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RESTRICTIONS ON THE POWERS OF *n*TH-ALUTHGE TRANSFORMS OF *w*-HYPONORMAL OPERATORS

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Abstract

Powers of operators from any class, including the normal operators, are not in general members of the same class. For instance, if T is a class (A) operator, then T^2 is not necessarily a class (A) operator, but if T is an invertible class (A) operator, all of its powers happen to be class (A) operators. Unlike in class (A) operators, every power of a w-hyponormal operator is a w-hyponormal operator. In this paper, we investigate the normality of the powers of generalized Aluthge transforms of w-hyponormal operators and that of the generalized Aluthge transforms of the powers of w-hyponormal operators.

Introduction

One interesting problem in the operator theory is to investigate some conditions under which certain operators are normal. Several mathematicians have paid attention to this problem, see [16, 20, 22, 24, 25, 27, 28, 37, 39-42, 46] and references therein. One of interesting articles, which presents some results about this topic is that of Stampfli [49]. He showed, among other things, that for a hyponormal operator A, if A^n is normal for some positive

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integer n, then A is normal. The problem had already been considered in the case when $n = 2$ by Putnam [47]. The results were generalized later to the other classes of operators by a number of authors, for instance, Embry [45], Radjavi and Rosenthal [48] and Duggal [44]. There is another point of view about this issue via spectrum of an operator. In [49], it is proved that if the spectrum of a hyponormal operator contains only a finite number of limited points or has zero area, then the operator is normal. Using Aluthge transform, this aspect is generalized to p-hyponormal and log-hyponormal operators [41]. In fact, if T is p-hyponormal or log-hyponormal, then $\widetilde{T}_2$ is hyponormal. The singular value decomposition [16, 27, 41], leads to another form of decomposition, popularly referred to as the polar decomposition. That is, every operator T can be written as $T = UP$, where U is unitary and P is positive. In this type of decomposition, the positive part P is unique, U is unique if T is invertible and the kernels of both T, U and P are the same. It is well known that T is normal iff U and P commute. However, given any two normal operators, say $A, B \in B(H)$, it is known that in general, AB is not normal. The question of characterizing those pairs of normal operators for which the products become normal has been solved for finite dimensional spaces by F. R. Gantmaher and M. G. Klein in 1930, [16, 30], and for compact normal operators by N. A. Wiegmann in 1949, [27]. Actually, in the aforementioned cases, the normality of AB is equivalent to that of BA. A more general result by Kittaneh [39], implies that, it is sufficient that AB be normal and compact to obtain that BA is also normal.

All log-hyponormals constitute a subclass of p-hyponormals and p-hyponormal operators extend hyponormal operators only when $p \in (0, 1]$. Otherwise, by the highly cerebrated Lowner-Heinz inequality, that is if A, B are any two positive operators such that $A \leq B$, then $A^r \leq B^r$, $\forall r \in (0, 1]$, it follows easily that every p-hyponormal is a q-hyponormal whenever $p \geq q$. Thus, the class of semi-hyponormal operators is larger than the class of p-hyponormals. However, if $p \geq 1$, then the inclusion series, viz., hyponormal $\subset$ p-hyponormal, might be reversed. And thus, some p-hyponormal operators might become hyponormal operators. However, in almost all extensions of p-hyponormal or hyponormal operators, such as the w-hyponormals, class (A), class $A(s, t)$, class $A(k)$, class wA, class $wA(s, t)$, etc., this reverse in the inclusion series is not exhibited. In particular, class $A(s_1, t_1) \subset$ class $A(s_2, t_2)$ and class $wA(s_1, t_1) \subset$ class $wA(s_2, t_2)$ for every pair of positive-real pairs s_1, s_2 and t_1, t_2 such that $s_1 \leq s_2$ and $t_1 \leq t_2$. Before finding conditions which restrict all powers of a given operator from a particular non-normal operator class to normal operators - a process which in some cases become imponderable -, it is good, at least to know which of these operator classes are closed under non-zero power scaling. That is, if an operator T is a member of class, say $\{Z\}$, is $T^n \in \{Z\}$, for every positive integer n?

The question of classifying T^n after identifying the class of T, has been looked into by a number of researchers. After introducing w-hyponormal operators, Aluthge and Wang [2] observed that if T is a w-hyponormal operator, then T^n is also a w-hyponormal if T is invertible. To extend this observation, Chō et al. [12] showed that T^2 is a w-hyponormal whenever T is a w-hyponormal under a weaker condition that $\ker(T) = 0$. These two results were later generalized by Yanagida [23], who proved that if T is a class $wA(s, t)$ operator, then T^n is a class $wA(s/n, t/n)$ operator which led to the wonderful conclusion that T^n is a w-hyponormal operator even when T is not an invertible w-hyponormal operator. Yamazaki [20, 21] showed that if T is a log-hyponormal, then T^n is also a log-hyponormal and that if T is a class $AI(s, t)$ operator, then T^n is a class $AI(s/n, t/n)$ operator. Ito [19] proved that if T is a p-hyponormal, then T^n is a $\min[1, 1/p]$-hyponormal. He extended this result on p-hyponormal, to class (A) and paranormal operators and concluded that; if T is a class (A) operator, then T^n is a class (A) operator under the condition that T is invertible, and that; if T is a paranormal operator, then T^n is also paranormal and if T is an invertible paranormal operator, then T^{-1} is also a paranormal operator.

n-power normal and n-power quasinormal operators have been studied extensively by several authors. Some of these authors include Mecheri [37], Jibril [34, 35], Bala [28], Jeon et al. [36] and Ahmed [25]. Bala [28] was among the earliest researchers to study quasinormal operators. He proved that every normal operator is a quasinormal operator and gave some examples of quasinormal operators which are not normal. Mecheri [37] extended normal operators to n-power normal operators. This class of operators is not an extension of the quasinormal operators, neither is the class of quasinormal operators an extension of this class. However, Jibril [34, 35] studied the relationships between quasinormal, n-power normal and n-power quasinormal operators. He proved that if an operator is n-power normal, for some positive integer n, then the nth-power of this operator is normal and converse. In general, he proved that every n-power normal operator is an n-power quasinormal operator. Thus, the class of n-power quasinormal operators is very large. Conditions under which members of this class are relaxed to n-power normal were studied by Mecheri [37]. In this paper, he proved that the kernel condition does not hold in general in this class but every n-power quasinormal operator is n-power normal, for the same integer n, whenever this condition is satisfied. Recently, Ahmed [25] generalized results by Mecheri and Jibril. Amongst other beautiful observations, he proved that if any operator and its adjoint are both n-power quasinormal, then the nth-power of such an operator is normal; if any operator and its square are both n-power quasinormal and if in addition, such an operator is in the class of 3-power quasinormal, then its square happens to be a quasinormal operator and he also, in the same paper, gave an example of a 2-power quasinormal operator which is not 3-power quasinormal. As far as the spectra of these operators are concerned, Ahmed also proved that if an operator is 2-power quasinormal, then such an operator is also normal provided zero is an isolated point in its spectrum.

In the quest to come up with an easier method of conquering complicated operators, some analysts have, instead of 'braking', transformed them into other forms with common major properties but easier to handle.

Recently, Aluthge et al. [2, 3, 13] generalized the polar decomposition by transforming an operator T into another operator $\widetilde{T}$ called the *Aluthge transform* of *T*. That is, an operator $\widetilde{T} = |T|^{1/2} + |T|^{1/2}$ is called the *Aluthge transformation* of *T* whose polar decomposition is $T = U|T|$, where $|T| = (T^*T)^{1/2}$. More precisely, $\widetilde{T} = |T|^{1/2} + |T|^{1/2}$ is called the *first Aluthge transform* and $\widetilde{\widetilde{T}} = \widetilde{T}_2 = |\widetilde{T}|^{1/2} + |\widetilde{T}|^{1/2}$ is called the *second Aluthge transform* of *T*. In general, $\widetilde{T}_n = |\widetilde{T}_{n-1}|^{1/2} + |\widetilde{T}_{n-1}|^{1/2}$, $\forall n \in N$ is called the *nth-Aluthge transform* of *T*, where $\widetilde{T}_n = \widetilde{U}_n|\widetilde{T}_n|$ is the polar decomposition of $\widetilde{T}_n$. It follows that $\|\widetilde{T}_{n-1}\| \geq \|\widetilde{T}_n\|$, $\forall n \in J^+$ in general. Note that $T = |\widetilde{T}_0|^{1/2} + |\widetilde{T}_0|^{1/2}$ is called the *null Aluthge transform* of *T*.

Aluthge transforms of different classes of operators have been studied extensively. For instance, it is well known [3, 23], that the Aluthge transform of an operator 'improves' *p*-hyponormality of an operator for $p \leq 1$ since if T is *p*-hyponormal for $1/2 \leq p < 1$, then $\widetilde{T}$ is hyponormal and if T is *p*-hyponormal for $0 < p < 1/2$, then $\widetilde{T}$ is $p + 1/2$-hyponormal. It is also known that if an operator *T* is *w*-hyponormal, then $\widetilde{T}$ is semi-hyponormal, $\widetilde{\widetilde{T}}$ is hyponormal and if $\widetilde{T}$ happens to be a normal operator, then its counterpart *T* is normal.

For both *p*- and log-hyponormal operators [7, 21] the kernel condition holds. But this condition is violated in general by *w*-hyponormal operators. However, many spectral properties satisfied by *p*- and log-hyponormal operators [21, 23] are inherited by *w*-hyponormal operators. Aluthge and Wang [2] studied the spectral properties of *w*-hyponormal operators. In this paper, they gave a characterization for an operator, say *T* to be *w*-hyponormal. In the same paper, they proved that if a *w*-hyponormal operator *T* satisfies the inequality $|\widetilde{T}| \geq |T|$, then *T* is also paranormal. In other words, they showed that every *w*-hyponormal operator is paranormal. Using Furuta's inequality, they also showed the following:

(i) if a w-hyponormal operator T satisfies the kernel condition and in addition if its first Aluthge transform is normal, then this T is also normal [2, Corollary 3];

(ii) every square of an invertible w-hyponormal operator is also w-hyponormal [3, Theorem 5.2];

(iii) for w-hyponormal operators, the kernel condition does not hold in general;

(iv) the non-zero points of the approximate and joint spectra of a w-hyponormal operator are identical [2, 3].

Chō et al. [12] generalized these results and proved that (i) holds without the kernel condition, (ii) holds without the invertibility if $\ker(T) = 0$ and there exists a w-hyponormal operator T such that $\ker(T^*) \not\subset \ker(T)$ and $\ker(T) \not\subset \ker(T^*)$.

In [22], we looked at some conditions under which the product of any two operators (each pair picked from either the class of n-power normal or that of n-power quasinormal operators) becomes normal even when the normality of the said operators is not necessarily clear.

In [24], we proved that all nth-Aluthge transforms of any operator have equal spectra, and in the case of w-hyponormal operators, we proved that every nth-Aluthge transform is spectraloid. In [52], we extended results obtained in [22] and [24]. Amongst other observations, we proved that:

Theorem A [52]**.** *Let* A, B *be any two commuting operators such that* A^{2^m} *and* B^{2^n} *are both normal for some* $m, n \in J^+$. *Then* $(A^{2^p} B^{2q})$ *is also a normal operator*, $\forall p, q \in J^+$, *where* $p \leq m$ *and* $q \leq n$, *if* $0 \notin W(A)$ *and* $0 \notin W(B)$.

Theorem B [52]**.** *Let* A, B *be any two w-hyponormal operators*. *If* $\widetilde{A}_n$ *and* $\widetilde{B}_n$ *are normal for some* $n \in J^+$ *and* $[A, B] = 0$, *then the following hold*:

(1) $[\widetilde{A}_n, \widetilde{B}_n] = 0$,

(2) $[\widetilde{A}_n^*, \widetilde{B}_n^*] = 0$,

(3) $(\widetilde{AB})_n$ *and* $(\widetilde{BA})_n$ *are normal operators*.

We seek herein to extend the two results above. In fact, our major task is to compare the variations between the powers of the *n*th-Aluthge transform with the *n*th-Aluthge transform of the powers of *w*-hyponormal operators.

Notations and Definitions

In what follows by an operator we mean a bounded linear transformation of a Hilbert space H into itself. Let $B(H)$ be the Banach algebra of all bounded linear operators on *H*. If $T \in B(H)$, T is said to be *self-adjoint* if and only if $T^* = T$, *unitary* if and only if $T^*T = TT^* = 1$, an *isometry* if and only if $T^*T = 1$, *normal* if and only if $T^*T = TT^*$, *quasinormal* if and only if $T(T^*T) = (T^*T)T$, a *projection* if and only if $T^2 = T$ and $T^* = T$, a *hyponormal* if and only if $T^*T \geq TT^*$, where $A \geq B$ means $A - B \geq 0$ for self-adjoint operators *A* and *B*.

We will denote the spectrum, the spectral radius, the numerical range and the numerical radius by $\sigma(T)$, $r(T)$, $W(T)$ and $w(T)$, respectively. $\sigma(T) = \{\lambda : (T - \lambda I) \text{ is not invertible}\}$, where λ is a complex number. $W(T) = \{\langle Tx, x\rangle : \| x \| = 1\}$ for *x* in *H*.

A complex number λ is said to be in the *point spectrum* $\sigma_p(T)$ of T if there is a non-zero vector x for which $(T - \lambda)x = 0$. If in addition $(T^\star - \overline{\lambda})x = 0$, then λ is said to be in the *normal point spectrum* $\sigma_{np}(T)$ of *T*. Thus, $\lambda \ni \sigma_{np}(T)$ if there is an eigenvector *x* corresponding to λ which is a normal eigenvector. λ is said to be in the *approximate point spectrum*, $\sigma_a(T)$ of *T* if there is a sequence $\{x_n\}$ of unit vectors for which $(T - \lambda)x_n$

$\to 0$. If in addition $(T^{\star} - \overline{\lambda})x_n \to 0$, then λ is said to be in the *normal approximate point spectrum* $\sigma_{na}(T)$ of *T*.

An operator $T \in B(H)$, is called (α, β)-*normal*, $0 \le \alpha \le 1 \le \beta$, if $\alpha^2 T^* T \le TT^* \le \beta^2 T^* T$, *n-power normal*, for a positive integer *n*, if $T^n T^* = T^* T^n$, *n-power quasinormal* if $T^n T^* T = T^* T^{n+1}$, $\forall n \in J^+$.

T is said to be *p-hyponormal* if $(T^{\star}T)^p - (TT^{\star})^p \ge 0$, for $0 < p \le 1$, *q-hyponormal* if $(T^{\star}T)^q - (TT^{\star})^q \ge 0$, for $0 < q \le p$,

quasihyponormal if $T^{\star}(T^{\star}T) - (TT^{\star})T \ge 0$,

p-quasihyponormal if $T^{\star}[(T^{\star}T)^p - (TT^{\star})^p]T \ge 0$, for $0 < p \le 1$,

q-quasihyponormal if $T^{\star}[(T^{\star}T)^q - (TT^{\star})^q]T \ge 0$, for $0 < q \le p$,

k-quasihyponormal if $T^{\star k}[(T^{\star}T) - (TT^{\star})]T^k \ge 0$, for $k \ge 1$.

log-hyponormal if *T* is invertible and $\log T^{\star}T \ge \log TT^{\star}$.

Absolute-(s, t)-*paranormal* if $\| \, |T|^s |T^{\star}|^t x \|^t \ge \| \, |T^{\star}|^t x \|^{s+t}$ for every unit vector *x* in *H*, where $s \ge 0$ and $t \ge 0$.

T is said to be *normaloid* if $\| T \| = r(T)$ (spectral radius of *T*) and (p, k)-*quasihyponormal* for a positive number $0 < p \le 1$ and positive integer *k*, if $T^{\star k}[(T^{\star}T)^p - (TT^{\star})^p]T^k \ge 0$.

The operator $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is called *Aluthge transformation* of an operator *T* whose polar decomposition is $T = U|T|$, where $|T| = (T^{\star}T)^{\frac{1}{2}}$.

T is a *w*-hyponormal operator if $|\widetilde{T}^*| \le |T| \le |\widetilde{T}|$,

T is in class *A* if $|T^2| \ge |T|^2$,

T belongs to class $A(k)$ for $k \geq 0$ if $(|T^\star||T|^{2k}|T^\star|)^{\frac{1}{k+1}} \geq (|T^\star|)^2$,

T belongs to class $wA(s, t)$ if for $s \geq 0$ and $t \geq 0$,

$$(|T^\star|^t|T|^{2s}|T^\star|^t)^{\frac{t}{s+t}} \geq (|T^\star|)^{2t} \text{ and } (|T|)^{2s} \geq (|T|^s|T^\star|^{2t}|T|^s)^{\frac{s}{s+t}}$$

and T is a class $A(s, t)$ operator if

$$(|T^\star|^t|T|^{2s}|T^\star|^t)^{\frac{t}{s+t}} \geq (|T^\star|)^{2t}.$$

Clearly, among the classes of operators discussed above, the following inclusions hold and are known to be proper:

(a) self-adjoint $\subset$ normal $\subset$ hyponormal $\subset$ p-hyponormal $\subset$ p-quasihyponormal $\subset$ (p, k)-quasihyponormal.

(b) hyponormal $\subset$ quasihyponormal $\subset$ k-quasihyponormal $\subset$ (p, k)-quasihyponormal.

p-hyponormal $\subset$ semi-hyponormal $\subset$ w-hyponormal $\subset$ wA $\subset$ class A $\subset$ class $A(k)$ $\subset$ class $A(s, t)$.

(c) p-hyponormal $\subset$ semi-hyponormal $\subset$ w-hyponormal $\subset$ wA $\subset$ class $wA(s, t)$ $\subset$ class $A(s, t)$.

(d) log-hyponormal $\subset$ w-hyponormal $\subset$ class $AI(s, t)$ $\subset$ class $wA(s, t)$ $\subset$ class $A(s, t)$.

Remark. To avoid confusions while talking about the nth-power and the nth-Aluthge transforms, we will use another positive integer k for the power. We would also like to assert that there is a difference between the nth-Aluthge transform of the kth power and the kth power of the nth-Aluthge transform of an operator, say T. In other words, $\widetilde{T^k}_n \neq \widetilde{T}_n^k$ for any pair of positive integers n, k. However, in the case of invertible w-hyponormal operators, the equality holds. But if the invertibility is dropped, at least $\widetilde{T}^k{}_n$ and $\widetilde{T}_n{}^k$ are spectraloid.

We want to prove the following theorems:

Theorem C. *If T is a w-hyponormal operator, then both $\widetilde{T}^k{}_n$ and $\widetilde{T}_n{}^k$ are spectraloid*, $\forall n, k \in J^+$.

Theorem D. *Let T be a w-hyponormal operator. If there exists a pair of positive integers k and n such that $(\widetilde{T}_n)^{2^k}$ is a normal operator, then the following are normal operators*:

(1) $(\widetilde{T}_{n_1})^{2^k}$, *for every positive integer n_1 less than n.*

(2) $(\widetilde{T}_n)^{2^{k_1}}$, *for each positive integer k_1 less than k, if* $0 \notin W(T)$.

(3) $(\widetilde{T}_{n_1})^{2^{k_1}}$, *for each pair of positive integers $k_1 \le k$ and $n_1 \le n$, if* $0 \notin W(T)$.

To achieve this, we will use the following well known results.

Preliminary Lemmas

Lemma 1.1 [2, 3]**.** *If T is w-hyponormal, then $\widetilde{T}$ is semi-hyponormal and $\widetilde{\widetilde{T}}$ is hyponormal.*

Lemma 1.2 [3]**.** *If T is an invertible w-hyponormal operator, then T^n is w-hyponormal for all positive integers n.*

Lemma 1.3 [12]**.** *Let T be a w-hyponormal operator. Then T^2 is w-hyponormal if* $\ker(T) = \{0\}$.

Lemma 1.4 [23]**.** *If T is a w-hyponormal operator, then T^n is also a w-hyponormal operator for every positive integer n.*

Lemma 1.5 [19]. *If T is a paranormal operator, then* T^n *is also a paranormal operator for every positive integer n and if T is invertible, then* T^{-1} *is also a paranormal operator.*

Lemma 1.6 [19]. *Let T be a class* (A) *operator. Then* T^n *is a class* (A) *operator if T is invertible. It follows that* T^{-1} *is also a class* (A) *operator.*

Lemma 1.7 [20, 21]. *Let T be a class* $AI(s, t)$ *operator for* $s, t \in (0, 1]$. *Then* T^n *is a class* $AI(s/n, t/n)$ *operator for all positive integers n.*

Lemma 1.8 [23]. *Let T be a class* $wA(s, t)$ *operator for* $s, t \in (0, 1]$. *Then* T^n *is a class* $wA(s/n, t/n)$ *operator for all positive integers n.*

Lemma 1.9 [2, 3]. *Let* $T = U|T|$ *be the polar decomposition of a w-hyponormal operator. If* $\widetilde{T}$ *is normal, then* $T = \widetilde{T}$. *That is T is normal.*

Lemma 1.10 [2]. *If T is a w-hyponormal operator, then*

$$\|\widetilde{T}\| = \|T\| = r(T).$$

Lemma 1.11 [2]. $\|\widetilde{T}_{n-1}\| = \|\widetilde{T}_n\|$ *if and only if T is normaloid.*

Lemma 1.12 [2, 3]. *For any operator T, the following are equivalent*:

(i) $\|T^k\| = \|T\|^k$,

(ii) $\|T\| = \|\widetilde{T}\|$.

Lemma 1.13 [24]. *If T is a w-hyponormal operator, then* $\|(\widetilde{T}_{n-1})\| = \|(\widetilde{T}_n)\|$ *for every natural number n and* $\|T^k\| = \|T\|^k$ *for every natural number k.*

Lemma 1.14 [24]. *Every w-hyponormal operator is spectraloid.*

Lemma 1.15 [2]. *For any operator T,* $w(\widetilde{T}) \leq w(T)$.

Lemma 1.16 [24]. *If T is a w-hyponormal operator, then* $r(\widetilde{T}_n) = w(\widetilde{T}_n)$.

Lemma 1.17 [31, 33]. *Every normal operator can be written in the form UP, where P is positive and U can be taken to be unitary such that* $UP = PU$ *and U commutes with all operators that commute with T and* T^*.

Lemma 1.18 [3]. *An operator T is normal iff the operator* $\widetilde{T}$ *is normal.*

Lemma 1.19 [24]. *An operator* T^n *is normal iff the operator T is n-power normal, for some positive integer n.*

Lemma 1.20 [34, 36]. *Let A be any operator. If* A^2 *is normal such that* $0 \notin W(A)$, *then A is also normal.*

Lemma 1.21 [33, 35]. *Let A be a normal operator. If* $0 \notin W(A)$, *then any other operator which commutes with A also commutes with* A^2.

Lemma 1.22 [24]. *If* $0 \notin W(A)$, *then* $0 \notin W(A^2)$.

Lemma 1.23 [24]. *If* $0 \notin W(A)$, *then* $0 \notin W(A^{2^n})$, $\forall n \in J^+$.

Lemma 1.24 [16]. *For any operator* $T \in B(H)$, $r(T^n) = r(T)^n$, *but* $w(T^n) \leq w(T)^n$, *for every positive integer n.*

Lemma 1.25 [24]. *Let* $A \in B(H)$ *be any operator such that* A^{2^n} *is a normal operator for some* $n \in J^+$. *Then* A^{2^m} *is also a normal operator,* $\forall m \in J^+$, *where* $m \leq n$, *if* $0 \notin W(A)$.

Lemma 1.26 [12]. *If T is a class* (A) *operator, then*

(a) $\sigma_{np}(T)\backslash\{0\} = \sigma_p(T)\backslash\{0\}$ *and*

(b) $\sigma_{na}(T)\backslash\{0\} = \sigma_a(T)\backslash\{0\}$.

Remark. Recall that Aluthge transform of an operator 'improves' *p*-hyponormality of an operator for $p \leq 1$ since if *T* is *p*-hyponormal for $1/2 \leq p < 1$, then $\widetilde{T}$ is hyponormal and if *T* is *p*-hyponormal for $0 < p$

$< 1/2$, then $\widetilde{T}$ is $p + 1/2$ -hyponormal. From Lemma 1.1, we have that if an operator T is w-hyponormal, then $\widetilde{T}$ is semi-hyponormal, $\widetilde{\widetilde{T}}$ is hyponormal, it follows easily that repeated Aluthge transformations of a w-hyponormal operator will result into a p-hyponormal operator and thus the class of all w-hyponormal operators is closed with respect to generalized Aluthge transformations. We also note from Lemma 1.18 that if $\widetilde{T}$ happens to be a normal operator, then its counterpart T is normal and conversely. Thus, repeated Aluthge transformations of a normal operator, begets another normal operator no matter the number of repetitions. However, if an operator T is normal, then T^n is not normal in general. It is also known that an operator T being not normal, does not imply that there cannot exist some positive integer n allowing T^n to be normal. In other words, normality of T^n, for some positive integer n does not imply that T is normal in general. The following theorem is in line with these observations:

Theorem 1.27. *Let T be a w-hyponormal operator. If there exists a pair of positive integers k, m such that $\widetilde{T}^k{}_n$ is a normal operator, then T is a k-power normal operator but T is not a normal operator in general.*

Proof. It suffices only to show that T is a k-power normal operator since there are a number of k-power normal operators which are not normal. In fact, from Lemma 1.19, we need to show that T^k is a normal operator. Now assume to the contrary, there does not exist a positive integer k such that T^k is a normal operator. By Lemma 1.4, T^k is a w-hyponormal operator. Thus, there exists a positive integer n such that $\widetilde{T}^k{}_n$ is a normal operator. But by Lemma 1.19, it follows that T^k is also normal. Thus, T is a k-power normal operator.

Remark. When does the normality of $\widetilde{T}^k{}_n$ imply the normality of T? To answer this question, we prove the following result:

Theorem 1.28. *Let T be a w-hyponormal operator such that $0 \notin W(T)$.*

If there exists a pair of positive integers k, m such that $\widetilde{T}^{2^k}{}_n$ *is a normal operator*, *then T is a normal operator*.

Proof. From the proof of Theorem 1.27 above, T^{2^k} is a normal operator. By Lemma 1.25, T^{2^m} is also a normal operator for every positive integer $m \leq k$. Thus, if $m = 1$, then we have that T^2 is also normal. Consequently, the normality of T follows from the fact that $0 \notin W(T)$.

Theorem 1.29. *If T is an invertible w-hyponormal operator*, *then both* $\widetilde{T}^k{}_n$ *and* $\widetilde{T}_n{}^k$ *are also invertible operators*, $\forall n, k \in J^+$. *Consequently*, $\widetilde{T}^k{}_n = \widetilde{T}_n{}^k$.

Proof. We first note that the invertibility of T implies that T^k is also an invertible w-hyponormal operator for any positive integer k. If T^k is invertible, then by all its Aluthge transforms are the same operator, thus invertible. That is, $\widetilde{T}^k{}_n$ is invertible and $\widetilde{T}^k{}_n = T^k$, $\forall n, k \in J^+$.

Now assume T is an invertible w-hyponormal operator. Then $\widetilde{T}_n$ is also an invertible w-hyponormal operator and $\widetilde{T}_n = T$, $\forall n \in J^+$. Thus, all of its powers are invertible. Hence, $\widetilde{T}_n{}^k$ is also an invertible operator, and $\widetilde{T}_n{}^k = T^k$, $\forall k \in J^+$.

Remark. The equality in Theorem 1.29 holds since T is invertible. Generally, if T is a w-hyponormal operator, then $\widetilde{T}^k{}_n \neq \widetilde{T}_n{}^k$, for every pair of positive integers n and k. However, the two operators at least, share the same spectra as the following theorem shows:

Theorem 1.30. *If T is a w-hyponormal operator*, *then* $\sigma(\widetilde{T}^k{}_n) = \sigma(\widetilde{T}_n{}^k)$, $\forall n, k \in J^+$.

Remark. In order to prove the theorem above, we first prove the following Corollary 1.34 which follows immediately from the following well known results:

Lemma 1.31 [29, 41]**.** *If* $\lambda \in \sigma(T)$, *then* $|\lambda| \le r(T)$.

Remark. We note that Lemma 1.31 confirms the fact that the spectrum of any operator is bounded from above by its spectral radius. Thus, any number larger than the spectral radius of any operator will be found in the resolvent of the operator in question.

Lemma 1.32 [29, 41]**.** *If* $\lambda \in \sigma(T)$, *then for each positive integer k, we have* $\lambda^k \in \sigma(T^k)$.

Remark. By Lemma 1.32 above, it is easy to conclude that the spectrum of any operator is always contained in the spectrum resulting from raising the said operator to any power. That is, $\sigma(T) \subset \sigma(T^k)$. In general, the following result is well known and follows as a consequence of the spectral mapping theorem:

Lemma 1.33 [29, 41]**.** *If* $\lambda \in \sigma(T)$, *then* $f(\lambda) \in \sigma(f(T))$ *for any polynomial f.*

Remark. Before we prove Theorem 1.30, we first prove the following corollary:

Corollary 1.34. *If* $r(T) = r$, *then* $r(T^k) = r^k$.

Proof. Assume that $\lambda \in \sigma(T)$ is the scaler with the largest positive value $|\lambda|$. Then $r(T) = |\lambda|$ and $\lambda^k \in \sigma(T^k)$. It follows that $|\lambda^k|$ is the largest positive value in $\sigma(T^k)$, if not, then there exists another number, say $n \ge k$ such that $|\lambda^n|$ is the largest positive value in $\sigma(T^k)$, which contradicts our selection of $\lambda \in \sigma(T)$. Consequently, $|\lambda^k| = r(T^k)$. Thus, from Lemma 1.31, $r(T^k) = r^k$.

Proof of Theorem 1.30. For any operator, all of its Aluthge transforms have the same spectra, [24]. Thus, $\sigma(\widetilde{T}^k{}_n) = \sigma(T^k)$.

On the other edge, $\sigma(\widetilde{T}_n{}^k)$ includes $\sigma\widetilde{T}_n$. But $\sigma\widetilde{T}_n = \sigma(T)$ and $\sigma(T) \subset \sigma(T^k)$. Now letting $r(T) = r$, then $r(T^k) = r^k$, $r(\widetilde{T}_n) = r$ and thus $r(\widetilde{T}_n{}^k) = r^k$. Hence $\sigma(\widetilde{T}^k{}_n) = \sigma(\widetilde{T}_n{}^k)$, $\forall n, k \in J^+$.

Proof of Theorem C. We only need to prove that $\widetilde{T}^k{}_n$ is spectraloid, since from Theorem 1.30 above, $\widetilde{T}^k{}_n$ and $\widetilde{T}_n{}^k$ have the same spectra, $\forall n$, $k \in J^+$. We first note that T^k is also a *w*-hyponormal operator, by Lemma 1.4. And from Lemma 1.14, $\widetilde{T}^k{}_n$ is spectraloid for any pair of positive integers *n*, *k*.

Theorem 1.35. *Let T be a w-hyponormal operator. If there exists a pair of positive integers k and n such that* $\widetilde{T}^k{}_n$ *is a normal operator, then* $\widetilde{T}^{k^2}{}_n$ *is a normal operator for every positive integer n.*

Proof. By Lemma 1.4, T^k is a *w*-hyponormal operator. And by Theorem 1.27, if there exists a positive integer *n* such that the *n*th-Aluthge transform of T^k is normal, then T^k is also a normal operator. Thus, from the property of normal operators, T^{k^2} is a normal operator. Hence, every Aluthge transform of T^{k^2} is normal.

Proof of Theorem D(1). From Lemma 1.18, we know that $\widetilde{T}_{n_1}$ is normal for each positive integer n_1 less than *n* whenever $\widetilde{T}_n$ is normal. Thus, $(\widetilde{T}_{n_1})^2$ is also normal. Generally, $(\widetilde{T}_{n_1})^{2^k}$ is also a normal operator for each positive integer *k*.

Proof of Theorem D(2). If $0 \notin W(T)$, then $0 \notin W(\widetilde{T})$. Moreover, $0 \notin W(\widetilde{T}_n)$. From Lemma 1.25, it follows that $0 \notin W(\widetilde{T}_n{}^{2^k})$. From Theorem 1.35, we have that $(\widetilde{T}_n)^{2^{k_1}}$ is normal for each positive integer k_1 less than *k*, since $(\widetilde{T}_n)^{2^k}$ is a normal operator for any positive integer *n*.

Proof of Theorem D(3). From the proof of (1) above, $\widetilde{T}_{n_1}$ is normal for each positive integer n_1 less than *n*. Thus, $(\widetilde{T}_{n_1})^{2^k}$ is also normal for any positive integer *k*. And from the proof of (2), $(\widetilde{T}_{n_1})^{2^{k_1}}$ is normal for each pair of positive integers $k_1 \leq k$ and $n_1 \leq n$.

Remark. *w*-hyponormal operators contain all self-adjoint and all unitary operators. In the following last observation, we will use the spectra of *n*th-Aluthge transforms of a *k*th power of a *w*-hyponormal operator to tell under what circumstances are such operators restricted to either self-adjoint or unitary operators.

Theorem 1.36. *If T is a w-hyponormal operator such that* $0 \notin W(T)$, *then*

(i) $\widetilde{T}^k{}_n$ *is self-adjoint if* $\sigma(\widetilde{T}^k{}_n) \subset R$,

(ii) $\widetilde{T}^k{}_n$ *is positive if* $\sigma(\widetilde{T}^k{}_n) \subset [0, \infty)$,

(iii) $\widetilde{T}^k{}_n$ *is unitary if* $\sigma(\widetilde{T}^k{}_n)$ *is a unit circle*.

Proof. Let *T* be a *w*-hyponormal operator such that zero is an isolated point in the numerical range of *T*. It follows that *T* is invertible. Thus, every power of such an operator is also a *w*-hyponormal operator. That is, T^k is *w*-hyponormal for every positive integer *k*. But every Aluthge transform of *w*-hyponormal operator is also a *w*-hyponormal operator since Aluthge transformation reduces *w*-hyponormality to hyponormality. Thus, $\widetilde{T}^k{}_n$ is a *w*-hyponormal for any pair of positive integers *n* and *k*. The rest of the proof follows from Lemma 1.25.

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