Far East Journal of Mathematical Sciences (FJMS)

Published Online: February 2014
Available online at http://pphmj.com/journals/fjms.htm
Volume 84, Number 1, 2014, Pages 33-46

ON CONVERGENCE THEOREMS OF THE FIXED POINTS FOR POINTWISE CONVERGENT MAPS IN AN IFMS

Jong Seo Park

Department of Mathematics Education
Institute of Mathematics Education
Chinju National University of Education
Jinju 660-756, South Korea
e-mail: parkjs@cue.ac.kr

Abstract

Shen et al. [14] proved the convergence of the sequence of fixed points for some sequences of contraction mappings satisfying certain conditions in fuzzy metric space. In this paper, we define some properties and obtain the convergence of the fixed points sequence for pointwise convergent sequences of contraction mapping and IFM satisfying certain conditions in IFMS.

1. Introduction

Kramosil and Michalek [4] introduced the concept of fuzzy metric space. Grabiec [2] extended fixed point theorems of Banach and Edelstein to fuzzy metric spaces in the sense of Kramosil and Michalek. Kutukcu et al. [5] extended fixed point theory to other types of fuzzy metric space in recent years, and Shen et al. [14] proved the convergence of the sequence of fixed points for some sequences of contraction mappings satisfying certain

Received: September 11, 2013; Revised: October 10, 2013; Accepted: October 24, 2013
2010 Mathematics Subject Classification: 46S40, 47H10, 54H25.
Keywords and phrases: convergence, fixed point, contraction map, locally compact, compact.

conditions in fuzzy metric space. Park et al. [12] proved the existence of fixed point for a nonexpansive mapping of intuitionistic fuzzy metric space (shortly, IFMS) and the intuitionistic Banach fixed point theorem in complete IFMS, respectively. Also, Park [9, 10] extended fixed point theory to others types of IFMS.

In 2013, Park [11] studied some common fixed point theorem in IFMS, and proved a fixed point theorem for a pair of *k*-weakly commuting mappings in IFMS. Also, Park et al. [8] extended some common fixed point theorem for five maps to *M*-fuzzy metric spaces.

In this paper, we define some properties and obtain the convergence of the fixed points sequence for pointwise convergent sequences of contraction mapping and IFM satisfying certain conditions in IFMS.

2. Preliminaries

In this section, we recall some definitions, properties and known results in the IFMS as following:

Let us recall (see [13]) that a continuous *t*-norm is an operation $* : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (a) $*$ is commutative and associative, (b) $*$ is continuous, (c) $a * 1 = a$ for all $a \in [0, 1]$, (d) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ $(a, b, c, d \in [0, 1])$. Also, a continuous *t*-conorm is an operation $\Diamond : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (a) $\Diamond$ is commutative and associative, (b) $\Diamond$ is continuous, (c) $a \Diamond 0 = a$ for all $a \in [0, 1]$, (d) $a \Diamond b \ge c \Diamond d$ whenever $a \le c$ and $b \le d$ $(a, b, c, d \in [0, 1])$.

Definition 2.1 ([7])**.** The 5-tuple $(X, M, N, *, \Diamond)$ is said to be an *intuitionistic fuzzy metric space* (shortly, *IFMS*) if X is an arbitrary set, $*$ is a continuous *t*-norm, $\Diamond$ is a continuous *t*-conorm and *M*, *N* are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions; for all $x, y, z \in X$ such that

(a) $M(x, y, t) > 0,$

(b) $M(x, y, t) = 1$ if and only if $x = y,$

(c) $M(x, y, t) = M(y, x, t),$

(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s),$

(e) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous,

(f) $N(x, y, t) > 0,$

(g) $N(x, y, t) = 0$ if and only if $x = y,$

(h) $N(x, y, t) = N(y, x, t),$

(i) $N(x, y, t) \lozenge N(y, z, s) \geq N(x, z, t + s),$

(j) $N(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous.

Note that *M*, *N* is called an *IFM* on *X*. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between *x* and *y* with respect to *t*, respectively.

In this paper, *X* is considered to be the IFMS with the following condition: for all $x, y \in X$ and $t > 0,$

$$\lim_{t \to \infty} M(x, y, t) = 1, \quad \lim_{t \to \infty} N(x, y, t) = 0. \tag{2.1}$$

Definition 2.2. Let *X* be an IFMS. Then

(a) A sequence $\{x_n\}$ is said to *converge* to *x* in *X*, denoted by $x_n \to x$, if and only if $\lim_{n \to \infty} M(x_n, x, t) = 1,$ $\lim_{n \to \infty} N(x_n, x, t) = 0$ for all $t > 0,$ that is, for each $r \in (0, 1)$ and $t > 0,$ there exists an $n_0 \in \mathbf{N}$ such that $M(x_n, x, t) > 1 - r,$ $N(x_n, x, t) < r$ for all $n \geq n_0.$

(b) A sequence $\{x_n\} \subset X$ is a *G-Cauchy sequence* if and only if for all $p > 0$ and $t > 0,$

$$\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1, \quad \lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0.$$

(c) The IFMS X is called *G-complete* if every G-Cauchy sequence is convergent in X.

Lemma 2.3 ([7])**.** *Let X be a G-complete IFMS. If there exists a number $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,*

$$M(Tx, Ty, kt) \geq M(x, y, t), \quad N(Tx, Ty, kt) \leq N(x, y, t),$$

then T has a unique fixed point.

Definition 2.4. Let X be an IFMS and let $\{T_n\}$ be a sequence of self-mappings on X. $T_0 : X \to X$ is a given map. The sequence $\{T_n\}$ is said to *converge pointwise* to T_0 if for each $r \in (0, 1)$ and $t > 0$, there exists an $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ and $x \in X$,

$$M(T_n x_0, T_0 x_0, t) > 1 - r, \quad N(T_n x_0, T_0 x_0, t) < r.$$

Definition 2.5. Let X be an IFMS and let $\{T_n\}$ be a sequence of self-mappings on X. $T_0 : X \to X$ is a given map. The sequence $\{T_n\}$ is said to *converge uniformly* to T_0 if for each $r \in (0, 1)$ and $x_0 \in X$, there exists an $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ and $x \in X$,

$$M(T_n x, T_0 x, t) > 1 - r, \quad N(T_n x, T_0 x, t) < r.$$

Definition 2.6. Let X be an IFMS. A sequence of self-mappings $\{T_n\}$ is *uniformly equicontinuous* if for each $r \in (0, 1)$, there exists an $\varepsilon \in (0, 1)$ such that for every $x, y \in X$, $n \in \mathbf{N}$ and $s, t > 0$, $M(x, y, s) > 1 - \varepsilon$, $N(x, y, s) < \varepsilon$ implies

$$M(T_n x, T_n y, t) > 1 - r, \quad N(T_n x, T_n y, t) < r.$$

Definition 2.7. Let X be an IFMS. The open ball $B(x, r, t)$ and closed ball $B[x, r, t]$ with center $x \in X$ and radius r, $0 < r < 1$, $t > 0$, respectively, are defined as follows:

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r,\ N(x, y, t) < r\},$$

$$B[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r,\ N(x, y, t) \leq r\}.$$

Definition 2.8. An IFMS X is a *compact space* if $(X, \tau_{M,N})$ is a compact topological space, where $\tau_{M,N}$ is a topology induced by the intuitionistic fuzzy metric M, N.

Lemma 2.9 ([9])**.** (a) *Every open*(*closed*) *ball is an open*(*closed*) *set*.

(b) *Every closed subset A of a compact IFMS X is compact*.

Lemma 2.10. *Let X be an IFMS and let* $\{T_n\}$ *be a sequence of self-mappings on X,* $T_0 : X \to X$ *be a contraction mapping of X and A be a compact subset of X. If* $\{T_n\}$ *converges pointwise to* T_0 *in A and it is a uniformly equicontinuous sequence, then the sequence* $\{T_n\}$ *converges uniformly to* T_0 *in A.*

Proof. For each $\bar{r} \in (0, 1)$, we choose an appropriate r such that $(1-r) * (1-r) * (1-r) > 1-\bar{r}$ and $r \lozenge r \lozenge r < \bar{r}$. Since $\{T_n\}$ is uniformly equicontinuous, there exists $\varepsilon \in (0, 1)$ with $\varepsilon \le r$ such that if $M(x, y, s) > 1-\varepsilon$, $N(x, y, s) < \varepsilon$, then $M(T_n x, T_n y, t) > 1-r$, $N(T_n x, T_n y, t) < r$ for every $x, y \in X$, $s, t > 0$ and $n \in \mathbf{N}$. For ε, we fix $s > 0$. Define $\beta = \{B(x, \varepsilon, s) : x \in A\}$. By Lemma 2.3, β is a family of open sets of A. Clearly, β constitutes an open covering of A. That is, $A \subset \bigcup_{i=1}^{m} B(x_i, \varepsilon, s)$. Since A is compact, there exist $x_1, x_2, \ldots, x_m \in A$ such that $A \subset \bigcup_{i=1}^{m} B(x_i, \varepsilon, s)$. For every $x_i \in A$ $(i = 1, 2, \ldots, m)$, since $\{T_n\}$ converges pointwise to T_0 in A, there exist $n_i \in \mathbf{N}$ $(i = 1, 2, \ldots, m)$ for $r \in (0, 1)$ such that $M(T_n x_i, T_0 x_i, t) > 1-r$, $N(T_n x_i, T_0 x_i, t) < r$ for all $n \ge n_i$. Putting $n^* = \max\{n_i : i = 1, 2, \ldots, m\}$, then n^* depends only on r. For $x \in X$, there is an $i_0 \in \{i = 1, 2, \ldots, m\}$ such that $x \in B(x_{i_0}, \varepsilon, s)$. Hence, we have that if $M(x, x_{i_0}, s) > 1-\varepsilon$, $N(x, x_{i_0}, s) < \varepsilon$, then $M(T_n x, T_n x_{i_0}, t) > 1-r$, $N(T_n x, T_n x_{i_0}, t) < r$ for all $n \in \mathbf{N}$. Thus, for all $n \ge n^*$,

$$
\begin{aligned}
& M(T_n x, T_0 x, 2t + ks) \\
\geq\ & M(T_n x, T_n x_{i_0}, t) * M(T_n x_{i_0}, T_0 x, t + ks) \\
\geq\ & M(T_n x, T_n x_{i_0}, t) * M(T_n x_{i_0}, T_0 x_{i_0}, t) * M(T_0 x_{i_0}, T_0 x, ks) \\
\geq\ & M(T_n x, T_n x_{i_0}, t) * M(T_n x_{i_0}, T_0 x_{i_0}, t) * M(x_{i_0}, x, s) \\
\geq\ & (1 - r) * (1 - r) * (1 - \varepsilon) \\
\geq\ & (1 - r) * (1 - r) * (1 - r) > 1 - \bar{r}, \\
& N(T_n x, T_0 x, 2t + ks) \\
\leq\ & N(T_n x, T_n x_{i_0}, t) \diamond N(T_n x_{i_0}, T_0 x, t + ks) \\
\leq\ & N(T_n x, T_n x_{i_0}, t) \diamond N(T_n x_{i_0}, T_0 x_{i_0}, t) \diamond N(T_0 x_{i_0}, T_0 x, ks) \\
\leq\ & N(T_n x, T_n x_{i_0}, t) \diamond N(T_n x_{i_0}, T_0 x_{i_0}, t) \diamond N(x_{i_0}, x, s) \\
\leq\ & r \diamond r \diamond \varepsilon \leq r \diamond r \diamond r < \bar{r}.
\end{aligned}
$$

Hence, the sequence $\{T_n\}$ converges uniformly to T_0 in A. □

Definition 2.11. An IFMS X in which every point has a compact neighborhood is called *locally compact*.

Definition 2.12. Let X be an IFMS with IFM M_0, N_0 and let $\{M_n\}$ and $\{N_n\}$ be sequences of IFM on X.

(a) The sequence $\{M_n\}$ is said to *upper semiconverge uniformly* to M_0 if for each $r \in (0, 1)$ and $t > 0$, there exists an $n_0 \in \mathbf{N}$ such that $M_n(x, y, t) \geq M_0(x, y, t)$ and $\dfrac{M_0(x, y, t)}{M_n(x, y, t)} > 1 - r$ for all $n \geq n_0$, $x, y \in X$.

(b) The sequence $\{N_n\}$ is said to *lower semiconverge uniformly* to N_0 if for each $r \in (0, 1)$ and $t > 0$, there exists an $n_0 \in \mathbf{N}$ such that $N_n(x, y, t) \leq N_0(x, y, t)$ and $\dfrac{N_0(x, y, t)}{N_n(x, y, t)} < r$ for all $n \geq n_0$, $x, y \in X$.

3. Main Results

Theorem 3.1. *Let X be a G-complete IFMS and let* $\{T_n\}$ *be a sequence of self-mappings on* X, *where t-norm* $a * b = \min\{a, b\}$ *and t-conorm* $a \lozenge b = \max\{a, b\}$. T_0 *is a contraction mapping of X, that is, there exists* $k_0 \in (0, 1)$ *such that* $M(T_0x, T_0y, k_0t) \geq M(x, y, t)$ *and* $N(T_0x, T_0y, k_0t) \leq N(x, y, t)$ *for all* $x, y \in X$, $t > 0$, *and satisfying* $T_0x_0 = x_0$. *If there exists at least a fixed point* x_n *for each* T_n $(n \in \mathbf{N})$ *and the sequence* $\{T_n\}$ *converges uniformly to* T_0, *then* $x_n \to x_0$.

Proof. Suppose that $x_n \nrightarrow x_0$, there exist $t_0 > 0$, $r_0 \in (0, 1)$ such that for any $n \in \mathbf{N}$, there is a $k(n) > n$ satisfying $M(x_{k(n)}, x_0, t_0) < 1 - r_0$ and $N(x_{k(n)}, x_0, t_0) > r_0$. Fixed a number $h \in (k_0, 1)$, from (2.1), we can find $p \in \mathbf{N}$ for $t_0 > 0$ such that $M\left(x_n, x_0, t_0\left(\frac{h}{k_0}\right)^p\right) > 1 - r_0$ and $N\left(x_n, x_0, t_0\left(\frac{h}{k_0}\right)^p\right) < r_0$ for any $n \in \mathbf{N}$. Since the sequence $\{T_n\}$ converges uniformly to T_0, we can make n_0 sufficiently large such that $M(T_nx_n, T_0x, t) > 1 - r_0$ and $N(T_nx_n, T_0x, t) < r_0$ for all $n \geq n_0$, $t > 0$. Now for $n \geq n_0$, we have

$$\begin{aligned}
1 - r_0 &> M(x_{k(n)}, x_0, t_0) \\
&= M(T_{k(n)}x_{k(n)}, T_0x_0, t_0) \\
&\geq M(T_{k(n)}x_{k(n)}, T_0x_{k(n)}, (1-h)t_0) * M(T_0x_{k(n)}, T_0x_0, ht_0) \\
&\geq M(T_{k(n)}x_{k(n)}, T_0x_{k(n)}, (1-h)t_0) * M\left(x_{k(n)}, x_0, \frac{ht_0}{k_0}\right) \\
&\geq M(T_{k(n)}x_{k(n)}, T_0x_{k(n)}, (1-h)t_0) * M\left(T_{k(n)}x_{k(n)}, T_0x_{k(n)}, t_0\frac{(1-h)h}{k_0}\right)
\end{aligned}$$

$$* M\left(x_{k(n)},\ x_0,\ t_0\left(\frac{h}{k_0}\right)^2\right)$$

$$\geq \cdots$$

$$\geq M(T_{k(n)}x_{k(n)}, T_0x_{k(n)}, (1-h)t_0) * M\left(T_{k(n)}x_{k(n)}, T_0x_{k(n)}, t_0\,\frac{(1-h)h}{k_0}\right)$$

$$* \cdots * M\left(x_{k(n)},\ x_0,\ t_0\left(\frac{h}{k_0}\right)^p\right)$$

$$\geq (1 - r_0) * (1 - r_0) * \cdots * (1 - r_0) = 1 - r_0,$$

$$r_0 < N(x_{k(n)},\ x_0,\ t_0)$$

$$= N(T_{k(n)}x_{k(n)},\ T_0x_0,\ t_0)$$

$$\leq N(T_{k(n)}x_{k(n)},\ T_0x_{k(n)},\ (1-h)t_0) \lozenge N(T_0x_{k(n)},\ T_0x_0,\ ht_0)$$

$$\leq N(T_{k(n)}x_{k(n)},\ T_0x_{k(n)},\ (1-h)t_0) \lozenge N\left(x_{k(n)},\ x_0,\ \frac{ht_0}{k_0}\right)$$

$$\leq N(T_{k(n)}x_{k(n)},\ T_0x_{k(n)},\ (1-h)t_0) \lozenge N\left(T_{k(n)}x_{k(n)},\ T_0x_{k(n)},\ t_0\,\frac{(1-h)h}{k_0}\right)$$

$$\lozenge N\left(x_{k(n)},\ x_0,\ t_0\left(\frac{h}{k_0}\right)^2\right)$$

$$\leq \cdots$$

$$\leq N(T_{k(n)}x_{k(n)},\ T_0x_{k(n)},\ (1-h)t_0) \lozenge N\left(T_{k(n)}x_{k(n)},\ T_0x_{k(n)},\ t_0\,\frac{(1-h)h}{k_0}\right)$$

$$\lozenge \cdots \lozenge N\left(x_{k(n)},\ x_0,\ t_0\left(\frac{h}{k_0}\right)^p\right)$$

$$\leq r_0 \lozenge r_0 \lozenge \cdots \lozenge r_0 = r_0.$$

Therefore, this is a contradiction. Hence $x_n \to x_0$. □

Theorem 3.2. *Suppose that X is a locally compact IFMS. Let $\{T_n\}$ be a sequence of self-mappings on X and let $T_0 : X \to X$ be a contraction mapping. If the following conditions are satisfied*:

(a) T_n^m *is a contraction mapping for a certain* $m = m(n)$,

(b) $\{T_n\}$ *converges pointwise to* T_0 *and* $\{T_n\}$ *is a uniformly equicontinuous,*

(c) $T_n x_n = x_n$, $x = 0, 1, 2, \ldots,$

then the sequence $\{x_n\}$ *converges to* x_0.

Proof. We can choose $r \in (0, 1)$ for each $\varepsilon \in (0, 1)$ such that $(1-r) * (1-r) \geq 1-\varepsilon$ and $r \lozenge r \leq \varepsilon.$ Assume that r is sufficiently small for given $x_0 \in X$ such that

$$U(x_0, r) = \{x : M(x, x_0, t) \geq 1-r,\ N(x, x_0, t) \leq r\}$$

is a compact subset of X. Since $\{T_n\}$ is uniformly equicontinuous and pointwise convergent on $U(x_0, r)$, by Lemma 2.10, $\{T_n\}$ converges uniformly to T_0 on the compact subset $U(x_0, r)$. Then for that r, there exists $n_\varepsilon \in \mathbf{N}$ such that $M(T_n x, T_0 x, (1-k_0)t) > 1-r$ and $N(T_n x, T_0 x, (1-k_0)t) < r$ for all $n \geq n_\varepsilon$, $t > 0$ and $x \in U(x_0, r)$. Also, since T_0 is a contraction mapping, we have $M(T_0 x, T_0 y, k_0 t) \geq M(x, y, t)$ and $N(T_0 x, T_0 y, k_0 t) \leq N(x, y, t)$ for all $x, y \in U(x_0, r)$. Thus, for all $n \geq n_\varepsilon$ and $x \in U(x_0, r)$, we can obtain

$$\begin{aligned} M(T_n x, x_0, t) &= M(T_n x, T_0 x_0, t) \\ &\geq M(T_n x, T_0 x, (1-k_0)t) * M(T_0 x, T_0 x_0, k_0 t) \\ &\geq M(T_n x, T_0 x, (1-k_0)t) * M(x, x_0, t) \\ &\geq (1-r) * (1-r) \geq 1-\varepsilon, \end{aligned}$$

$$\begin{aligned} N(T_n x,\ x_0,\ t) &= N(T_n x,\ T_0 x_0,\ t) \\ &\le N(T_n x,\ T_0 x,\ (1-k_0)t) \lozenge N(T_0 x,\ T_0 x_0,\ k_0 t) \\ &\le N(T_n x,\ T_0 x,\ (1-k_0)t) \lozenge N(x,\ x_0,\ t) \\ &\le r \lozenge r \le \varepsilon. \end{aligned}$$

Hence, for all $n \ge n_\varepsilon$, $U(x_0, r)$ is an invariant set for T_n^m. Since T_n^m is a contraction mapping for a certain positive integer $m = m(n)$, the fixed point x_n of T_n is contained in the set $U(x_0, r)$. By definition of $U(x_0, r)$, we have $M(x_n, x_0, t) \ge 1 - r$ and $N(x_n, x_0, t) \le r$ for all $n \ge n_\varepsilon$. Therefore, $x_n \to x_0$. □

Lemma 3.3. *Suppose that X is a G-complete IFMS. Let A be a compact subset of X, where t-norm* $a * b = \min\{a, b\}$ *and t-conorm* $a \lozenge b = \max\{a, b\}$, *and let* $\{M_n\}$ *and* $\{N_n\}$ *be sequences of IFM,* $\{T_n\}$ *be a sequence of self-mappings on X. If they satisfy the following conditions*:

(a) $\{M_n\}$ *upper semiconverges uniformly to* M_0,

(b) $\{N_n\}$ *lower semiconverges uniformly to* N_0,

(c) T_n *is a contraction mapping for the IFM* M_n *and* N_n, $n = 0, 1, 2, \ldots$,

(d) $\{T_n\}$ *converges pointwise to* T_0,

then $\{T_n\}$ *converges uniformly to* T_0 *in A with IFM* M_0 *and* N_0.

Proof. We can choose $r \in (0, 1)$ for each $\varepsilon \in (0, 1)$ such that $(1-r) * (1-r) \ge 1 - \varepsilon$ and $r \lozenge r \le \varepsilon$. From (a) and (b), there exists $n_r \in \mathbf{N}$ such that $M_n(x, y, t) \ge M_0(x, y, t)$, $\dfrac{M_0(x, y, t)}{M_n(x, y, t)} > 1 - r$, $N_n(x, y, t) \le N_0(x, y, t)$ and $\dfrac{N_0(x, y, t)}{N_n(x, y, t)} < r$ for all $n \ge n_r$, $t > 0$. Choose $x, y \in X$ such that

$M_0(x, y, t) > 1 - r$ and $N_0(x, y, t) < r$ for each $t > 0$. Then we have for all $n \geq n_r$,

$$
\begin{aligned}
M_0(T_n x, T_n y, t) &= \frac{M_0(x, y, t)}{M_n(x, y, t)} * M_n(T_n x, T_n y, t) \\
&\geq (1 - r) * M_n(T_n x, T_n y, t) \\
&\geq (1 - r) * M_n\left(x, y, \frac{1}{k_n}\right), (k_n \in (0, 1)) \\
&\geq (1 - r) * M_0\left(x, y, \frac{1}{k_n}\right) \\
&\geq (1 - r) * (1 - r) > 1 - \varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
N_0(T_n x, T_n y, t) &= \frac{N_0(x, y, t)}{N_n(x, y, t)} \lozenge N_n(T_n x, T_n y, t) \\
&\leq r \lozenge N_n(T_n x, T_n y, t) \\
&\leq r \lozenge N_n\left(x, y, \frac{1}{k_n}\right), (k_n \in (0, 1)) \\
&\leq r \lozenge N_0\left(x, y, \frac{1}{k_n}\right) \\
&\leq r \lozenge r < \varepsilon.
\end{aligned}
$$

Therefore, the sequence $\{T_n\}$ is uniformly equicontinuous in A with IFM M_0 and N_0. Also, by (d), since $\{T_n\}$ is pointwise convergent and A is a compact subset of X, we have $\{T_n\}(n \geq n_r)$ converges uniformly to T_0 in A from Lemma 2.10. Thus, $\{T_n\}$ converges uniformly to T_0 in A with IFM M_0 and N_0. □

Theorem 3.4. *Suppose that X is a locally compact IFMS, where t-norm $a * b = \min\{a, b\}$ and t-conorm $a \lozenge b = \max\{a, b\}$. If $\{M_n\}$, $\{N_n\}$ and $\{T_n\}$ satisfy the following conditions*:

(a) $\{M_n\}$ *upper semiconverges uniformly to* M_0,

(b) $\{N_n\}$ *lower semiconverges uniformly to* N_0,

(c) T_n *is a contraction mapping for the IFM* M_n *and* N_n, $n = 0, 1, 2, \ldots$,

(d) $\{T_n\}$ *converges pointwise to* T_0,

(e) $T_n x_n = x_n$, $n = 0, 1, 2, \ldots$,

then the fixed points $\{x_n\}$ *of* $\{T_n\}$ *converge to the fixed point* x_0 *of* T_0.

Proof. We can choose $r \in (0, 1)$ for each $\varepsilon \in (0, 1)$ such that $(1-r) * (1-r) \geq 1-\varepsilon$ and $r \diamond r \leq \varepsilon$. Also, we may make r sufficiently small for each $x_0 \in X$ such that $U(x_0, r) = \{x : M(x, x_0, t) \geq 1-r, N(x, x_0, t) \leq r\}$ is compact in X for each $t > 0$. By Lemma 3.3, $\{T_n\}$ converges uniformly to T_0 in $U(x_0, r)$ with respect to the IFM M_0 and N_0. Thus, for every $x \in X$, there exists an $n_r \in \mathbf{N}$ such that $M_0(T_n x, T_0 x, t) \geq 1-r$ and $N(T_n x, T_0 x, t) \leq r$ for all $n \geq n_r$, $t > 0$. Therefore, we have for all $x \in U(x_0, r)$ and $n \geq n_r$,

$$\begin{aligned} M_0(T_n x, x_0, (1+k_0)t) &\geq M_0(T_n x, T_0 x, t) * M_0(T_0 x, x_0, k_0 t) \\ &\geq M_0(T_n x, T_0 x, t) * M_0(T_0 x, T_0 x_0, k_0 t) \\ &\geq M_0(T_n x, T_0 x, t) * M_0(x, x_0, t) \\ &\geq (1-r) * M_0\left(x, y, \frac{1}{k_n}\right) \\ &\geq (1-r) * (1-r) \geq 1-\varepsilon, \end{aligned}$$

$$\begin{aligned} N_0(T_n x, x_0, (1+k_0)t) &\leq N_0(T_n x, T_0 x, t) \diamond N_0(T_0 x, x_0, k_0 t) \\ &\leq N_0(T_n x, T_0 x, t) \diamond N_0(T_0 x, T_0 x_0, k_0 t) \\ &\leq N_0(T_n x, T_0 x, t) \diamond N_0(x, x_0, t) \\ &\leq r \diamond r \leq \varepsilon. \end{aligned}$$

Hence, $U(x_0, r)$ is an invariant set in X with M_0 and N_0. From (c), since T_n is a contraction mapping in $U(x_0, r)$ with IFM M_n and N_n, we know that the fixed point is included in $U(x_0, r)$. Therefore, we can obtain $M_0(x_n, x_0, t) \geq 1 - r$ and $N_0(x_n, x_0, t) \leq r$. Since r is sufficiently small, $x_n \to x_0$. □

Theorem 3.5. *Let X be a compact IFMS, where t-norm* $a * b = \min\{a, b\}$, *t-conorm* $a \lozenge b = \max\{a, b\}$. *Suppose that* $\{M_n\}$, $\{N_n\}$ *and* $\{T_n\}$ *satisfy the following conditions*:

(a) $\{M_n\}$ *upper semiconverges uniformly to* M_0,

(b) $\{N_n\}$ *lower semiconverges uniformly to* N_0,

(c) T_n *is a contraction mapping for the IFM* M_n *and* N_n, $n = 0, 1, 2, \ldots$,

(d) $\{T_n\}$ *converges pointwise to* T_0.

If T_n $(n \in \mathbf{N})$ *has a fixed point* x_n *and there is a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *which converges to* x_0, *then* $T_0 x_0 = x_0$.

Proof. Let $U(x_0, r)$ denote the closure of the set $\{x_{n_k}\}$. By Lemma 2.9, we know that $U(x_0, r)$ is a compact set. From Lemma 3.3, $\{T_{n_k}\}$ converges uniformly to T_0 in $U(x_0, r)$ with M_0 and N_0. Clearly, $\{T_{n_k} x_{n_k}\}$ converges to $T_0 x_0$. Hence $T_0 x_0 = x_0$. □

References

[1] S. Chauhan, Sumitra and B. D. Pant, Coincidence and common fixed point theorems in intuitionistic fuzzy metric spaces, Far East J. Math. Sci. (FJMS) 79(1) (2013), 25-48.

[2] M. Grabiec, Fixed point in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988), 385-389.

[3] Z. H. Jiao, W. X. Yang, Z. F. Zhang and C. H. Jin, Fixed point and common fixed point theorems in intuitionistic fuzzy metric spaces, Mohu Xitong yu Shuxue 26(5) (2012), 59-66.

[4] J. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975), 326-334.

[5] S. Kutukcu, D. Turkoglu and C. Yildiz, Common fixed points of compatible maps of type (β) on fuzzy metric spaces, Commun. Korean Math. Soc. 21(1) (2006), 89-100.

[6] F. Merghadi, A related fixed point theorem in *n* intuitionistic fuzzy metric spaces, J. Concr. Appl. Math. 11(1) (2013), 121-135.

[7] J. H. Park, J. S. Park and Y. C. Kwun, A common fixed point theorem in the intuitionistic fuzzy metric space, Advances in Natural Comput. Data Mining, Proc. 2nd ICNC and 3rd FSKD, 2006, pp. 293-300.

[8] J. H. Park, J. S. Park and Y. C. Kwun, Fixed points in M-fuzzy metric spaces, Fuzzy Optim. Decis. Mak. 7 (2008), 305-315.

[9] J. S. Park, Fixed point theorem for common property (E.A.) and weak compatible functions in intuitionistic fuzzy metric space, Int. J. F. I. S. 11(3) (2011), 149-153.

[10] J. S. Park, On fixed point theorem of weak compatible maps of type (β) in complete intuitionistic fuzzy metric space, Int. J. F. I. S. 11(1) (2011), 38-43.

[11] J. S. Park, Some common fixed point theorems for the weakly commuting maps on intuitionistic fuzzy metric space, Far East J. Math. Sci. (FJMS) 76(1) (2013), 97-104.

[12] J. S. Park, S. Y. Kim and H. J. Kang, A common fixed point theorem in the intuitionistic fuzzy metric space, J. K. F. I. S. 16(3) (2006), 321-325.

[13] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 314-334.

[14] Y. H. Shen, D. Qiu and W. Chen, On convergence of fixed points in fuzzy metric spaces, Abstr. Appl. Anal. 2013 (2013), Article ID 135202, 6pp.