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INDEPENDENT AND VERTEX COVERING NUMBER ON STRONG PRODUCT OF COMPLETE BIPARTITE GRAPHS

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Abstract

Let $\alpha(G)$ and $\beta(G)$ be the independent number and vertex covering number, respectively. The strong product $G_1 \boxtimes G_2$ of graphs of G_1 and G_2 has vertex set $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \boxtimes G_2) = \{(u_1v_1)(u_2v_2) \mid [u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)] \cup [u_1 = u_2$ and $v_1v_2 \in E(G_2)] \cup [u_1u_2 \in E(G_1)$ and $v_1 = v_2]\}$. In this paper, we determine generalization of independent number and vertex covering number on strong product of complete bipartite graphs and any simple graph.

1. Introduction

In this paper, graphs must be simple graphs which can be the trivial graph. Let G_1 and G_2 be graphs. The strong product of graphs G_1 and G_2,

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denoted by $G_1 \boxtimes G_2$, is the graph with $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \boxtimes G_2) = \{(u_1v_1)(u_2v_2) \mid [u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)] \cup [u_1 = u_2$ and $v_1v_2 \in E(G_2)] \cup [u_1u_2 \in E(G_1)$ and $v_1 = v_2]\}$. There are some properties about strong product of graphs. We recall these here.

Proposition 1. *Let* $H = G_1 \boxtimes G_2 = (V(H), E(H))$. *Then*

(i) $|V(H)| = |V(G_1)||V(G_2)|$,

(ii) $|E(H)| = 2|E(G_1)||E(G_2)| + |V(G_1)||E(G_2)| + |V(G_2)||E(G_1)|$,

(iii) *for every* $(u, v) \in V(H)$,

$$d_H((u, v)) = d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u) + d_{G_2}(v).$$

Theorem 2. *Let* G_1 *and* G_2 *be connected g graphs. Then the graph* $H = G_1 \boxtimes G_2$ *is connected if and only if* G_1 *or* G_2 *contains an odd cycle*.

Next, we give the definitions about some graph parameters. A subset *U* of the vertex set $V(G)$ of *G* is said to be an *independent set* of *G* if the induced subgraph $G[U]$ is a trivial graph. An independent set of *G* with maximum number of vertices is called a *maximum independent set* of *G*. The number of vertices of a maximum independent set of *G* is called the *independent number* of *G*, denoted by $\alpha(G)$.

A vertex of graph *G* is said to cover the edges incident with it, and a vertex cover of a graph *G* is a set of vertices covering all the edges of *G*. The minimum cardinality of a vertex cover of a graph *G* is called the *vertex covering number* of *G*, denoted by $\beta(G)$.

By definitions, it is clear that $\alpha(K_{m,n}) = \max\{m, n\}$ and $\beta(K_{m,n}) = \min\{m, n\}$.

Next, we get that general form for graph of strong product of $K_{m,n}$ and a simple graph.

Proposition 3. *Let G be a connected graph of order p, $V(G) = \{v_j / j = 1, 2, \ldots, p\}$ and $V(K_{m,n}) = \{u_i / i = 1, 2, \ldots, m+n\}$, the graph of*

$$K_{m,n} \boxtimes G \cong \bigcup_{i=1}^{m} H_i \cup \bigcup_{i=1}^{m+n} R_i \cup \bigcup_{j=1}^{p} S_j; \quad H_i = \bigcup_{j=m+1}^{m+n} H_{i,j},$$

where $V(H_{i,j}) = W_i \cup W_j$, $W_i = \{(u_i, v_1) \cdot (u_i, v_2), \ldots, (u_i, v_p)\}$, $W_j = \{(u_j, v_1), (u_j, v_2), \ldots, (u_j, v_p)\}$; $i < j$ and $E(H_{i,j}) = \{(u_i, v)(u_j, w)/vw \in E(G)\}$. $V(R_i) = W_i$ and $E(R_i) = \{(u_i, v)(u_i, w)/vw \in E(G)\}$. $V(S_j) = \{(u_1, v_j), (u_2, v_j), \ldots, (u_{m+n}, v_j)\}$ and $E(S_j) = \{(u, v_j)(w, v_j)/uw \in E(K_{m,n})\}$.

Moreover, if G has no odd cycle, then each $H_{i,j}$ has exactly two connected components isomorphic to G.

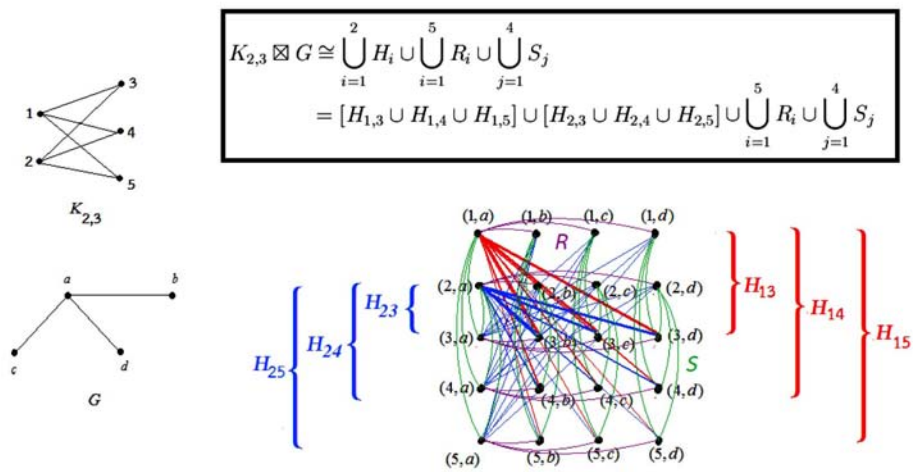

Figure 1. The graph of $K_{2,3} \boxtimes G$.

2. Vertex Covering Number of the Graph of $K_{m,n} \boxtimes G$

We now state proposition and prove lemma before stating our main results. We begin this section by giving Lemma 4 which shows character of vertex covering set.

Lemma 4. *Let*

$$K_{m,n} \boxtimes G \cong \bigcup_{i=1}^{m} H_i \cup \bigcup_{i=1}^{m+n} R_i \cup \bigcup_{j=1}^{p} S_j; \quad H_i = \bigcup_{j=m+1}^{m+n} H_{i,j}.$$

Then $\beta(H_{i,j}) = 2\beta(R_i) = 2\beta(G)$ *and* $\beta(S_j) = 1$.

Proof. Suppose G has no odd cycle, by Proposition 3, we have that $H_{i,j}$ each contains 2 components isomorphic to G. So $\gamma(H_{i,j}) = 2\gamma(G)$.

If G has odd cycle, for each $H_{i,j}$, let $E^* = \{e^* / e^*$ be one edge on each odd cycles of $G\}$, $H_{i,j}^* = H_{i,j} - \{(u_i, x)(u_{i+1}, y) / e^* = xy \in E^*\}$ and let VD be a minimum dominating set of G. We get $H_{i,j}^* \cong 2(G/E^*)$ then

$$\beta(H_{i,j}^*) = 2\beta(G/E^*) = \begin{cases} 2[\beta(G) + |E^*|], & \text{if } e^* = xy, \text{then } x \in VD \\ & y \notin VD \text{ and } y \text{ is not adjecent} \\ & \text{with vertex } z \in VD \text{ or not} \\ & d_G(y) = 1, \\ 2\beta(G), & \text{otherwise.} \end{cases}$$

When we add e^* comeback, in the case $\beta(G/E^*) = \beta(G) + |E^*|$ be not impossible because the end vertices of edge e^* are in dominating set of G/E^*, so $\beta(H_{i,j}) = \beta(H_{i,j}^*) - |E^*|$.

Hence $\beta(H_{i,j}) = 2\beta(G) = 2\beta(R_i)$ (since $R_i \cong G$). From $S_j \cong K_n$, we get $\beta(S_j) = \beta(K_n) = 1$. □

Next, we establish Theorem 5 for a minimum vertex covering number of $K_{m,n} \boxtimes G$.

Theorem 5. *Let G be connected graph order m. Then*

$$\beta(K_{m,n} \boxtimes G) \geq T = \begin{cases} n\beta(G) + mp; & m < n, \\ m\beta(G) + np; & m > n. \end{cases}$$

Proof. Let

$$V(K_{m,n}) = \{u_i / i = 1, 2, ..., m+n\}, \ \ V(G) = \{v_j / j = 1, 2, ..., p\},$$

$$W_i = \{(u_i, v_j) \in V(K_{m,n} \boxtimes G) / j = 1, 2, ..., p\}, \ \ i = 1, 2, ..., m+n$$

and since $\beta(K_{m,n}) = \max\{m, n\}$. Let the minimum vertex covering set of $K_{m,n}$, G be

$$VD_1 = \begin{cases} \{u_1, u_2, ..., u_m\}; & m < n, \\ \{u_{m+1}, u_{m+2}, ..., u_{m+n}\}; & m > n, \end{cases}$$

$VD_2 = \{v_1, v_2, ..., v_k\}_2$, respectively.

For H_1, by Lemma 4 we have $\beta(H_{1,j}) = 2\beta(G)$, $j = m+1, m+2, ..., m+n$. From every $H_{1,j}, H_{1,k}$; $j \neq k$, $k = m+1, m+2, ..., m+n$ have $\beta(G)$ common vertices in their vertex covering set which is in W_1. So the vertex covering set of H_1 be in $W_1 \cup \bigcup_{j=m+1}^{m+n} W_j$.

Similarly, for the vertex covering t set of $H_2, H_3, ..., H_m$ have $\beta(G)$ common vertices in their vertex covering set which is in $W_2, W_3, ..., W_m$, respectively. But the independent set of $H_2 - W_2, H_3 - W_3, ..., H_m - W_m$ are subset of the independent set of H_1, then vertex covering set of $\bigcup_{i=1}^{m} H_i$ is $\bigcup_{i=1}^{m+n} A_i \subset \bigcup_{i=1}^{m+n} W_i$, where $A_i = \{(u_i, v_k) / v_k \in VD_2\}$.

For each R_i, it is clear that vertex covering set of $\bigcup_{i=1}^{m+n} R_i$ is $\bigcup_{i=1}^{m+n} A_i$.

And for $S_j \cong K_{m,m}$, we have vertex covering set of $\bigcup_{v_a \in VD_2} S_a$ is in $\bigcup_{i=1}^{m+n} A_i$. But for S_b, $v_b \notin VD_2$, we have vertex covering set of $\bigcup_{v_b \notin VD_2} S_b$ is $\bigcup_{v_b \notin VD_2} B_b \subset \bigcup_{v_b \notin VD_2} S_b$, where $B_b = \{(u_i, v_b / u_i \in VD_1)\}$.

Therefore,

$$\beta(K_{m,n} \boxtimes G) \geq \begin{cases} n\beta(G) + mp; & m < n, \\ m\beta(G) + np; & m > n. \end{cases}$$

On the other hand, we get another vertex covering set of $K_{m,n} \boxtimes G$ is $K \cup M$,

$$K = \begin{cases} W_1, W_2, \ldots, W_m, & m < n \\ W_{m+1}, W_{m+2}, \ldots, W_{m+n}, & m > n \end{cases} \text{ and } M = \{(u_k, v) \in W_k / v \in VD_2\},$$

then $|K| = p \min\{m, n\}$ and $|M| = [(m+n) - \min\{m, n\}]\beta(G)$.

Hence

$$\beta(K_{m,n} \boxtimes G) \geq T = \begin{cases} n\beta(G) + mp; & m < n, \\ m\beta(G) + np; & m > n. \end{cases}$$

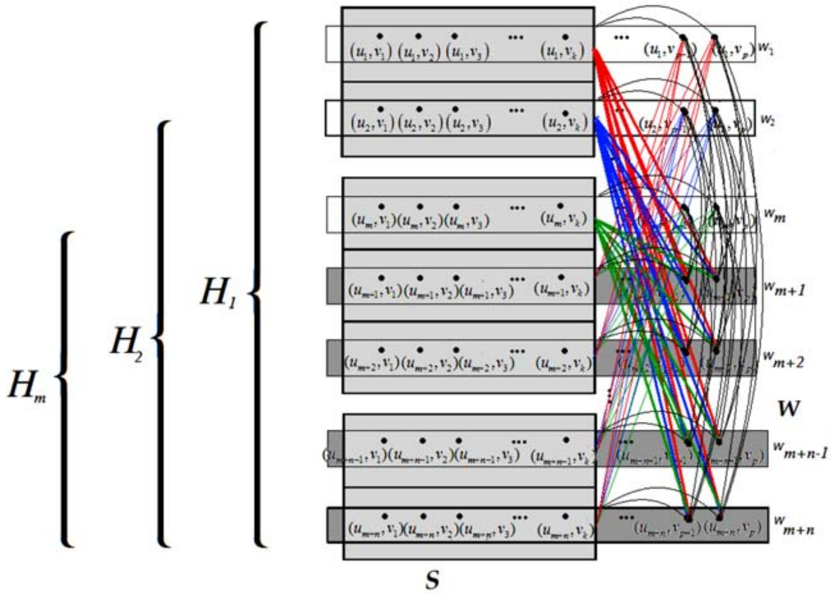

Figure 2. The region of W, S where $m > n$.

Suppose that $\beta(K_{m,n} \boxtimes G) < T$, then there exists at least one edge in W (or S) is not incident with vertices in W (or S), $W = \{(u, v_k)/v_k \in VD_2\}$ and $S = \{(u_h, v)/u_h \in VD_1\}$. It is not true.

Hence

$$\beta(K_{m,n} \boxtimes G) = \begin{cases} n\beta(G) + mp; & m < n, \\ m\beta(G) + np; & m > n. \end{cases} \qquad \square$$

3. Independent Number of the Graph of $K_{m,n} \boxtimes G$

We begin this section by giving Lemma 6 that shows a relation of independent number and vertex covering number.

Lemma 6 [2]**.** *Let G be a simple graph with order n. Then* $\alpha(G) + \beta(G) = n$.

Next, we establish Theorem 7 for independent number of $K_{m,n} \boxtimes G$.

Theorem 7. *Let G be connected graph order p. Then*

$$\alpha(K_{m,n} \boxtimes G) = \begin{cases} n\beta(G) + mp; & m < n, \\ m\beta(G) + np; & m > n. \end{cases}$$

Proof. In case $m < n,$ by Theorem 5 and Lemma 6, we can also show that

$$\alpha(K_{m,n} \boxtimes G) + \beta(m, n \boxtimes G) = nm$$

$$pm + n\alpha(G) + \beta(K_{m,n} \boxtimes G) = nm$$

$$\beta(K_{m,n} \boxtimes G) = nm - pm - n\alpha(G)$$

$$= (n - p)m - n\alpha(G).$$

Similarly, in case $m > n,$ we get

$$\beta(K_{m,n} \boxtimes G) = (m - p)n - m\alpha(G). \qquad \square$$

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