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$M_{\beta}^{(i,\, j)}$-CONTINUOUS FUNCTIONS

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Abstract

In this paper, $M_{\beta}^{(i,\, j)}$-continuous functions on a biminimal structure space have been introduced. Some characterizations of such functions are given.

1. Introduction

In 1983, Abd El-Monsef et al. [3] introduced and investigated β-open sets and β-continuity in topological spaces. Recently, Noiri and Popa [6] have introduced the concepts of weak β-continuity and almost β-continuity.

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In 1997, Noiri and Nasef [5] have studied fundamental properties of almost β-continuous functions. In 2009, Noiri and Popa [7] introduced the notion of minimal structure. Also, they introduced the notions of *m*-open sets and *m*-closed sets and characterized those sets using *m*-closure and *m*-interior operators, respectively. The notion of biminimal structure spaces was introduced by Boonpok [1] in 2010. Also, he studied $m_X^1 m_X^2$-closed sets and $m_X^1 m_X^2$-open sets in biminimal structure spaces. Moreover, Boonpok [2] introduced the notion of *M*-continuous functions on biminimal structure spaces and studied some characterizations and several properties of such functions. In this paper, we study the new notions of weak form of continuous functions on a biminimal structure space. And we investigate characterizations of such functions.

2. Preliminaries

In this section, we recall some basic definitions and notation of minimal structure space in [4]. Let X be a nonempty set and $\mathscr{P}(X)$ the power set of X. A subfamily m_X of $\mathscr{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X if $\varnothing \in m_X$ and $X \in m_X$. The pair (X, m_X) is called an *m-space*. Each member of m_X is called m_X*-open* and the complement of an m_X-open set is called m_X*-closed*. For a minimal structure m_X on X and $A \subseteq X$, the closure of A on m_X, denoted by $m_X\text{-Cl}(A)$, is the intersection of all m_X-closed sets containing A, i.e.,

$$m_X\text{-Cl}(A) = \bigcap \{F : X - F \in m_X \text{ and } A \subseteq F\},$$

and the interior of A on m_X, denoted by $m_X\text{-Int}(A)$, is the union of all m_X-open sets contained in A, i.e.,

$$m_X\text{-Int}(A) = \bigcup \{U : U \in m_X \text{ and } U \subseteq A\}.$$

We see that $A \subseteq m_X\text{-Cl}(A)$ and $m_X\text{-Int}(A) \subseteq A$. Moreover, $m_X\text{-Cl}(A) = A$ if $X - A \in m_X$ and $m_X\text{-Int}(A) = A$ if $A \in m_X$. It is easy to observe that m_X-Cl and m_X-Int are monotonic, i.e., if $A \subseteq B \subseteq X$, then $m_X\text{-Cl}(A) \subseteq m_X\text{-Cl}(B)$ and $m_X\text{-Int}(A) \subseteq m_X\text{-Int}(B)$. Furthermore, m_X-Cl and m_X-Int are idempotent, i.e., $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl}(A)$ and $m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int}(A)$. The closures and interiors on m_X have a complementary relationship as follows: $m_X\text{-Cl}(X - A) = X - (m_X\text{-Int}(A))$ and $m_X\text{-Int}(X - A) = X - (m_X\text{-Cl}(A))$. If A is a subset of an m-space (X, m_X), then $x \in m_X\text{-Cl}(A)$ if and only if $W \cap A \neq \emptyset$ for every $W \in m_X$ such that $x \in W$.

A minimal structure m_X on a nonempty set X is said to have *property* $\mathscr{B}$ if the union of any family of subsets belonging to m_X belongs to m_X. It can verify that m_X has property $\mathscr{B}$ if and only if $m_X\text{-Int}(V) = V$ implies $V \in m_X$ if and only if $m_X\text{-Cl}(F) = F$ implies $X - F \in m_X$. In [8], if m_X is a minimal structure on X satisfying property $\mathscr{B}$, then $m_X\text{-Int}(A) \in m_X$ and $m_X\text{-Cl}(A)$ is m_X-closed.

Now, we review some basic definitions and notation of biminimal structure spaces in [1] and [4]. Let X be a nonempty set and let m_X^1 and m_X^2 be minimal structures on X. A triple (X, m_X^1, m_X^2) is called a *biminimal structure space* (briefly *bi m-space*). For any subset A of a biminimal structure space (X, m_X^1, m_X^2), the closure and interior of A with respect to m_X^i are denoted by $m_X^i\text{-Cl}(A)$ and $m_X^i\text{-Int}(A)$, respectively, for $i = 1, 2$. Let (X, m_X^1, m_X^2) be a biminimal structure space and let $i, j \in \{1, 2\}$ be such that $i \neq j$. A subset A of X is said to be $m_X^{(i,\,j)}$-β-*open* (resp. $m_X^{(i,\,j)}$-*regular open*, $m_X^{(i,\,j)}$-*semi open*, $m_X^{(i,\,j)}$-*preopen*, $m_X^{(i,\,j)}$-α-*open*) [2] if $A \subseteq m_X^i\text{-Cl}(m_X^j\text{-Int}(m_X^i\text{-Cl}(A)))$ (resp. $A = m_X^i\text{-Int}(m_X^j\text{-Cl}(A))$, $A \subseteq m_X^i\text{-Cl}(m_X^j\text{-Int}(A))$, $A \subseteq m_X^i\text{-Int}(m_X^j\text{-Cl}(A))$, $A \subseteq m_X^i\text{-Int}(m_X^j\text{-Cl}(m_X^i\text{-}$

$\mathrm{Int}(A))))$. The complement of a $m_X^{(i,j)}$-β-open (resp. $m_X^{(i,j)}$-regular open, $m_X^{(i,j)}$-semi open, $m_X^{(i,j)}$-preopen, $m_X^{(i,j)}$-α-open) set is called $m_X^{(i,j)}$-β-*closed* (resp. $m_X^{(i,j)}$-*regular closed*, $m_X^{(i,j)}$-*semi closed*, $m_X^{(i,j)}$-*preclosed*, $m_X^{(i,j)}$-α-*closed*).

Theorem 2.1 [2]. *Let* (X, m_X^1, m_X^2) *be a biminimal structure space and A be a subset of X. Let* $i, j \in \{1, 2\}$ *be such that* $i \neq j$. *Then*

(1) *A is* $m_X^{(i,j)}$-*regular closed if and only if* $A = m_X^i\text{-}\mathrm{Cl}(m_X^j\text{-}\mathrm{Int}(A))$.

(2) *A is* $m_X^{(i,j)}$-*semi closed if and only if* $m_X^i\text{-}\mathrm{Int}(m_X^j\text{-}\mathrm{Cl}(A)) \subseteq A$.

(3) *A is* $m_X^{(i,j)}$-*preclosed if and only if* $m_X^i\text{-}\mathrm{Cl}(m_X^j\text{-}\mathrm{Int}(A)) \subseteq A$.

(4) *A is* $m_X^{(i,j)}$-α-*closed if and only if* $m_X^i\text{-}\mathrm{Cl}(m_X^j\text{-}\mathrm{Int}(m_X^i\text{-}\mathrm{Cl}(A))) \subseteq A$.

(5) *A is* $m_X^{(i,j)}$-β-*closed if and only if* $m_X^i\text{-}\mathrm{Int}(m_X^j\text{-}\mathrm{Cl}(m_X^i\text{-}\mathrm{Int}(A))) \subseteq A$.

Proposition 2.2. *Let* (X, m_X^1, m_X^2) *be a biminimal structure space and let* $i, j \in \{1, 2\}$ *be such that* $i \neq j$. *Let* $\{A_\gamma : \alpha \in J\}$ *be a family of subsets of X. Then*

(1) *If* A_γ *is* $m_X^{(i,j)}$-β-*open for all* $\gamma \in J$, *then* $\bigcup_{\gamma \in J} A_\gamma$ *is* $m_X^{(i,j)}$-β-*open*.

(2) *If* A_γ *is* $m_X^{(i,j)}$-β-*closed for all* $\gamma \in J$, *then* $\bigcap_{\gamma \in J} A_\gamma$ *is* $m_X^{(i,j)}$-β-*closed*.

Definition 2.3. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X and let $i, j \in \{1, 2\}$ be such that $i \neq j$. The $m_X^{(i,j)}$-β-closure of A, denote by $m_X^{(i, j)}$-$\mathrm{Cl}_{\beta}(A)$, is defined to be intersection of all $m_X^{(i, j)}$-β-closed sets containing A. The $m_X^{(i, j)}$-β-interior of A, denote by $m_X^{(i, j)}$-$\mathrm{Int}_{\beta}(A)$, is defined to be union of all $m_X^{(i, j)}$-β-open sets contained in A.

Proposition 2.4. *Let* (X, m_X^1, m_X^2) *be a biminimal structure space and let A be a subset of X. Let* $i, j \in \{1, 2\}$ *be such that* $i \neq j$. *Then the following statements are evident*:

(1) $x \in m_X^{(i, j)}$-$\mathrm{Cl}_{\beta}(A)$ *iff* $U \cup A \neq \emptyset$ *for every* $m_X^{(i, j)}$-β-*open set U containing x*.

(2) $X - m_X^{(i, j)}$-$\mathrm{Int}_{\beta}(A) = m_X^{(i, j)}$-$\mathrm{Cl}_{\beta}(X - A)$.

(3) $X - m_X^{(i, j)}$-$\mathrm{Cl}_{\beta}(A) = m_X^{(i, j)}$-$\mathrm{Int}_{\beta}(X - A)$.

(4) $m_X^{(i, j)}$-$\mathrm{Int}_{\beta}(A)$ *is* $m_X^{(i, j)}$-β-*open*.

(5) $m_X^{(i, j)}$-$\mathrm{Cl}_{\beta}(A)$ *is* $m_X^{(i, j)}$-β-*closed*.

(6) *A is* $m_X^{(i, j)}$-β-*open if and only if* $A = m_X^{(i, j)}$-$\mathrm{Int}_{\beta}(A)$.

(7) *A is* $m_X^{(i, j)}$-β-*closed if and only if* $A = m_X^{(i, j)}$-$\mathrm{Cl}_{\beta}(A)$.

3. $M_X^{(i, j)}$-continuous Functions

In this section, we introduce the concept of $M_X^{(i, j)}$-continuous functions on a biminimal structure space. Furthermore, we give some properties of such functions. Next, we introduce the weak forms of $M_X^{(i, j)}$-continuous functions on a biminimal structure space and study the properties of these functions. Throughout this section, let $i, j \in \{1, 2\}$ be such that $i \neq j$.

Definition 3.1. Let (X, m_X^1, m_X^2) and (Y, m_Y^1, m_Y^2) be biminimal structure spaces. A function $f : (X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$ is said to be $M_\beta^{(i,j)}$-*continuous* at $x \in X$ if for each $V \in m_Y^i$ containing $f(x)$, there exists an $m_X^{(i,j)}$-β-open set U containing x such that $f(U) \subseteq V$.

A function $f : (X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$ is said to be $M_\beta^{(i,j)}$-*continuous* if for all $x \in X$, f is $M_\beta^{(i,j)}$-continuous at x.

A function $f : (X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$ is said to be *pairwise* M_β-*continuous* if f is $M_\beta^{(1,2)}$-continuous and $M_\beta^{(2,1)}$-continuous.

Example 3.2. Let $X = \{a, b, c\}$ and $Y = \{u, v, w\}$. Consider m-structures on X and Y as follows:

$$m_X^1 = \{\varnothing, \{a\}, \{b\}, X\}, \quad m_X^2 = \{\varnothing, \{a, c\}, \{b, c\}, X\}$$

and

$$m_Y^1 = \{\varnothing, \{u\}, \{v\}, \{w\}, Y\}, \quad m_Y^2 = \{\varnothing, \{u, v\}, \{u, w\}, \{v, w\}, Y\}.$$

Define $f : (X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$ as follows:

$$f(a) = u, \ f(b) = v \text{ and } f(c) = u.$$

Then f is pairwise M_β-continuous.

Theorem 3.3. *For a function* $f : (X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$, *the following properties are equivalent*:

(1) *f is $M_\beta^{(i,j)}$-continuous.*

(2) $f^{-1}(V)$ *is* $m_X^{(i,j)}$-β-*open in X for every* $V \in m_Y^i$.

(3) $f(m_X^{(i,j)}\text{-Cl}_\beta(A)) \subseteq m_Y^i\text{-Cl}(f(A))$ *for every subset A of X.*

(4) $m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(B)) \subseteq f^{-1}(m_Y^i\text{-Cl}(B))$ *for every subset B of Y.*

(5) $f^{-1}(m_Y^i\text{-Int}_\beta(B)) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(B))$ *for every subset B of Y.*

(6) $f^{-1}(F)$ *is* $m_X^{(i,j)}$-β-*closed in X for every* m_Y^i-*closed subset F of Y.*

Proof. (1) $\Rightarrow$ (2) Let $V \in m_Y^i$ and let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is $M_\beta^{(i,j)}$-continuous, there exists an $m_X^{(i,j)}$-β-open subset U of X containing x such that $f(U) \subseteq V$. Since U is $m_X^{(i,j)}$-β-open, we have $x \in m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(V))$. Thus $f^{-1}(V) = m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(V))$. Hence $f^{-1}(V)$ is $m_X^{(i,j)}$-β-open in X.

(2) $\Rightarrow$ (1) Let $x \in X$ and let V be an m_Y^i-open subset of Y containing $f(x)$. By (2), $f^{-1}(V)$ is $m_X^{(i,j)}$-β-open in X containing x. Put $U = f^{-1}(V)$. Then U is an $m_X^{(i,j)}$-β-open subset U of X containing x such that $f(U) \subseteq V$. Thus f is $M_\beta^{(i,j)}$-continuous at x.

(2) $\Rightarrow$ (3) Let A be any subset of X. Let $x \in m_X^{(i,j)}\text{-Cl}_\beta(A)$ and $V \in m_Y^i$ containing $f(x)$. By (2), $f^{-1}(V)$ is $m_X^{(i,j)}$-β-open in X containing x. Since $x \in m_X^{(i,j)}\text{-Cl}_\beta(A)$, $f^{-1}(V) \cap A \neq \varnothing$. Then $\varnothing \neq f(f^{-1}(V) \cap A) \subseteq f(f^{-1}(V)) \cap f(A) \subseteq V \cap f(A)$. Thus $f(x) \in m_Y^i\text{-Cl}(f(A))$, and so $x \in f^{-1}(m_Y^i\text{-Cl}(f(A)))$. Hence, $m_X^{(i,j)}\text{-Cl}_\beta(A) \subseteq f^{-1}(m_Y^i\text{-Cl}(f(A)))$. Then $f(m_X^{(i,j)}\text{-Cl}_\beta(A)) \subseteq m_Y^i\text{-Cl}(f(A))$.

(3) $\Rightarrow$ (4) Let B be any subset of Y. By (3), we have

$$f(m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(B))) \subseteq m_Y^i\text{-Cl}(f(f^{-1}(B))).$$

Hence, $m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(B)) \subseteq f^{-1}(m_Y^i\text{-Cl}(B))$.

(4) $\Rightarrow$ (5) Let B be any subset of Y. By (4), we have

$$m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(Y-B)) \subseteq f^{-1}(m_Y^i\text{-Cl}(Y-B)).$$

Hence, $f^{-1}(m_Y^i\text{-Int}(B)) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(B))$.

(5) $\Rightarrow$ (6) Let F be any m_Y^i-closed subset of Y. Then $m_Y^i\text{-Int}(Y-F) = Y-F$. By (5), $f^{-1}(m_Y^i\text{-Int}(Y-F)) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(Y-F))$. Then $m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(F)) \subseteq f^{-1}(F)$. Hence, $f^{-1}(F)$ is $m_X^{(i,j)}$-β-closed in X.

(6) $\Rightarrow$ (2) It is obvious. □

Definition 3.4. Let (X, m_X^1, m_X^2) and (Y, m_Y^1, m_Y^2) be biminimal structure spaces. A function $f:(X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$ is said to be *almost* $M_\beta^{(i,j)}$*-continuous* at $x \in X$ if for each $V \in m_Y^i$ containing $f(x)$, there exists an $m_X^{(i,j)}$-β-open set U containing x such that $f(U) \subseteq m_Y^i\text{-Int}(m_Y^j\text{-Cl}(V))$.

A function $f:(X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$ is said to be *almost* $M_\beta^{(i,j)}$*-continuous* if for all $x \in X$, f is almost $M_\beta^{(i,j)}$-continuous at x.

A function $f:(X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$ is said to be *pairwise almost* M_β*-continuous* if f is almost $M_\beta^{(1,2)}$-continuous and almost $M_\beta^{(2,1)}$-continuous.

From the above definitions, every $M_\beta^{(i,j)}$-continuous function is almost $M_\beta^{(i,j)}$-continuous but the converse is not always true.

Example 3.5. Let $X = \{a, b, c\}$ and $Y = \{u, v, w\}$. Consider m-structures on X and Y as follows:

$$m_X^1 = \{\varnothing, \{a\}, \{b\}, X\}, \quad m_X^2 = \{\varnothing, \{a, c\}, \{b, c\}, X\}$$

and

$$m_Y^1 = \{\varnothing, \{u\}, \{v\}, \{w\}, Y\}, \quad m_Y^2 = \{\varnothing, \{u, v\}, \{v, w\}, Y\}.$$

Define $f : (X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$ as follows:

$$f(a) = u, \ f(b) = u \ \text{ and } \ f(c) = v.$$

Then f is almost $M_\beta^{(1,2)}$-continuous but it is not $M_\beta^{(1,2)}$-continuous.

Theorem 3.6. *For a function* $f : (X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$, *where* m_Y^i *has property* $\mathscr{B}$ *for* $i = 1, 2$, *the following properties are equivalent*:

(1) *f is almost* $M_\beta^{(i,j)}$ *-continuous.*

(2) $f^{-1}(V) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(V))))$ *for every* m_Y^i*-open set V of Y.*

(3) $m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(m_Y^i\text{-Cl}(m_Y^j\text{-Int}(F)))) \subseteq f^{-1}(F)$ *for every* m_Y^i*-closed subset F of Y.*

(4) $m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(m_Y^i\text{-Cl}(m_Y^j\text{-Int}(m_Y^i\text{-Cl}(B))))) \subseteq f^{-1}(m_Y^i\text{-Cl}(B))$ *for every subset B of Y.*

(5) $f^{-1}(m_Y^i\text{-Int}(B)) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(m_Y^i\text{-Int}(B)))))$ *for every subset B of Y.*

(6) $f^{-1}(V)$ *is* $m_X^{(i,j)}$-β-*open in X for every* $m_Y^{(i,j)}$*-regular open subset V of Y.*

(7) $f^{-1}(F)$ *is* $m_X^{(i,j)}$-β-*closed in X for every* $m_Y^{(i,j)}$*-regular closed subset V of Y.*

Proof. (1) $\Rightarrow$ (2) Let V be any m_Y^i-open subset of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is almost $M_\beta^{(i,j)}$-continuous at x, there exists an $m_X^{(i,j)}$-β-open subset U of X containing x such that $f(U) \subseteq m_Y^i\text{-Int}(m_Y^j\text{-Cl}(V))$. Thus $U \subseteq f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(V)))$. Hence,

$$x \in m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(V)))).$$

This implies that $f^{-1}(V) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(V))))$.

(2) $\Rightarrow$ (3) Let F be an m_Y^i-closed subset of Y. Since $Y - F$ is m_Y^i-open, by (2), we obtain that $f^{-1}(Y - F) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(Y - F))))$. This implies that $m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(m_Y^i\text{-Cl}(m_Y^j\text{-Int}(F)))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) It follows from the fact that m_Y^i has property $\mathscr{B}$.

(4) $\Rightarrow$ (5) Let B be any subset of Y. By (4), we have

$$m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(m_Y^i\text{-Cl}(m_Y^j\text{-Int}(m_Y^i\text{-Cl}(Y - B))))) \subseteq f^{-1}(m_Y^i\text{-Cl}(Y - B)).$$

This implies

$$f^{-1}(m_Y^i\text{-Int}(B)) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(m_Y^i\text{-Int}(B))))).$$

(5) $\Rightarrow$ (6) Let V be any $m_Y^{(i,j)}$-regular open subset of Y. Since m_Y^i has property $\mathscr{B}$, V is m_Y^i-open. By (5), we get that $f^{-1}(V) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(V))$. This implies that $f^{-1}(V)$ is $m_X^{(i,j)}$-β-open in X.

(6) $\Rightarrow$ (7) Let F be any $m_Y^{(i,j)}$-regular closed subset of Y. Since $Y - F$ is an $m_Y^{(i,j)}$-regular open set in Y, by (6), we have $f^{-1}(Y - F)$ is $m_X^{(i,j)}$-β-open in X. Then $f^{-1}(F)$ is $m_X^{(i,j)}$-β-closed in X.

(7) $\Rightarrow$ (1) Let $x \in X$ and let V be any m_Y^i-open set containing $f(x)$. Since $m_Y^i\text{-Cl}(m_Y^j\text{-Int}(Y - V))$ is an $m_Y^{(i,j)}$-regular closed set in Y, by (7),

$$f^{-1}(m_Y^i\text{-Cl}(m_Y^j\text{-Int}(Y - V))) \text{ is an } m_X^{(i,j)}\text{-}\beta\text{-closed set in } X.$$

This implies that $f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(V)))$ is an $m_X^{(i,j)}$-β-open set in X containing $f(x)$. Set $U = f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(V)))$. Then U is an $m_X^{(i,j)}$-β-open set containing x and $f(U) \subseteq m_Y^i\text{-Int}(m_Y^j\text{-Cl}(V))$. Thus f is almost $M_\beta^{(i,j)}$-continuous at x. Then f is almost $M_\beta^{(i,j)}$-continuous. □

Theorem 3.7. *For a function* $f : (X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$, *where* m_Y^i *has property* $\mathscr{B}$ *for* $i = 1, 2,$ *the following properties are equivalent*:

(1) *f is almost* $M_\beta^{(i,j)}$-*continuous*.

(2) $m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(U)) \subseteq f^{-1}(m_Y^i\text{-Cl}(U))$ *for every* $m_Y^{(i,j)}$-β-*open subset U of Y*.

(3) $m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(U)) \subseteq f^{-1}(m_Y^i\text{-Cl}(U))$ *for every* $m_Y^{(i,j)}$-*semi open subset U of Y*.

(4) $f^{-1}(U) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(U))))$ *for every* $m_Y^{(i,j)}$-*preopen subset U of Y*.

Proof. (1) $\Rightarrow$ (2) Let U be any $m_Y^{(i,j)}$-β-open subset of Y. Then $m_Y^i\text{-Cl}(U)$ is $m_Y^{(i,j)}$-regular closed. Since f is almost $M_\beta^{(i,j)}$-continuous, by Theorem 3.6(7), $f^{-1}(m_Y^i\text{-Cl}(U))$ is $m_X^{(i,j)}$-β-closed in X. This implies that $m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(U)) \subseteq f^{-1}(m_Y^i\text{-Cl}(U))$.

(2) $\Rightarrow$ (3) It follows from the fact that every $m_Y^{(i,j)}$-semi open set is $m_Y^{(i,j)}$-β-open.

(3) $\Rightarrow$ (1) Let F be any $m_Y^{(i,j)}$-regular closed subset of Y. Since m_Y^i has property $\mathscr{B}$, F is m_Y^i-closed. Furthermore, F is $m_Y^{(i,j)}$-semi open in Y. By (3), we have $m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(F)) \subseteq f^{-1}(F)$. This implies that $f^{-1}(F)$ is $m_X^{(i,j)}$-β-closed in X. Thus, by Theorem 3.6(7), f is almost $M_\beta^{(i,j)}$-continuous.

(1) $\Rightarrow$ (4) Let U be an $m_Y^{(i,j)}$-preopen subset of Y. Then $m_Y^i\text{-Int}(m_Y^j\text{-Cl}(U))$ is $m_Y^{(i,j)}$-regular open in Y. Since f is almost $M_\beta^{(i,j)}$-continuous, by Theorem 3.6(6), $f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(U)))$ is $m_X^{(i,j)}$-β-open in X. Since U is $m_Y^{(i,j)}$-preopen in Y,

$$f^{-1}(U) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(m_Y^i\text{-Int}(m_Y^j\text{-Cl}(U)))).$$

(4) $\Rightarrow$ (1) Let U be any $m_Y^{(i,j)}$-regular open subset of Y. Then U is $m_Y^{(i,j)}$-preopen in Y. By (4), $f^{-1}(U) \subseteq m_X^{(i,j)}\text{-Int}_\beta(f^{-1}(U))$. This implies $f^{-1}(U)$ is $m_X^{(i,j)}$-β-open in X. Hence, by Theorem 3.6(6), f is almost $M_\beta^{(i,j)}$-continuous. □

Definition 3.8. Let (X, m_X^1, m_X^2) and (Y, m_Y^1, m_Y^2) be biminimal structure spaces. A function $f : (X, m_X^1, m_X^2) \to (Y, m_Y^1, m_Y^2)$ is said to be *weakly* $M_\beta^{(i,j)}$*-continuous* at $x \in X$ if for each $V \in m_Y^i$ containing $f(x)$, there exists an $m_X^{(i,j)}$-β-open set U containing x such that $f(U) \subseteq m_Y^j\text{-Cl}(V)$.

A function $f : (X,\, m_X^1,\, m_X^2) \to (Y,\, m_Y^1,\, m_Y^2)$ is said to be *weakly* $M_\beta^{(i,\, j)}$*-continuous* if for all $x \in X$, f is almost $M_\beta^{(i,\, j)}$-continuous at x.

A function $f : (X,\, m_X^1,\, m_X^2) \to (Y,\, m_Y^1,\, m_Y^2)$ is said to be *pairwise weakly* M_β*-continuous* if f is weakly $M_\beta^{(1,\, 2)}$-continuous and weakly $M_\beta^{(2,1)}$-continuous.

From the above definitions, every almost $M_\beta^{(i,\, j)}$-continuous function is weakly $M_\beta^{(i,\, j)}$-continuous but the converse is not always true.

Example 3.9. Let $X = \{a,\, b,\, c\}$ and $Y = \{u,\, v,\, w\}$. Consider m-structures on X and Y as follows:

$$m_X^1 = \{\varnothing,\, \{a\},\, \{b\},\, X\}, \quad m_X^2 = \{\varnothing,\, \{a,\, c\},\, \{b,\, c\},\, X\}$$

and

$$m_Y^1 = \{\varnothing,\, \{v\},\, \{w\},\, Y\},\, m_Y^2 = \{\varnothing,\, \{u,\, v\},\, \{w\},\, Y\}.$$

Define $f : (X,\, m_X^1,\, m_X^2) \to (Y,\, m_Y^1,\, m_Y^2)$ as follows:

$$f(a) = u,\; f(b) = u \text{ and } f(c) = v.$$

Then f is weakly $M_\beta^{(1,\, 2)}$ -continuous but it is not almost $M_\beta^{(1,\, 2)}$-continuous.

Theorem 3.10. *For a function* $f : (X,\, m_X^1,\, m_X^2) \to (Y,\, m_Y^1,\, m_Y^2)$, *where* m_Y^i *has property* $\mathscr{B}$ *for* $i = 1,\, 2$, *the following properties are equivalent*:

(1) f *is weakly* $M_\beta^{(i,\, j)}$*-continuous*.

(2) $f^{-1}(V) \subseteq m_X^{(i,\, j)}\text{-Int}_\beta(f^{-1}(m_Y^j\text{-Cl}(V)))$ *for every* m_Y^i*-open set* V *of* Y.

(3) $m_X^{(i,\,j)}\text{-Cl}_\beta(f^{-1}(m_Y^j\text{-Int}(F))) \subseteq f^{-1}(F)$ *for every* m_Y^i*-closed set F of Y*.

(4) $m_X^{(i,\,j)}\text{-Cl}_\beta(f^{-1}(m_Y^j\text{-Int}(m_Y^i\text{-Cl}(A)))) \subseteq f^{-1}(m_Y^i\text{-Cl}(A))$ *for every subset A of Y*.

(5) $f^{-1}(m_Y^i\text{-Int}(A)) \subseteq m_X^{(i,\,j)}\text{-Int}_\beta(f^{-1}(m_Y^j\text{-Cl}(m_Y^i\text{-Int}(A))))$ *for every subset A of Y*.

Proof. (1) $\Rightarrow$ (2) Let V be an m_Y^i-open set of Y and $x \in f^{-1}(V)$. By (1), there exists an $m_X^{(i,\,j)}$-β-open set U containing x such that $f(U) \subseteq m_Y^j\text{-Cl}(V)$. This implies that $x \in m_X^{(i,\,j)}\text{-Int}_\beta(f^{-1}(m_Y^j\text{-Cl}(V)))$. Then $f(V) \subseteq m_X^{(i,\,j)}\text{-Int}_\beta(f^{-1}(m_Y^j\text{-Cl}(V)))$.

(2) $\Rightarrow$ (1) Let $x \in X$ and let V be an m_Y^i-open set containing $f(x)$. By (2), we have $x \in m_X^{(i,\,j)}\text{-Int}_\beta(f^{-1}(m_Y^j\text{-Cl}(V)))$. Thus there exists an $m_X^{(i,\,j)}$-β-open set U containing x such that $U \subseteq f^{-1}(m_Y^j\text{-Cl}(V))$. Then $f(U) \subseteq m_Y^j\text{-Cl}(V)$. Hence f is weakly $M_\beta^{(i,\,j)}$-continuous at x. This implies that f is weakly $M_\beta^{(i,\,j)}$-continuous.

(2) $\Rightarrow$ (3) Let F be any m_Y^i-closed subset of Y. Then $Y - F$ is an m_Y^i-open subset of Y. By (2), we have

$$f^{-1}(Y - F) \subseteq m_X^{(i,\,j)}\text{-Int}_\beta(f^{-1}(m_Y^j\text{-Cl}(Y - F))).$$

This implies that $m_X^{(i,\,j)}\text{-Cl}_\beta(f^{-1}(m_Y^j\text{-Int}(F))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) It follows from the fact that m_Y^i has property $\mathscr{B}$.

(4) $\Rightarrow$ (5) Let A be a subset of Y. From (4), it follows that

$$m_X^{(i,j)}\text{-Cl}_\beta(f^{-1}(m_Y^j\text{-Int}(m_Y^i\text{-Cl}(Y-A)))) \subseteq f^{-1}(m_Y^i\text{-Cl}(Y-A)).$$

This implies that $f^{-1}(m_Y^i\text{-Int}(A)) \subseteq m_Y^{(i,j)}\text{-Int}_\beta(f^{-1}(m_Y^j\text{-Cl}(m_Y^i\text{-Int}(A))))$.

(5) $\Rightarrow$ (2) It follows from the fact that $V = m_Y^i\text{-Int}(V)$ if V is m_Y^i-open.

$\square$

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