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AN ALTERNATIVE APPROACH TO MUNDICI NON SIMPLICITY CRITERION

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Abstract

In this short paper, we present an alternative approach to a classical non simplicity criterion for AF $\mathcal{C}^*$-algebras proved by Mundici in [6]. The results of [6] are based on model theoretical tools. Our aim here is to make use of a group theoretical approach to Petri nets as presented in [2]. This algebraic viewpoint, besides providing a simpler access to Mundici criterion, allows to treat different types of Petri nets in a unified way.

1. Introduction

It is well-known that model theory is a deep subject that encompasses a large range of applications in pure and applied mathematics, see, e.g., the excellent book by Manin [5]. Nevertheless, it seems that a large number of mathematicians are not fully acquainted with its basic notions, so our approach, based on Petri nets [7], aims to be more palatable, though following the same guideline of Mundici original result, that reads

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Mundici theorem. *Let A be an AF* $\mathcal{C}^*$*-algebra with lattice ordered* K_0. *Assume there exists a theory* $\Theta \in \theta(A)$ *such that the set* $\tilde{\Theta}$ *of consequences of* Θ *is recursively enumerable but not recursive. Then A is not simple.*

2. Basic Definitions

Definition. A set K with an associative law + possessing an identity e and with the property $a + b = e \Rightarrow a = b = e$ for all a, b in K is called a *cone* and is denoted by $K := (K, e, +)$. If + is commutative K is said *commutative* or *abelian*.

Example. Let A be a non-empty set and denote by A^* the set of all finite sequences or strings on A. Then the system $\mathcal{K} = (A^*, \varepsilon, \cdot)$ is a cone (non-abelian), where ε denotes the empty string and $\cdot$ the concatenation of strings in A^*.

Definition. Let $\mathcal{K}_1 = (K_1, e_1, +_1)$, $\mathcal{K}_2 = (K_2, e_2, +_2)$ be cones. A cone homomorphism F from $\mathcal{K}_1$ to $\mathcal{K}_2$ denoted by $F : \mathcal{K}_1 \to \mathcal{K}_2$ is a mapping $F : K_1 \to K_2$ such that $F(e_1) = e_2$ and such that the diagram

$$\begin{tikzcd} K_1 \times K_2 \arrow[r, "F \times F"] \arrow[d, "+_1"'] & K_2 \times K_2 \arrow[d, "+_2"] \\ K_1 \arrow[r, "F"'] & K_2 \end{tikzcd}$$

commutes.

The image $IM(F)$ of a cone homomorphism is a cone. The cones with cone homomorphisms define a category. An isomorphism of cones is an isomorphism in this category. A bijective mapping $F : K_1 \to K_2$ defines a cone isomorphism if for F and F^{-1} the diagram is commutative and $e_2 = F(e_1)$, $e_1 = F^{-1}(e_2)$.

Definition. Let $\mathcal{K}_1 = (K_1, e_1, +_1)$, $\mathcal{K}_2 = (K_2, e_2, +_2)$ be cones with

$K_1 \subset K_2$. The cone $\mathcal{K}_1$ is called a *subcone* of $\mathcal{K}_2$ if the natural injection of K_1 in K_2 defines a cone homomorphism from $\mathcal{K}_1$ to $\mathcal{K}_2$.

Example. For $n \geq 2$ let be given the cones $\mathcal{K}_0 := (\{0, 1, ..., n\}, 0, \max_0)$ and $\mathcal{K}_1 := (\{1, 2, ..., n\}, 1, \max_1)$. The mapping $F : K_0 \to K_1$ defined by $F(\nu) := \nu - 1$, for $\nu = 1, ..., n$ defines a cone homomorphism F, which maps the cone $\mathcal{K}_1$ to the cone $\mathcal{K}' := F(\mathcal{K}_1) := (F(K_1), F(1), \max_0)$ with $F(K_1) \subset K_0$. Since the natural injection of $F(K_1)$ in K_0 defines a homomorphism from $F(K_1)$ to K_0 the cone $\mathcal{K}'$ is a subcone of $\mathcal{K}_0$.

The injective mapping $F_1 : K_1 \to K_0$ defined by $F_1(\nu) := \nu$, for $\nu = 1, ..., n$ defines a cone homomorphism from $\mathcal{K}_1$ to $\mathcal{K}_0$. The image $IM(F_1)$ is the cone $F_1(\mathcal{K}_1) = (F_1(K_1), F_1(1), \max_1)$ such that $F_1(K_1) \subset K_0$. Since the natural injection does not define a homomorphism the cone $F_1(\mathcal{K}_1)$ is not a subcone of $\mathcal{K}_0$.

Remark. Let be given a family $\{\mathcal{K}_i \mid i \in I\}$, I an index set, of subcones of a cone $\mathcal{K}$. Then the intersection $\bigcap_{i \in I} \mathcal{K}_i := \left(\bigcap_{i \in I} K_i, e_{\mathcal{K}}, +_{\mathcal{K}}\right)$ is also a subcone of $\mathcal{K}$.

Example. Consider the cones $\mathcal{K}_0 := (\mathbb{N}_0, 0, +)$ and $\mathcal{K}_1 := (\mathbb{N}, 1, *)$. The mapping $F : \mathbb{N} \to \mathbb{N}_0$ defined by $F(n) := \alpha(n, 1) + \alpha(n, 2) + \cdots + \alpha(n, \beta_n)$ for all $n \in \mathbb{N}$ defines a surjective cone homomorphism $F : \mathcal{K}_1 \to \mathcal{K}_0$. Here $n = p_{n,1}^{\alpha(n, 1)} p_{n,2}^{\alpha(n, 2)} \cdots p_{n,\beta_n}^{\alpha(n, \beta_n)}$ denotes the unique factorization of n into primes. The mapping $F_1 : \mathbb{N}_0 \to \mathbb{N}$ defined by

$$F_1(n) := \begin{cases} np, & \text{if } n > 0, \\ 1, & \text{if } n = 0 \end{cases}$$

defines an injective cone homomorphism from $\mathcal{K}_0$ to $\mathcal{K}_1$; here p denotes a given prime.

Example. Let be given a set $A := \{a_1, a_2, ..., a_n\}, \ n \geq 1,$ and consider the cones $\mathcal{K}_0 := (\mathbb{N}_0, 0, +)$ and $\mathcal{K}_1 := (A^*, \varepsilon, \cdot)$. The mapping $L : A^* \to \mathbb{N}$ defined by

$$L(\omega) := \begin{cases} 0, & \text{if } \omega = \varepsilon, \\ k, & \text{if } \omega = a_1 a_2 \cdots a_k, \ k \geq 1 \end{cases}$$

defines a surjective cone homomorphism $h : \mathcal{K}_1 \to \mathcal{K}_0$. $L(\omega)$ is called the *length* of the string $\omega \in A^*$.

The system $\mathcal{K}_n := (\mathbb{N}_0^n, 0, +)$ endowed with the usual addition in $\mathbb{N}_0^n$ and the zero vector as the identity element is an abelian cone. The mapping $p : A^* \to \mathbb{N}_0^n$ defined by $p(\omega) := (p_1(\omega), p_2(\omega), ..., p_n(\omega))$ defines a surjective cone homomorphism $p : \mathcal{K}_1 \to \mathcal{K}_0$, where $p_\nu(\omega)$ denotes the number of occurrences of a_ν in the string ω, $\nu = 1, 2, ..., n$. The function p is called the *Parikh vector* of the set A.

Remark. Let be given a non-empty set A and a cone $\mathcal{K}$. For any mapping F from A into $\mathcal{K}$ there is a unique extension of F to a cone homomorphism

$$F^* : (A^*, \varepsilon, \cdot) \to \mathcal{K}$$

such that diagram below is commutative where i denotes the natural injection.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "F"'] & A^* \arrow[d, "F^*"] \\ & K_{\mathcal{K}} \end{tikzcd}$$

In fact, if $\omega = a_1 a_2 \cdots a_n \in A^*$, then we must have $F^*(\omega) = F(a_1)F(a_2)$ $\cdots F(a_n)$. So F^* is unique. If F^* is defined in this way, then F^* defines a cone homomorphism which satisfies $F = F^* \circ i$.

Now, it is well-known that a group homomorphism h from a group $\mathcal{G}_1$ into a group $\mathcal{G}_2$ is injective if and only if $\ker(h) := \{a \in G_1 \,|\, h(a) = e_2\}$

$= \{e_1\}$. In general, this is not true for cone homomorphisms. In fact, consider the cones $(\mathbb{N}_0^2, 0, +)$ and $(\mathbb{N}_0 \setminus \{1\}, 0, +)$. Consider the cone homomorphism $F(x, y) = 2x + 3y$. Since $2x + 3y = 0$ for $x, y \in \mathbb{N}_0$ if and only if $x = 0$ and $y = 0$, one has $\ker(F) = (0, 0)$. However, F is not injective since it maps $(3, 0)$ and $(0, 2)$ into 6.

Definition. A system $\mathcal{C} := (G, e, +, K)$ is called a *cone group* if $\mathcal{K} = (K, e, +)$, $\mathcal{G} = (G, e, +)$ is a group and $K \subset \mathcal{G}$.

Example. (a) Let $\mathcal{V}_A$ be the set of all mappings $\psi : A \to \mathbb{Z}$ such that $\psi(a) = 0$ for almost all $a \in A$, i.e. $\psi(a) = 0$ with exception of a finite number of points. Then $(\mathcal{V}_A, 0, +, (\mathcal{V}_A)_0^+)$ is an abelian cone group, here + is the pointwise addition. (b) Let $\mathcal{F}_A$ be the free group generated by the non-empty set A. The system $(\mathcal{F}_A, \varepsilon, \cdot, A^*)$ is a non-abelian cone group.

Definition. Let $\mathcal{C}_1 := (G_1, e_1, +_1, K_1)$, $\mathcal{C}_2 := (G_2, e_2, +_2, K_2)$ be cone groups. A cone group homomorphism, short a CG-homomorphism, from $\mathcal{C}_1$ to $\mathcal{C}_2$ denoted by $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a group homomorphism $F : (G_1, e_1, +_1) \to (G_2, e_2, +_2)$ such that $F(K_1) \subset K_2$.

The image $IM(F)$ of a CG-homomorphism is a cone group. The cone groups with the CG-homomorphisms define a category. CG-isomorphims of cone groups are isomorphisms in this category. Morphisms in this category which are set-theoretically bijective need not to be isomorphisms. For instance, consider the cone groups $(\mathbb{Z}, 0, +, \mathbb{N}_0 \setminus \{1\})$ and $(\mathbb{Z}, 0, +, \mathbb{N}_0)$. Then the set-theoretic identity map is a bijective CG-homomorphism but not a CG-isomorphism.

An isomorphism $F : \mathcal{C}_1 \to \mathcal{C}_2$ in this category must satisfy $F(K_1) = K_2$. Conversely, a group isomorphism $F : G_1 \to G_2$ satisfying $F(K_1) = K_2$ must be a CG-isomorphism, because F and F^{-1} are both cone homomorphisms.

Definition. Let $\mathcal{C}_1 = (G_1, e_1, +_1, K_1)$, $\mathcal{C}_2 = (G_2, e_2, +_2, K_2)$ be cone groups with $G_1 \subset G_2$. The cone group $\mathcal{C}_1$ is called a *subcone group* of $\mathcal{C}_2$ if the natural injection of G_1 in G_2 defines a CG-homomorphism from $\mathcal{C}_1$ to $\mathcal{C}_2$.

Remark. Let $F : (G, e, +, K) \to (F_1, e_1, +_1, K_1)$ be a CG-homomorphism. Then $F : (G, e, +, \leq_K) \to (G_1, e_1, +_1, \leq_{K_1})$ is an isotone homomorphism. In fact, it suffices to note that *F* is isotone. Consider the elements $a, b \in G$ with $a \leq_K b$. Then one has the following equivalences:

$$\begin{aligned} a \leq_K b &\Leftrightarrow b - a \in K \\ &\Leftrightarrow F(b - a) \in K_1 \\ &\Leftrightarrow F(b) -_1 F(a) \in K_1 \\ &\Leftrightarrow F(a) \leq_{K_1} F(b). \end{aligned}$$

Consequently, *F* is an isotone mapping. Analogously, let $F : (G, e, +, \leq) \to (G_1, e_1, +_1, \leq_1)$ be an isotone group homomorphism. Then $F : (G, e, +, G_e^+) \to (G_1, e_1, +_1, (G_1)_{e_1}^+)$ is a CG-homomorphism. Here it suffices to show that *F* maps cone elements on cone elements. Let *a* be an element $\in G^+$. Then one has

$$\begin{aligned} a \in G^+ &\Leftrightarrow e \leq a \\ &\Leftrightarrow F(e) \leq_1 F(a) \\ &\Leftrightarrow e_1 \leq_1 F(a) \\ &\Leftrightarrow F(a) \in (G_1)_{e_1}^+. \end{aligned}$$

Definition. An *abelian Petri net* is a system $\mathcal{P} = (\mathcal{C}, T, \text{Pre}, \text{Post})$ consisting of an abelian cone group $\mathcal{C} = (G, e, + K)$, a non-empty set of transitions T, mappings $\mathrm{Pre}: T \to K$ and $\mathrm{Post}: T \to K$. The cone group $\mathcal{C}$ is called the *net group*. An element $m \in K$ is called *marking* or *state* of the Petri net.

Example. Consider the system $\mathcal{P}$ consisting of the cone group $\mathcal{C} = (\mathbb{R}^n, 0, +, (\mathbb{R}_0^+)^n)$, addition and identity 0 in its usual meaning, the set of transitions $T := \mathbb{R}_0^+ \times T_0$, where $T_0 := \{\tau_1, \tau_2, ..., \tau_k\}$, the mappings Pre and Post defined by

$$\mathrm{Pre}(t) := \mathrm{Pre}(r, \tau) := r \cdot pr(\tau),$$

$$\mathrm{Post}(t) := \mathrm{Post}(r, \tau) := r \cdot po(\tau),$$

where the functions *pr* and *po* map T_0 into the cone $(\mathbb{R}_0^+)^n$. The system is an abelian Petri net with an uncountable set of transitions.

Example. Consider the system $\mathcal{P}$ consisting of the cone group $\mathcal{C} = (\mathbb{Z}^2, (0, 0), +, K)$, addition and identity in its usual meaning and the cone given by $K := \{(x, y) \in \mathbb{Z}^2 \mid x > 0 \vee (x = 0) \text{ and } y \geq 0\}$, the set of transitions T, the mappings Pre and Post. This system is an abelian Petri net. The net group is totally ordered by the cone K. Note that we can define Petri nets morphisms in a natural way.

Definition. A transition $t \in T$ of a Petri net $\mathcal{P} = ((\mathcal{G}, e, +, K), T, \mathrm{Pre}, \mathrm{Post})$ is enabled with respect to a marking $m \in K$ if:

(a) one of the following equivalent conditions holds:

- $m - \mathrm{Pre}(t) \in K$;
- $\mathrm{Pre}(t) \leq_K m$ (here $\leq_K$ denotes the partial order induced by K on G).

(b) $\mathrm{Pre}(t) \neq e$ or $\mathrm{Post}(t) \neq e$.

Definition. Let be given a Petri net $\mathcal{P}$ and a transition t which is enabled with respect to the marking m. We say that t *fires* whenever m is transformed into the marking $\delta(t, m) := m - \operatorname{Pre}(t) + \operatorname{Post}(t)$. The partially defined mapping $\delta : K \times T \to K$, is well defined iff t is enabled with respect to the marking m.

Definition. Given a Petri net $\mathcal{P} = ((\mathcal{G}, e, +, K), T, \operatorname{Pre}, \operatorname{Post})$ and a marking $m_0 \in K$, we say a *word of transitions* $\omega = t_1 t_2 \cdots t_\ell \in T^*$ *is enabled* with respect to m_0, whenever

t_1 is enabled with respect to m_0

t_2 is enabled with respect to $m_1 := \delta(m_0, t_1)$

$\vdots$

t_ℓ is enabled with respect to $m_{\ell-1} := \delta(m_{\ell-2}, t_{\ell-1})$.

An enabled word of transitions $\omega = t_1 t_2 \cdots t_\ell$ can fire, in this case the marking m_0 is transformed in $\delta(m_0, \omega) := \delta(m_{\ell-1}, t_\ell)$. Thus one gets the partial mapping $\delta^* : K \times T^* \to K$, where

$$\delta^*(m, \omega) = \begin{cases} m, & \text{if } \omega = \varepsilon, \\ \delta(m, t), & \text{if } \omega = t \text{ and } t \text{ is enabled with respect to } m, \\ \delta(\delta^*(m, \omega_1), t), & \text{if } \omega = \omega_1 t \text{ and } t \\ & \text{is enabled with respect to } \delta^*(m, \omega_1). \end{cases}$$

Definition. Given a Petri net $\mathcal{P} = ((\mathcal{G}, e, +, K), T, \operatorname{Pre}, \operatorname{Post})$ and a marking $m_0 \in K$. We say a marking $m \in K$ is *reachable* from marking m_0 if there is $\omega \in T^*$ such that $m = \delta(m_0, \omega)$. The set $\mathcal{E}_{\mathcal{P}}(m_0) = \mathcal{E}(m_0) := \{m \in K \mid m \text{ is reachable from } m_0\}$ is called the *reachable set* with respect to $\mathcal{P}$ and m_0.

3. Main Theorems

We shall need the following classical result.

Koenig Infinity Lemma. *Let $V_0, V_1, \ldots$ be an infinite sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that every vertex v in a set V_n, with $n \geq 1$ has a neighbour $f(v)$ in V_{n-1}. Then G contains an infinite path $v_0 v_1 \cdots v_n \in V_n$ for all n.*

Proof. See, e.g., Diestel [4]. □

Theorem 1. (a) *Given $\mathcal{P} = ((\mathcal{G}, e, +, K), T, Pre, Post)$ and a marking m_0. If there are $m_1, m_2 \in K$ and $\omega_1, \omega_2 \in T^*$ such that*

$$m_1 = \delta(m_0, \omega_1) \text{ and } m_2 = \delta(m_1, \omega_2) > m_1,$$

then $\mathcal{E}_{\mathcal{P}}(m_0)$ has infinitely many elements.

(b) *If a Petri net $\mathcal{P} = ((\mathcal{G}, e, +, K), T, Pre, Post)$ has a $*$-cone, that is, if each sequence in K contains a monotone increasing subsequence and the reachable set $\mathcal{E}(m_0)$ has infinitely many elements, then there exist m_1, m_2 in $K_{\mathcal{P}}$ and ω_1, ω_2 in T^* such that*

$$m_1 = \delta(m_0, \omega_1) \text{ and } m_2 = \delta(m_1, \omega_2) > m_1.$$

Proof. If $\omega \in T^*$ is enabled with respect to $m' \in K$, so is ω enabled with respect to $m \geq m'$. Setting

$$\Delta(\omega_2) := \sum_{t \in T} [\mathrm{Post}(t) - \mathrm{Pre}(t)] p_t(\omega_2)$$

follows from our hypothesis that

$$\Delta(\omega_2) = m_2 - m_1 > e.$$

Hence, ω_2 is also enabled with respect to m_2. Thus, there is

$$m_3 := \delta(m_2, \omega_2) = m_2 + \Delta(\omega_2).$$

This implies that $m_2 < m_3$. By induction we get that $\mathcal{E}(m_0)$ contains

$$m_0 < m_1 < m_2 < \cdots.$$

That proves (a).

For (b), define an infinite sequence $V_0, V_1, \ldots$ of pairwise, disjoint, non-empty, finite subsets $\mathcal{E}(m_0)$ by setting

$$V_k := \{\delta(m_0, \omega) \in \mathcal{E}(m_0) \,|\, \text{length}\,(\omega) = k \text{ and } k \text{ minimal}\}.$$

Now define a graph on the union of the V_k 's taking directed edges

$$(m', m) \in V_{k-1} \times V_k,$$

where $m = \delta(m', t)$. Each element m in V_k is joined to an element from V_{k-1}. It follows from Koenig Infinity Lemma that there exists an infinite path $m_0 m_1 \cdots$ with $m_k \in V_k$, for each $k \in \mathbb{N}$.

By hypothesis, $K_{\mathcal{P}}$ is a $*$-cone, so the sequence (m_k) contains a weak increasing subsequence, for which – since $m'_k \neq m'_{k+1}$ – holds:

$$m'_0 = m_0 < m'_1 < m'_2 < \cdots.$$

Therefore, for suitable $\omega_1, \omega_2 \in T^*$

$$m'_1 = \delta(m_0, \omega_1) \text{ and } m'_2 = \delta(m'_1, \omega_2) > m'_1. \qquad \square$$

Theorem 2 (Alternative Mundici Criterion)**.** *Let $\mathfrak{A}$ be an AF $\mathcal{C}^*$-algebra with lattice ordered K_0. Assume there exists a Petri net $\mathcal{P}$ such that a set $\mathcal{S}$ of words of transitions of $\mathcal{P}$ is recursively enumerable but not recursive. Then $\mathfrak{A}$ is not simple.*

Proof. According to Badouel et al., see [1], the cone K_0 allows us to canonically associate to a given Petri net a so-called MV-algebra (introduced by Chang in [3]). On the other hand, this object is exactly the algebraic tool one needs to follows mutatis mutandis Mundici's reasoning, where set $\mathcal{S}$ plays the role of the set $\tilde{\Theta}$ of consequences of the theory Θ. $\square$

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