, 4, \ldots, n)$ when $n$ is even, from which one may deduce that $\hat{A}(\alpha)$ is irreducible. Let $\hat{D}_\alpha = diag(d_1, \ldots, d_n)$. Then $d_i = \alpha_i a_{i,i+1} a_{i+1,i} > 0$, $i = 1, \ldots, n-1$. It is easy to see that the splittings

$$\begin{aligned} \hat{A}(\alpha) &= (I - L - \hat{L}_\alpha) - [\hat{D}_\alpha + (I + S(\alpha))U - S(\alpha)] \\ &= [I - L - S_\alpha L] - [(I + S(\alpha))U - S(\alpha)] \end{aligned}$$

are both $M$-splittings of an irreducible nonsingular $M$-matrix with

$$[\hat{D}_\alpha + (I + S(\alpha))U - S(\alpha)] > [(I + S(\alpha))U - S(\alpha)].$$

Then the inequality (1.5) follows from Lemma 3.1.

**Remark.** By the example given in Section 1 we know that the condition $a_{i,i+1}a_{i+1,i} > 0$, $i = 1, \ldots, n-1$ cannot be omitted in Theorem 3.2. The same case to the inequality (1.6).

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