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THE EXTENDED SPACE OF A STATIONARY MULTISCALE SYSTEM

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Abstract

Let S be a Schur multiplier in a multiscale system. Then there is a quasi-coisometric linear system whose state space is the extension space $\hat{\mathcal{D}}(S)$ of the de Branges-Rovnyak space $\mathcal{H}(S)$ and the transfer function is S.

1. Introduction

In this paper, the relationships between the theory of non-stationary linear systems indexed by the integers and the theory of stationary multiscale systems indexed by a homogeneous tree are explained. Stationary multiscale systems were first introduced by Basseville et al. [5, 6]. In the non-stationary case, we replace the Hardy space by space of upper triangular Hilbert Schmidt operators, complex variables by the bilateral backward shift operators [2, 3]. This paper presents a construction of a stationary multiscale

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system indexed by a homogeneous tree using the methods first introduced by de Branges and Rovnyak.

We first review some of definitions using in the theory of multiscale systems introduced by Basseville, Benveniste, Nikoukhah and Willsky.

A homogeneous tree $\mathcal{T}$ of order $q \geq 2$ is an infinite acyclic, undirected, connected graph such that every node of $\mathcal{T}$ has exactly $(q+1)$ branches. For each nodes $s, t \in \mathcal{T}$, the notion of distance $d(s, t)$ is the number of edges along the shortest path between s and t and denote $s \wedge t$ by the first of their common node.

Using these notation, define the partial order $\preceq$ by

$$s \preceq t \text{ if } d(s, s \wedge t) \leq d(t, s \wedge t)$$

and the equivalence relation $\asymp$ by

$$s \asymp t \text{ if } d(s, s \wedge t) = d(t, s \wedge t).$$

The primitive shift operator $\hat{\bar{\gamma}} : \mathcal{T} \to \mathcal{T}$ is defined by

$$t\hat{\bar{\gamma}} \preceq t, \quad d(t\hat{\bar{\gamma}}, t) = 1, \quad \forall t \in \mathcal{T}.$$

Let $l_2(\mathcal{T})$ be the Hilbert space of square summable sequences indexed by the nodes of a homogeneous tree $\mathcal{T}$ of order $q \geq 2$:

$$l_2(\mathcal{T}) = \left\{ f : \mathcal{T} \to \mathbb{C} : \| f \|_{l_2}^2 = \sum_{t \in \mathcal{T}} | f(t) |^2 < \infty \right\}.$$

Define the upward shift operator $\bar{\gamma} : l_2(\mathcal{T}) \to l_2(\mathcal{T})$ by

$$\bar{\gamma} f(t) = \frac{1}{\sqrt{q}} f(t\hat{\bar{\gamma}}).$$

Then $\bar{\gamma}$ is an isometry but not unitary since it is not surjective. Denote the adjoint operator of $\bar{\gamma}$ by γ.

Let $\mathbf{X}(\mathcal{T})$ denote the C^*-algebra of bounded linear operators on $l_2(\mathcal{T})$. A multiscale system is a linear system of the form

$$y = Su, \text{ where } u, y \in l_2(\mathcal{T}), \tag{1.1}$$

where the transfer function S is in $\mathbf{X}(\mathcal{T})$. The multiscale linear system (1.1) is said to be *causal* if

$$(u(t) = 0, \forall t \preceq t_0) \to (y(t) = 0, \forall t \preceq t_0)$$

and it is said to be *stationary* if for every tree isometry $\hat{\tau}$ with

$$t\hat{\tau}\hat{\bar{\gamma}} \asymp t, \quad \forall t \in \mathcal{T},$$

the corresponding operator $\tau : l_2(\mathcal{T}) \to l_2(\mathcal{T})$ defined by $\tau f(t) = f(t\hat{\tau})$ commutes with the transfer function S.

In [4], it has been shown that a causal and stationary multiscale system can be represented by a series which is convergent in the operator norm. The following definitions and results were announced in [4].

Define

$$\sigma_n = \bar{\gamma}^n \gamma^n, \sigma_0 = I, \omega_0 = I - \sigma_1, \text{ and } \omega_n = \sigma_n - \sigma_{n+1}, \quad n \in \mathbb{Z}_+.$$

Then we have $\bar{\gamma}\omega_n = \omega_{n+1}\bar{\gamma}$ and $\omega_n\bar{\gamma}^k = 0$ if $n < k$. Define a Banach algebra

$$\mathbf{U}(\mathcal{T}) = \overline{\operatorname{span}}_{\mathbb{C}}\{\bar{\gamma}^n \sigma_m : n, m = 0, 1, 2...\},$$

where the closure is taken in the pointwise sense and define the commutative $\mathbb{C}^*$ algebra $\mathbb{K}$ as

$$\mathbb{K} = \left\{ \mathbf{c} = \sum_{k=0}^{\infty} c_k \omega_k : c_k \in \mathbb{C}, \sup_k |c_k| < \infty \right\}$$

with the usual operator norm $\|\mathbf{c}\|_{op} = \sup_k |c_k|$. Let $S \in \mathbf{X}(\mathcal{T})$. From

Theorems 4.2 and 4.3 in [1], we can characterize a causal and stationary multiscale linear system.

$S \in \mathbf{U}(\mathcal{T})$ if and only if

$$S = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{s}_k, \text{ where } \| \mathbf{s}_k \|_{op} \le \| S \|_{op}.$$

The multiscale system $y = Su$ is both causal and stationary if and only if $S \in \mathbf{U}(\mathcal{T})$.

2. Multiscale Systems and Point Evaluations

In the stationary multiscale case, we consider the Hardy space as the following space. Define a Hilbert space

$$\mathbb{K}_2 = \left\{ \mathbf{c} = \sum_{k=0}^{\infty} c_k \omega_k \in \mathbb{K} : \sum_{k=0}^{\infty} | c_k |^2 < \infty \right\}$$

with the inner product

$$\langle \mathbf{c}, \mathbf{d} \rangle_{\mathbb{K}_2} = \sum_{k=0}^{\infty} \bar{d}_k c_k, \text{ where } \mathbf{c} = \sum_{k=0}^{\infty} c_k \omega_k \text{ and } \mathbf{d} = \sum_{k=0}^{\infty} d_k \omega_k.$$

Define the Hilbert space

$$\mathbf{U}_2(\mathcal{T}) = \left\{ F = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{f}_k : \mathbf{f}_k \in \mathbb{K}_2, \| F \|^2_{\mathbf{U}_2(\mathcal{T})} = \sum_{k=0}^{\infty} \| \mathbf{f}_k \|^2_{\mathbb{K}_2} < \infty \right\}$$

with the inner product

$$\langle F, G \rangle_{\mathbf{U}_2(\mathcal{T})} = \sum_{k=0}^{\infty} \langle \mathbf{f}_k, \mathbf{g}_k \rangle_{\mathbb{K}_2}, \text{ where } F = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{f}_k \text{ and } G = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{g}_k.$$

Then the space $\mathbf{U}_2(\mathcal{T})$ is contained contractively in $\mathbf{U}(\mathcal{T})$.

Now exploit the left and right point evaluation on $\mathbf{U}(\mathcal{T})$. Let $\mathbf{c} \in \mathbb{K}$. Define

$$\mathbf{c}^{[n]} = (\mathbf{c}\gamma)^n \overline{\gamma}^n,\ \mathbf{c}^{\langle n \rangle} = \overline{\gamma}^n (\gamma\mathbf{c})^n,\ \mathbf{c}^{[0]} = \mathbf{c}^{\langle 0 \rangle} = I,$$

$$\mathbf{c}^{(n)} = \gamma^n \mathbf{c} \overline{\gamma}^n,\ \mathbf{c}^{(-n)} = \overline{\gamma}^{-n} \mathbf{c} \gamma^{-n} \text{ and } \mathbf{c}^{(0)} = \mathbf{c}$$

for a non-negative integer n. Then

$$\mathbf{c}^{*[n]} = \mathbf{c}^{[n]*}, \quad \mathbf{c}^{*\langle n \rangle} = \mathbf{c}^{\langle n \rangle *},$$

$$\mathbf{c}^{[n]} = (\mathbf{c}\gamma)^n \overline{\gamma}^n = \gamma^n (\overline{\gamma}\mathbf{c})^n,$$

$$\mathbf{c}^{\langle n \rangle} = \overline{\gamma}^n (\gamma\mathbf{c})^n = (\mathbf{c}\overline{\gamma}^n)\gamma^n,$$

$$\gamma^m \mathbf{c}^{(n)} = \mathbf{c}^{(n+m)} \gamma^m, \quad \mathbf{c}^{[n+1]} = \mathbf{c}^{[n]} \mathbf{c}^{(n)},$$

where

$$\overline{\mathbf{c}} = \mathbf{c}^* = \sum_{k=0}^{\infty} \overline{c}_k \omega_k.$$

For $c \in \mathbb{K}$ with $\limsup_{n\to\infty} \| \mathbf{c}^{[n]} \|^{\frac{1}{n}} < 1$, the series

$$\sum_{k=0}^{\infty} \mathbf{c}^{[k]} \mathbf{s}_k \quad \text{and} \quad \sum_{k=0}^{\infty} \mathbf{s}_k^{(-n)} \mathbf{c}^{\langle n \rangle}$$

are absolutely convergent in $\mathbb{K}$. Define the space

$$\mathbb{D}(\mathcal{T}) = \{\mathbf{c} \in \mathbb{K} : \rho(c) = \limsup_{n\to\infty} \| \mathbf{c}^{[n]} \|^{\frac{1}{n}} < 1\}.$$

Let $F = \sum_{k=0}^{\infty} \overline{\gamma}^k \mathbf{f}_k \in \mathbf{U}_2(\mathcal{T})$ and $\mathbf{c} \in \mathbb{D}(\mathcal{T})$. The left point evaluation of F at $\mathbf{c}$ is defined by

$$F^{\wedge}(\mathbf{c}) = \sum_{k=0}^{\infty} \mathbf{c}^{[k]} \mathbf{f}_k = \sum_{k=0}^{\infty} (\mathbf{c}\gamma)^k \overline{\gamma}^k \mathbf{f}_k$$

and the right point evaluation of F at $\mathbf{c}$ is defined by

$$F^{\triangle}(\mathbf{c}) = \sum_{k=0}^{\infty} \mathbf{f}_k^{(-k)} \mathbf{c}^{\langle k \rangle} = \sum_{k=0}^{\infty} \bar{\gamma}^k \mathbf{f}_k (\gamma \mathbf{c})^k.$$

The space $\mathbf{U}_2(\mathcal{T})$ is a reproducing kernel space with respect to the left and right point evaluations (Theorem 4.2 [4]).

Theorem 2.1. *Let* $F \in \mathbf{U}_2(\mathcal{T})$ *and* $\mathbf{c} \in \mathbb{D}(\mathcal{T})$. *For any* $\mathbf{k} \in \mathbb{K}_2$, *the identities*

$$\langle F, K_{\wedge}^{\mathbf{c}} \mathbf{k} \rangle_{\mathbf{U}_2(\mathcal{T})} = \langle F^{\wedge}(\mathbf{c}), \mathbf{k} \rangle_{\mathbb{K}_2}, \quad \langle F, \mathbf{k} K_{\Delta}^{\mathbf{c}} \rangle_{\mathbf{U}_2(\mathcal{T})} = \langle F^{\Delta}(\mathbf{c}), \mathbf{k} \rangle_{\mathbb{K}_2}$$

hold, *where*

$$K_{\wedge}^{\mathbf{c}} = (I - \bar{\gamma}\mathbf{c}^*)^{-1}, \quad K_{\Delta}^{\mathbf{c}} = (I - \mathbf{c}^* \bar{\gamma})^{-1} \in \mathbf{U}_2(\mathcal{T}).$$

We can apply similar argument on the space $\mathbf{L}(\mathcal{T})$, where

$$\mathbf{L}(\mathcal{T}) = \overline{\mathrm{span}}_{\mathbb{C}} \{\sigma_m \gamma^n : n, m = 0, 1, 2, \ldots\}.$$

Define the Hilbert space

$$\mathbf{L}_2(\mathcal{T}) = \left\{ G = \sum_{k=0}^{\infty} \mathbf{g}_k \gamma^k : \mathbf{g}_k \in \mathbb{K}_2, \| G \|_{\mathbf{L}_2(\mathcal{T})}^2 = \sum_{k=0}^{\infty} \| \mathbf{g}_k \|_{\mathbb{K}_2}^2 < \infty \right\}$$

with the inner product

$$\langle F, G \rangle_{\mathbf{L}_2(\mathcal{T})} = \sum_{k=0}^{\infty} \langle \mathbf{f}_k, \mathbf{g}_k \rangle_{\mathbb{K}_2}, \text{ where } F = \sum_{k=0}^{\infty} \mathbf{f}_k \gamma^k \text{ and } G = \sum_{k=0}^{\infty} \mathbf{g}_k \gamma^k.$$

Theorem 2.2. *Let* $G \in \mathbf{X}(\mathcal{T})$. *Then* $G \in \mathbf{L}(\mathcal{T})$ *if and only if it can be represented as a row-wise converging series*

$$G = \sum_{k=0}^{\infty} \mathbf{s}_k \gamma^k, \quad \mathbf{s}_k \in \mathbb{K}.$$

Proof. $G \in \mathbf{L}(\mathcal{T})$ if and only if $G^* \in \mathbf{U}(\mathcal{T})$. □

We can define the left and right point evaluation on $\mathbf{L}_2(\mathcal{T})$. Let $G = \sum_{k=0}^{\infty} \mathbf{g}_k \gamma^k$, where $\mathbf{g}_k \in \mathbb{K}_2$ for each k. Define the left point evaluation of G at $\mathbf{c}$ by

$$G^{\vee}(\mathbf{c}) = \sum_{k=0}^{\infty} \mathbf{g}_k \gamma^k (\bar{\gamma}\mathbf{c})^k$$

and the right point evaluation of F at $\mathbf{c}$ by

$$G^{\nabla}(\mathbf{c}) = \sum_{k=0}^{\infty} (\mathbf{c}\bar{\gamma})^k \mathbf{g}_k \gamma^k.$$

Theorem 2.3. *Let* $G \in \mathbf{L}_2(\mathcal{T})$ *and* $\mathbf{c} \in \mathbb{D}(\mathcal{T})$. *For any* $\mathbf{k} \in \mathbb{K}_2$, *the identities*

$$\langle G, \mathbf{k} K_{\vee}^{\mathbf{c}} \rangle_{\mathbf{L}_2(\mathcal{T})} = \langle G^{\vee}(\mathbf{c}), \mathbf{k} \rangle_{\mathbb{K}_2}, \quad \langle G, K_{\nabla}^{\mathbf{c}} \mathbf{k} \rangle_{\mathbf{L}_2(\mathcal{T})} = \langle G^{\nabla}(\mathbf{c}), \mathbf{k} \rangle_{\mathbb{K}_2}$$

hold where

$$K_{\vee}^{\mathbf{c}} = (I - \mathbf{c}^* \gamma)^{-1}, \quad K_{\nabla}^{\mathbf{c}} = (I - \gamma \mathbf{c}^*)^{-1} \in \mathbf{L}_2(\mathcal{T}).$$

Proof. Since $(\mathbf{c}\gamma)^k = \mathbf{c}^{[k]}\gamma^k$,

$$K_{\vee}^{\mathbf{c}} = (I - \mathbf{c}^* \gamma)^{-1} = \sum_{k=0}^{\infty} (\mathbf{c}^* \gamma)^k = \sum_{k=0}^{\infty} \mathbf{c}^{*[k]} \gamma^k$$

and

$$\langle G, \mathbf{k} K_{\vee}^{\mathbf{c}} \rangle_{\mathbf{L}_2(\mathcal{T})} = \sum_{k=0}^{\infty} \langle \mathbf{g}_k, \mathbf{k}\mathbf{c}^{*[k]} \rangle_{\mathbb{K}_2} = \left\langle \sum_{k=0}^{\infty} \mathbf{g}_k \mathbf{c}^{[k]}, \mathbf{k} \right\rangle_{\mathbb{K}_2}$$

$$= \langle G^{\vee}(\mathbf{c}), \mathbf{k} \rangle_{\mathbb{K}_2}.$$

Since $(\gamma\mathbf{c})^k = \gamma^k\mathbf{c}^{\langle k\rangle} = (\mathbf{c}^{\langle k\rangle})^{(k)}\gamma^k$,

$$K_{\nabla}^{\mathbf{c}} = (I - \gamma\mathbf{c}^*)^{-1} = \sum_{k=0}^{\infty}(\gamma\mathbf{c}^*)^k = \sum_{k=0}^{\infty}(\mathbf{c}^{*\langle k\rangle})^{(k)}\gamma^k$$

and

$$\langle G,\, K_{\nabla}^{\mathbf{c}}\mathbf{k}\rangle_{\mathbf{L}_2(\mathcal{T})} = \left\langle \sum_{k=0}^{\infty}\mathbf{g}_k^{(-n)}\mathbf{c}^{[k]},\, \mathbf{k}\right\rangle_{\mathbb{K}_2} = \langle G^{\nabla}(\mathbf{c}),\, \mathbf{k}\rangle_{\mathbb{K}_2}. \qquad \square$$

3. The Extension Space $\hat{\mathcal{D}}(S)$

In this section, we construct the extension space $\hat{\mathcal{D}}(S)$ associated with $\mathcal{H}(S)$ using the method introduced by de Branges and Rovnyak [8].

Definition 3.1. An operator $S \in \mathbf{U}(\mathcal{T})$ is called a *Schur multiplier* if $\| S \|_{op} \leq 1$.

Let S be a Schur multiplier and $F \in \mathbf{U}_2(\mathcal{T})$. Then the multiplication operator

$$M_S : \mathbf{U}_2(\mathcal{T}) \to \mathbf{U}_2(\mathcal{T})$$

defined by $M_S F = SF$ is contractive [1]. Hence, the de Branges-Rovnyak space $\mathcal{H}(S)$ also can be constructed in the multiscale cases [7, 8].

The space

$$\mathcal{H}(S) = \{F \in \mathbf{U}_2(\mathcal{T}) : k(F) < \infty\},$$

where

$$k(F) = \sup_{U \in \mathbf{U}_2(\mathcal{T})}\{\| F + SU \|_{\mathbf{U}_2(\mathcal{T})}^2 - \| U \|_{\mathbf{U}_2(\mathcal{T})}^2\}$$

is a Hilbert space with the inner product $\| F \|_{\mathcal{H}(S)}^2 = k_L(F)$.

Let $\mathbf{k} \in \mathbb{K}_2$ and $\mathbf{c} \in \mathbb{D}(\mathcal{T})$. From

$$
\begin{aligned}
M_S^* K_\wedge^c \mathbf{k} &= S^\wedge(\mathbf{c})^* K_\wedge^c \mathbf{k} \\
&= \sum_{j=0}^{\infty} \mathbf{s}_j^* \bar{\gamma}^j \left[\sum_{k=0}^{\infty} (\bar{\gamma}\mathbf{c}^*)^{j+k} \mathbf{k} \right] \\
&= \sum_{k=0}^{\infty} \bar{\gamma}^k \sum_{j=0}^{\infty} \mathbf{s}_j^{*(k)} \mathbf{c}^{*[j+k]} \mathbf{k},
\end{aligned}
$$

the reproducing kernel function $K_S^{\mathbf{c}}$ of the space $\mathcal{H}(S)$ is of the form

$$
\begin{aligned}
K_S^{\mathbf{c}} &= (I - SM_S^*) K_\wedge^c \\
&= K_\wedge^c - S\left(\sum_{k=0}^{\infty} \bar{\gamma}^k \sum_{j=0}^{k} \mathbf{s}_{k-j}^{(j)} \mathbf{c}^{*[k]} \right).
\end{aligned}
$$

For a Schur multiplier S, there exists a co-isometric linear system whose transfer function is S (Theorem 6.1 in [1]).

Theorem 3.2. *Let S be a Schur multiplier. Then there exists a linear system*

$$
V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H}(S) \\ \mathbb{K}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{H}(S) \\ \mathbb{K}_2 \end{pmatrix}
$$

defined by

$$(AF) = (F - F^\wedge(0))\gamma,$$

$$(B\mathbf{k}) = (S - S^\wedge(0))\mathbf{k}\gamma,$$

$$CF = F^\wedge(0),$$

and

$$D\mathbf{k} = S^\wedge(0)\mathbf{k}, \tag{3.1}$$

where

$$VV^* = \begin{pmatrix} \bar{\gamma}\gamma & 0 \\ 0 & I \end{pmatrix} \tag{3.2}$$

and S can be written by

$$S\mathbf{k} = D\mathbf{k} + \sum_{k=0}^{\infty} \bar{\gamma}^{n+1}(CA^n B\mathbf{k})^{n+1}.$$

A linear system V is called to be *quasi-coisometric* if V satisfies (3.2). The extension space $\hat{\mathcal{D}}(S)$ associated with $\mathcal{H}(S)$ can be constructed.

Let $F \in \mathcal{H}(S)$ and $A^*F = G$ and $B^*F = H$. Since the linear system (3.1) is co-isometric, the identities

$$[(AA^*)F + (BB^*)F] = F \quad \text{and} \quad (CA^*)F + (DB^*)F = 0$$

hold. Hence, we have

$$A^*F = F\bar{\gamma} - SH.$$

Set

$$F_0 = F, \quad F_n = A^*F_{n-1}, \quad H_0 = B^*F_0, \quad H_n = B^*F_{n-1} \text{ for } n \geq 1.$$

Then

$$F_n = F\bar{\gamma}^n - S(H_0\bar{\gamma}^{n-1} + \cdots + H_{n-1}) \in \mathcal{H}(S)$$

and

$$\begin{aligned} \| F_n \|^2_{\mathcal{H}(S)} &= \langle A^*F_{n-1}, A^*F_{n-1}\rangle_{\mathcal{H}(S)} \\ &= \langle (I - BB^*)F_{n-1}, F_{n-1}\rangle_{\mathcal{H}(S)} \\ &= \| F_{n-1} \|^2_{\mathcal{H}(S)} - \| H_{n-1} \|^2_{\mathbf{K}_2} \\ &= \| F \|^2_{\mathcal{H}(S)} - \sum_{k=0}^{n-1} \| H_k \|^2_{\mathbf{K}_2}. \end{aligned} \tag{3.3}$$

Let the extension space $\hat{\mathcal{D}}(S)$ associated with $\mathcal{H}(S)$ be the set of pairs (F, H), where $F \in \mathcal{H}(S)$ and $H = \sum_{k=0}^{\infty} H_k \gamma^k$ such that

$$F\gamma^n - S(H_0\overline{\gamma}^{n-1} + \cdots + H_{n-1}) \in \mathcal{H}(S)$$

and the sequence

$$\| F\overline{\gamma}^n - S(H_0\overline{\gamma}^{n-1} + \cdots + H_{n-1}) \|_{\mathcal{H}(S)}^2 + \sum_{k=0}^{n-1} \| H_k \|_{\mathbf{K}_2}^2$$

is finite for every non-negative integer *n*. Then $\hat{\mathcal{D}}_L(S)$ becomes a Hilbert space with the inner product

$$\| (F, H) \|_{\hat{\mathcal{D}}_L(S)}$$

$$= \lim_{n\to\infty}\left(\| F\overline{\gamma}^n - S(H_0\overline{\gamma}^{n-1} + \cdots + H_{n-1}) \|_{\mathcal{H}(S)}^2 + \sum_{k=0}^{n-1} \| H_k \|_{\mathbf{K}_2}^2 \right). \quad (3.4)$$

From (3.2) and (3.3), we have

$$\| (F, H) \|_{\hat{\mathcal{D}}(S)} = \| F \|_{\mathcal{H}(S)}. \quad (3.5)$$

The space $\hat{\mathcal{D}}(S)$ is a reproducing kernel space.

Theorem 3.3. *For each* $\mathbf{k} \in \mathbf{K}_2$,

$$(K_S^{\mathbf{c}}\mathbf{k}, (S^* - S^{*\vee}(\mathbf{c}))(I - \overline{\gamma}\mathbf{c})^{-1}\mathbf{k}\overline{\gamma}) \in \mathcal{D}(S)$$

and

$$\langle F^{\wedge}(\mathbf{c}), \mathbf{k}\rangle_{\mathbb{K}_2} = \langle (F, H), (K_S^{\mathbf{c}}\mathbf{k}, (S^* - S^{*\vee}(\mathbf{c}))(I - \overline{\gamma}\mathbf{c})^{-1}\mathbf{k}\overline{\gamma})\rangle_{\hat{\mathcal{D}}(S)}.$$

Proof. Let $\mathbf{k} \in \mathbf{K}_2$ and $F \in \mathbf{U}_2(\mathcal{T})$. First represent $M_S^*(K_{\wedge}^{c}\mathbf{k}\overline{\gamma}^n)$ as a series for each non-negative integer *n*. Since

$$M_S F = SF = \sum_{k=0}^{\infty} \overline{\gamma}^k \sum_{j=0}^{k} \mathbf{s}_{k-j}^{(j)} f_j$$

and

$$K_{\wedge}^{c}\mathbf{k}\bar{\gamma}^{n} = \sum_{k=0}^{\infty}\bar{\gamma}^{k}\mathbf{c}^{*[k]}\mathbf{k}\bar{\gamma}^{n} = \sum_{k=0}^{\infty}\bar{\gamma}^{k+n}(\mathbf{c}^{*[k]}\mathbf{k})^{(n)},$$

$$\begin{aligned}\langle SF,\, K_{\wedge}^{c}\mathbf{k}\bar{\gamma}^{n}\rangle_{\mathcal{H}_2} &= \sum_{k=0}^{\infty}\left\langle \sum_{j=0}^{k+n}\mathbf{s}_{k+n-j}^{(j)}f_j,\, (\mathbf{c}^{*[k]}\mathbf{k})^{(n)}\right\rangle \mathbf{K}_2 \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k+n}\langle f_j,\, \mathbf{s}_{k+n-j}^{*(j)}(\mathbf{c}^{*[k]}\mathbf{k})^{(n)}\rangle \mathbf{K}_2 \\ &= \sum_{k=0}^{n}\left\langle f_k,\, \sum_{j=0}^{\infty}\mathbf{s}_{j+n-k}^{*(k)}(\mathbf{c}^{*[j]}\mathbf{k})^{(n)}\right\rangle \mathbf{K}_2 \\ &\quad + \sum_{k=n+1}^{\infty}\left\langle f_k,\, \sum_{j=0}^{\infty}\mathbf{s}_{j}^{*(k)}(\mathbf{c}^{*[j+k-n]}\mathbf{k})^{(n)}\right\rangle \mathbf{K}_2.\end{aligned}$$

Hence,

$$M_S^*(K_{\wedge}^{c}\mathbf{k}\bar{\gamma}^{n}) = \sum_{k=0}^{\infty}\bar{\gamma}^{k}\mathbf{a}_k,$$

where

$$\mathbf{a}_k = \begin{cases}\sum_{j=0}^{\infty}\mathbf{s}_{j+n-k}^{*(k)}(\mathbf{c}^{*[j]}\mathbf{k})^{(n)}, & 0 \le k \le n,\\ \sum_{j=0}^{\infty}\mathbf{s}_{j}^{*(k)}(\mathbf{c}^{*[j+k-n]}\mathbf{k})^{(n)}, & n \le k.\end{cases}$$

For $0 \le k \le n-1$,

$$\begin{aligned}\bar{\gamma}^{k}\mathbf{a}_k &= \sum_{j=0}^{\infty}\mathbf{s}_{j+n-k}^{*}\bar{\gamma}^{k}(\mathbf{c}^{*[j]}\mathbf{k})^{(n)} \\ &= \sum_{j=0}^{\infty}\mathbf{s}_{j+n-k}^{*}(\mathbf{c}^{*[j]}\mathbf{k})^{(n-k)}\bar{\gamma}^{k}.\end{aligned}$$

For non-negative integer k,

$$\bar{\gamma}^{k+n}\mathbf{a}_{k+n} = \sum_{j=0}^{\infty} \mathbf{s}_j^{*(k+n)}(\mathbf{c}^{*[j+k]}\mathbf{k})^{(n)}$$

$$= \bar{\gamma}^k \sum_{j=0}^{\infty} \mathbf{s}_j^{*(k)}\mathbf{c}^{*[j+k]}\mathbf{k}\bar{\gamma}^n.$$

Then $(I - SM_S^*)(K_\wedge^c\mathbf{k}\bar{\gamma}^n) \in \mathcal{H}(S)$ and

$$(I - SM_S^*)(K_\wedge^c\mathbf{k}\bar{\gamma}^n)$$

$$= K_\wedge^c\mathbf{k}\bar{\gamma}^n - S\left(\sum_{k=0}^{n-1}\bar{\gamma}^k\mathbf{a}_k + \sum_{k=0}^{\infty}\bar{\gamma}^{k+n}\mathbf{a}_{k+n}\right)$$

$$= K_S^{\mathbf{c}}\mathbf{k}\bar{\gamma}^n - S\sum_{k=0}^{n-1}\left(\sum_{j=0}^{\infty}\mathbf{s}_{j+n-k}^*(\mathbf{c}^{*[j]}\mathbf{k})^{(n-k)}\right)\bar{\gamma}^k.$$

For each k with $0 \le k \le n-1$, the series $\sum_{j=0}^{\infty}\mathbf{s}_{j+n-k}^*(\mathbf{c}^{*[j]}\mathbf{k})^{(n-k)}$ is a coefficient of γ^{n-k-1} in the series $(S^* - S^{*\vee}(\mathbf{c}))(I - \bar{\gamma}\mathbf{c})^{-1}\mathbf{k}\bar{\gamma}$,

$$(S^* - S^{*\vee}(\mathbf{c}))(I - \bar{\gamma}\mathbf{c})^{-1}\mathbf{k}\bar{\gamma} = \sum_{k=0}^{\infty}\mathbf{s}_{k+1}^*\gamma^{k+1}\sum_{j=0}^{k}(\bar{\gamma}\mathbf{c}^*)^j\mathbf{k}\bar{\gamma}$$

$$= \sum_{k=0}^{\infty}\mathbf{s}_{k+1}^*\sum_{j=0}^{k}(\mathbf{c}^{*[j]}\mathbf{k})^{(k+j-1)}\gamma^{k-j}$$

$$= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{\infty}\mathbf{s}_{k+j+1}^*(\mathbf{c}^{*[j]}\mathbf{k})^{(k+1)}\right]\gamma^k.$$

Hence

$$(K_S^{\mathbf{c}}\mathbf{k}, (S^* - S^{*\vee}(\mathbf{c}))(I - \bar{\gamma}\mathbf{c})^{-1}\mathbf{k}\bar{\gamma}) \in \hat{\mathcal{D}}(S).$$

The reproducing kernel property in the space $\hat{\mathcal{D}}(S)$ holds since $K_S^{\mathbf{c}}$ is the reproducing kernel function of the space $\mathcal{H}(S)$. □

The extension space $\hat{\mathcal{D}}(S)$ associated with $\mathcal{H}(S)$ is the state space of a multiscale system.

Theorem 3.4. *The extension space* $\hat{\mathcal{D}}(S)$ *associated with* $\mathcal{H}(S)$ *is the state space of a multiscale system which is defined by*

$$\hat{V} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} : \begin{pmatrix} \hat{\mathcal{D}}(S) \\ \mathbb{K}_2 \end{pmatrix} \to \begin{pmatrix} \hat{\mathcal{D}}(S) \\ \mathbb{K}_2 \end{pmatrix}, \tag{3.6}$$

where

$$\hat{A}((F, H)) = ((F - F^{\wedge}(0))\gamma, H\gamma - S^* F^{\wedge}(0)),$$

$$\hat{B}\mathbf{k} = ((S - S^{\wedge}(0))\mathbf{k}\gamma, (I - S^* S^{\vee}(0))\mathbf{k}),$$

$$\hat{C}((F, H)) = F^{\wedge}(0), \text{ and}$$

$$\hat{D}\mathbf{k} = S^{\wedge}(0)\mathbf{k}.$$

The linear system satisfies the identity

$$\hat{V}\hat{V}^* \begin{pmatrix} (F, H) \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} (F\bar{\gamma}\gamma, (H - H^{\vee}(0))\bar{\gamma}\gamma + H^{\vee}(0)) \\ \mathbf{k} \end{pmatrix}.$$

Proof. Let $\mathbf{k} \in \mathbf{K}_2$. First claim that

$$((1 - SS^{\wedge}(0)^*)\mathbf{k}, (S^* - S^{\vee}(0)^*)\mathbf{k}\bar{\gamma}) \in \hat{\mathcal{D}}(S),$$

where $S^* = \sum_{k=0}^{\infty} \mathbf{s}_k^* \gamma^k$. In order to that represent $M_S^*(\mathbf{k}\bar{\gamma}^n)$ as a series.

$$\langle SF, \mathbf{k}\bar{\gamma}^n \rangle_{\mathcal{H}_2} = \sum_{k=0}^{n} \langle f_k, \mathbf{s}_{n-k}^{*(k)} \mathbf{k}^{(n)} \rangle_{\mathbf{K}}$$

$$= \left\langle F,\ \sum_{k=0}^{n} \overline{\gamma}^k \mathbf{s}_{n-k}^{*(k)} \mathbf{k}^{(n)} \right\rangle_{\mathcal{H}_2}$$

$$= \left\langle F,\ \sum_{k=0}^{n} \mathbf{s}_{n-k}^{*} \mathbf{k}^{(n-k)} \overline{\gamma}^k \right\rangle_{\mathcal{H}_2}.$$

Hence, we have

$$M_S^*(\mathbf{k}\overline{\gamma}^n) = \sum_{k=0}^{n} \mathbf{s}_{n-k}^* \mathbf{k}^{(n-k)} \overline{\gamma}^k = \sum_{k=0}^{n} \mathbf{s}_k^* \mathbf{k}^{(k)} \overline{\gamma}^{n-k}$$

and for each non-negative integer *n*,

$$(1 - M_S M_S^*)(\mathbf{k}\overline{\gamma}^n) = (1 - SS^\wedge(0)^*)\mathbf{k}\overline{\gamma}^n - S\left(\sum_{k=0}^{n-1} \mathbf{s}_{k+1}^* \mathbf{k}^{(k+1)} \overline{\gamma}^{(n-k-1)}\right)$$

is an element of $\mathcal{H}(S)$. Since $(I - SS^\wedge(0)^*)\mathbf{k} \in \mathcal{H}(S)$ and

$$(S^* - S^\vee(0)^*)\mathbf{k}\overline{\gamma} = \sum_{k=0}^{\infty} \mathbf{s}_{k+1}^* \mathbf{k}^{(k+1)} \gamma^n,$$

we have

$$((I - SS^\wedge(0)^*)\mathbf{k},\ (S^* - S^\vee(0)^*)\mathbf{k}\overline{\gamma}) \in \mathcal{D}(S).$$

We can rewrite $(I - M_S M_S^*)(\mathbf{s}_0\mathbf{k}\overline{\gamma}^n)$ as the following way:

$$(I - M_S M_S^*)(\mathbf{s}_0\mathbf{k}\overline{\gamma}^n)$$

$$= \mathbf{s}_0\mathbf{k}\overline{\gamma}^n - S\left(\sum_{k=0}^{n} \mathbf{s}_k^* (\mathbf{s}_0\mathbf{k})^{(k)} \overline{\gamma}^{(n-k)}\right)$$

$$= [S - S^\wedge(0)]\mathbf{k}\gamma\overline{\gamma}^{n+1} - S(I - \mathbf{s}_0^*\mathbf{s}_0)\mathbf{k}\overline{\gamma}^n + \left(\sum_{k=1}^{n} \mathbf{s}_k^* (\mathbf{s}_0\mathbf{k})^{(k)} \overline{\gamma}^{n-k}\right).$$

Hence

$$((S - S^\wedge(0))\mathbf{k}\gamma,\ (I - S^* S^\vee(0))\mathbf{k}) \in \hat{\mathcal{D}}(S).$$

Assume that $(F,\ H) \in \hat{\mathcal{D}}(S)$ and write $H = \sum_{k=0}^{\infty} \mathbf{h}_k \gamma^k$. Then $F\overline{\gamma} - \mathbf{h}_0 \in \mathcal{H}(S)$ and

$$F\overline{\gamma}^{n+1} - S(\mathbf{h}_0\overline{\gamma}^n + \cdots + \mathbf{h}_{n-1}\overline{\gamma} + \mathbf{h}_n)$$

$$= (F\overline{\gamma} - S\mathbf{h}_0)\overline{\gamma}^n - S(\mathbf{h}_1\overline{\gamma}^{n-1} + \cdots + \mathbf{h}_{n-1}\overline{\gamma} + \mathbf{h}_n).$$

Hence, we have

$$(F\overline{\gamma} - SH^\vee(0),\ (H - H^\vee(0))\overline{\gamma}) \in \hat{\mathcal{D}}(S).$$

Since

$$((I - SS^\wedge(0)^*)F^\wedge(0),\ (S^* - S^\vee(0)^*)F^\wedge(0)\overline{\gamma}) \in \hat{\mathcal{D}}(S),$$

$$F - [I - SS^\wedge(0)^*]F^\wedge(0) = [(F - F^\wedge(0))\gamma]\overline{\gamma} + SS^\wedge(0)^* F^\wedge(0) \in \mathcal{H}(S).$$

Since

$$H - [S^* - S^\vee(0)^*]F^\wedge(0)\overline{\gamma} = (H\gamma - S^* F^\wedge(0))\overline{\gamma} + S^\vee(0)^* F^\wedge(0)\overline{\gamma},$$

$$((F - F^\wedge(0))\gamma,\ H\gamma - S^* F^\wedge(0)) \in \hat{\mathcal{D}}(S).$$

Define the linear system by

$$\hat{A}((F,\ H)) = ((F - F^\wedge(0))\gamma,\ H\gamma - S^* F^\wedge(0)),$$

$$\hat{B}\mathbf{k} = ((S - S^\wedge(0))\mathbf{k}\gamma,\ (I - S^* S^\vee(0))\mathbf{k}),$$

$$\hat{C}((F,\ H)) = F^\wedge(0), \text{ and}$$

$$\hat{D}\mathbf{k} = S^\wedge(0)\mathbf{k}.$$

Then the adjoint of transformations are

$$\hat{A}^*((F, H)) = (F\overline{\gamma} - SH^{\vee}(0), (H - H^{\vee}(0))\overline{\gamma}),$$

$$\hat{B}^*((F, H)) = H^{\vee}(0),$$

$$\hat{C}^*\mathbf{k} = ((I - SS^{\wedge}(0)^*)\mathbf{k}, (S^* - S^{\vee}(0)^*)\mathbf{k}\overline{\gamma}), \text{ and}$$

$$\hat{D}^*\mathbf{k} = S^{*\vee}(0)k.$$

Hence, we have

$$\hat{V}\hat{V}^*\begin{pmatrix} (F, H) \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} (F\overline{\gamma}\gamma, (H - H^{\vee}(0))\overline{\gamma}\gamma + H^{\vee}(0)) \\ \mathbf{k} \end{pmatrix}. \qquad \square$$

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