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MATCHING NUMBER AND EDGE COVERING NUMBER ON TENSOR PRODUCT OF WHEELS

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Abstract

Let $\alpha'(G)$ and $\beta'(G)$ be the matching number and edge covering number, respectively. The tensor product $G_1 \otimes G_2$ of graphs G_1 and G_2 has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2) | u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$. In this paper, we determined generalization of matching and edge covering number on tensor product of wheel and any simple graph.

1. Introduction

In this paper, graphs must be simple graphs which can be trivial graph. Let G_1 and G_2 be graphs. The tensor product of graphs G_1 and G_2, denoted by $G_1 \otimes G_2$, is the graph that $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2) | u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$.

Next, we give the definitions about some graph parameters. A subset of the edge set E of G is said to be *matching* or an *independent edge set* of G, if no two distinct edges in M have a common vertex. A matching M is

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maximum matching in G if there is no matching M' of G with $|M'| > |M|$. The cardinality of maximum matching of G is called the *matching number* of G, denoted by $\alpha'(G)$.

An edge of graph G is said to cover the two vertices incident with it, and an edge cover of a graph G is a set of edges covering all the vertices of G. The minimum cardinality of an edge cover of a graph G is called the *edge covering number* of G, denoted by $\beta'(G)$.

By definitions of matching number, edge covering number, clearly that $\alpha'(W_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and $\beta'(W_n) = \left\lceil \frac{n}{2} \right\rceil$.

In [2], there are some properties about tensor product of graph. We recall here.

Proposition 1. *Let* $H = G_1 \otimes G_2 = (V(H), E(H))$. *Then*

(i) $n(V(H)) = n(V(G_1))n(V(G_2))$,

(ii) $n(E(H)) = 2n(E(G_1))n(E(G_2))$,

(iii) *for every* $(u, v) \in V(H)$, $d_H((u, v)) = d_{G_1}(u)d_{G_2}(v)$.

Note that for any graph G, we have $G_1 \otimes G_2 \cong G_2 \otimes G_1$.

Theorem 2. *Let* G_1 *and* G_2 *be connected graphs. Then the graph* $H = G_1 \otimes G_2$ *is connected if and only if* G_1 *or* G_2 *contains an odd cycle.*

Theorem 3. *Let* G_1 *and* G_2 *be connected graphs with no odd cycle. Then* $G_1 \otimes G_2$ *has exactly two connected components.*

Next, we get that general form of graph of tensor Product of W_n and a simple graph.

Proposition 4. *Let G be a connected graph of order m. Then the graph of* $W_n \otimes G$ *is*

$$\bigcup_{j=2}^{n} H_{1j} \cup \bigcup_{i=2}^{n-1} H_{i(i+1)} \cup H_{2n},$$

where $V(H_{ij}) = S_i \cup S_j$, $S_i = \{(i, 1), (i, 2), ..., (i, m)\}$ *and* $E(H_{ij}) = \{(i, u)(j, v)/uv \in E(G)\}$. *Moreover, if G has no odd cycle then each* H_{ij} *has exactly two connected components isomorphic to G.*

Example.

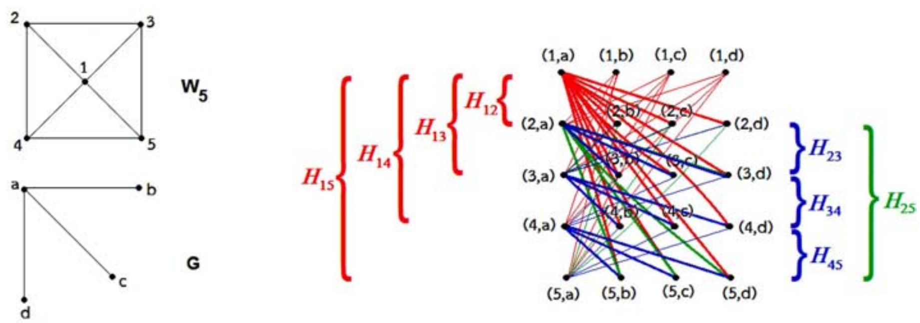

Figure 1. The graph of $W_5 \otimes G$.

2. Matching Number of the Graph of $W_n \otimes G$

We begin this section by giving the definition and theorem for alternating path and augmenting path, Lemma 7 that shows character of matching for each H_{ij}.

Definition 5. Given a matching *M*, an *M*-alternating path is a path that alternates between edges in *M* and edges not in *M*. An *M*-alternating odd path whose endpoints are unsaturated by *M* is an *M*-augmenting path.

Theorem 6. *A matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path.*

Next, we give Lemma 7 which shows the character of matching for each H_{ij}.

Lemma 7. *Let* $W_n \otimes G = \bigcup_{j=2}^{n} H_{1j} \cup \bigcup_{i=2}^{n-1} H_{i(i+1)} \cup H_{2n}$. *For each* H_{ij}, *then* $\alpha'(H_{ij}) = 2\alpha'(G)$.

Proof. Suppose G has no odd cycle, by Proposition 4 we get $H_{ij} = 2G$. So $\alpha'(H_{ij}) = 2\alpha'(G)$. If G has odd cycle, for each H_{ij}, vertex $(u_i, v) \in S_i$ and $(u_j, v) \in S_j$ have $d_{H_i}((u_i, v)) = d_{H_i}((u_j, v)) = d_G(v)$. Let $E^* = \{e_i / e_i$ is any one edge in each odd cycle C_i in G, $i = 1, 2, ..., l; |E^*| \leq l\}$ and let M be the maximum matching of G.

Now consider the tensor product $\bigcup_{j=2}^{n} H_{1j}^* \cup \bigcup_{i=2}^{n-1} H_{i(i+1)}^* \cup H_{2n}^* = W_n \otimes (G - E^*)$. We get $H_{ij}^* = 2(G - E^*)$, then

$$\alpha'(H_{ij}^*) = 2\alpha'(G - E^*) = \begin{cases} 2[\alpha'(G) - |\overline{E}|], & \text{if } \overline{E} = \{\overline{e}/\overline{e} \in M\}, \\ 2\alpha'(G), & \text{otherwise.} \end{cases}$$

Adding edges in E^* with $W_n \otimes (G - E^*)$, we get $\alpha'(H_i) = \alpha'(\overline{H_i}) + |\overline{E}|$. Hence $\alpha'(H_{ij}) = 2\alpha'(G)$. □

Next, we establish Theorem 8 for a matching number of $W_n \otimes G$.

Theorem 8. *Let G be connected graph of order m. Then*

$$\alpha'(W_n \otimes G) = \begin{cases} n\alpha'(G), & \text{if } n \text{ is even}, \\ n\alpha'(G) + |\overline{M}_2|, & \text{if } n \text{ is odd}, \end{cases}$$

where a matching $\overline{M}_2 = \{uv / u$ *is not matched in maximum matching* M_2 *in* G *and* $v \in N_G(u)\}$.

Proof. Let $V(W_n) = \{u_i, i = 1, 2, ..., n\}$, $V(G) = \{v_j, j = 1, 2, ..., m\}$, $S_i = \{(u_i, v_j) \in V(W_n \otimes G) / j = 1, 2, ..., m\}$, $i = 1, 2, ..., n$.

From $\alpha'(W_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and let $\alpha'(G) = k$. Assume that the maximum matching of W_n and G be

$$M_1 = \begin{cases} \{u_1 u_n, u_2 u_3, ..., u_{n-2} u_{n-1}\}, & \text{if } n \text{ is even,} \\ \{u_2 u_n 3, u_4 u_5, ..., u_{n-1} u_n\}, & \text{if } n \text{ is odd} \end{cases}$$

and $M_2 = \{v_j v_{j+1} / j = 1, 3, ..., 2k - 1\}$, respectively.

By Lemma 7, we have $\alpha'(H_{ij}) = 2\alpha'(G)$. Since $W_n \otimes G$ is

$$\bigcup_{j=2}^{n} H_{1j} \cup \bigcup_{i=2}^{n-1} H_{i(i+1)} \cup H_{2n}.$$

We get the matching of $W_n \otimes G$ to be

$$M = \begin{cases} \{M_{1n} \cup M_{23} \cup \ldots \cup M_{(n-2)(n-1)}\}, & \text{if } n \text{ is even,} \\ \{M_{23} \cup M_{45} \cup \ldots \cup M_{(n-1)(n)} \cup \overline{M}_2\}, & \text{if } n \text{ is odd,} \end{cases}$$

where $M_{ij} \subset E(H_{ij})$, $M_{ij} = \{(u_i, v_a)(u_j, v_b) / v_a v_b \in M_2\}$ and $\overline{M}_2 = \{uv / u$ is not matched in maximum matching in G and $v \in N_G(u)\}$.

Hence

$$\alpha'(W_n \otimes G) \geq \begin{cases} n\alpha'(G), & \text{if } n \text{ is even,} \\ n\alpha'(G) + |\overline{M}_2|, & \text{if } n \text{ is odd.} \end{cases}$$

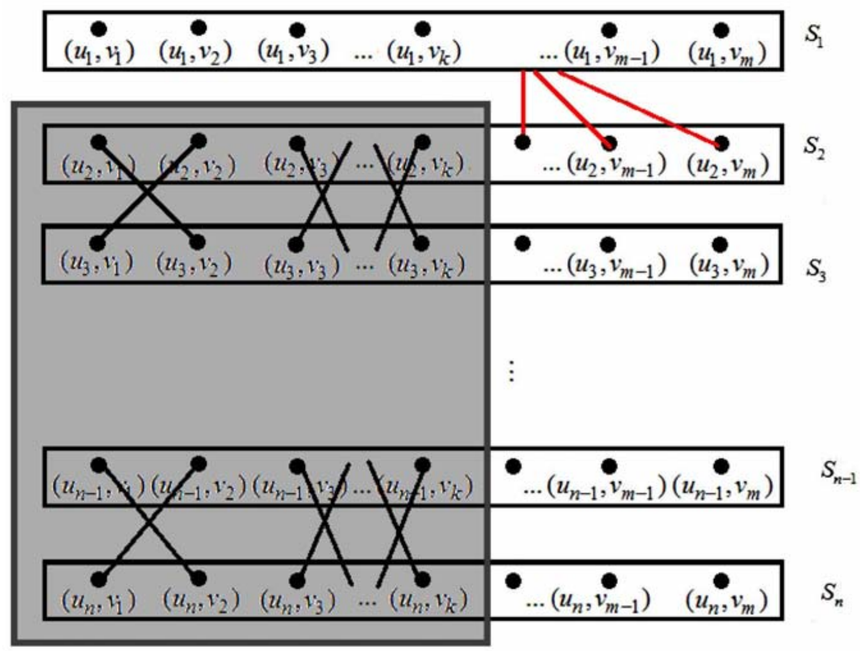

Figure 2. The matching M where n is odd.

Suppose that $\alpha'(W_n \otimes G) > n\alpha'(G)$, where n is even, then there exists a matching M^* is an augmenting path. That is not true because each edges in $W_n \otimes G$ either is in M, or adjacent to an edge of M. In the case n is odd, we have the same.

Hence

$$\alpha'(W_n \otimes G) = \begin{cases} n\alpha'(G), & \text{if } n \text{ is even,} \\ n\alpha'(G) + |\overline{M}_2|, & \text{if } n \text{ is odd.} \end{cases} \qquad \square$$

3. Edge Covering Number of the Graph of $W_n \otimes G$

We begin this section by giving Lemma 9 that shows a relation of matching number and edge covering number.

Lemma 9 [1]**.** *Let G be a simple graph with order n. Then* $\alpha'(G) + \beta'(G) = n$.

Next, we establish Theorem 10 for a minimum edge covering number of $W_n \otimes G$.

Theorem 10. *Let G be connected graph of order m. Then*

$$\beta'(W_n \otimes G) = \begin{cases} n\beta'(G), & \text{if } n \text{ is even}, \\ n\beta'(G) - |\overline{M}_2|, & \text{if } n \text{ is odd}, \end{cases}$$

where a matching $\overline{M}_2 = \{uv / u$ *is not matched in maximum matching* M_2 *in* G *and* $v \in N_G(u)\}$.

Proof. Let n be even, by Theorem 8 and Lemma 9, we can also show that

$$\alpha(W_n \otimes G) + \beta(W_n \otimes G) = mn$$

$$n\alpha(G) + \beta(W_n \otimes G) = mn$$

$$\beta(W_n \otimes G) = mn - n\alpha(G)$$

$$= n(m - \alpha(G))$$

$$= n\beta(G).$$

Let n be odd. Then we have

$$\alpha(W_n \otimes G) + \beta(W_n \otimes G) = mn$$

$$n\alpha(G) + |\overline{M}_2| + \beta(W_n \otimes G) = mn$$

$$\beta(W_n \otimes G) = mn - n\alpha(G) - |\overline{M}_2|$$

$$= n(m - \alpha(G)) - |\overline{M}_2|$$

$$= n\beta(G) - |\overline{M}_2|.$$

□

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