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STATISTICAL PROPERTIES OF AN ESTIMATOR FOR THE MEAN FUNCTION OF A COMPOUND CYCLIC POISSON PROCESS

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Abstract

The problem of estimating the mean function of a compound cyclic Poisson process is considered. An estimator of this mean function is constructed and investigated. We do not assume any parametric form for the intensity function except that it is periodic. Moreover, we consider the case when only a single realization of the Poisson process is observed in a bounded interval. Asymptotic bias and variance of the proposed estimator are computed, when the size of the interval indefinitely expands.

1. Introduction

We consider a non-homogeneous Poisson process N on $[0, \infty)$ with (unknown) locally integrable intensity function λ. The intensity function λ is

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assumed to be periodic with (known) period $\tau > 0$. We do not assume any (parametric) form of λ except that it is periodic, that is, the equality

$$\lambda(s + k\tau) = \lambda(s) \tag{1.1}$$

holds for all $s \in [0, \infty)$ and all $k \in \mathbf{Z}$, with $\mathbf{Z}$ denotes the set of integers. This condition of intensity function is also considered in [2] and [3].

Let $\{Y(t), t \geq 0\}$ be a process with

$$Y(t) = \sum_{i=1}^{N(t)} X_i, \tag{1.2}$$

where $\{X_i, i \geq 1\}$ is a sequence of independent and identically distributed random variables with mean μ and variance σ^2, which is also independent of the process $\{N(t), t \geq 0\}$. The process $\{Y(t), t \geq 0\}$ is called a *compound cyclic Poisson process*. The model presented in (1.2) is a generalization of the (well known) compound Poisson process, which assume that $\{N(t), t \geq 0\}$ is a homogeneous Poisson process. We refer to [1], [4], [5] and [6] for some applications of compound Poisson process.

Suppose that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of the cyclic Poisson process $\{N(t), t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Furthermore, suppose that for each data point in the observed realization $N(\omega) \cap [0, n]$, say i-th data point, $i = 1, 2, \ldots, N([0, n])$, its corresponding random variable X_i is also observed.

The mean function (expected value) of $Y(t)$, denoted by $\psi(t)$, is given by

$$\psi(t) = \mathbf{E}[N(t)]\mathbf{E}[X_1] = \Lambda(t)\mu$$

with $\Lambda(t) = \int_0^t \lambda(s)ds$. Let $t_r = t - \left\lfloor \frac{t}{\tau} \right\rfloor \tau$, where for any real number x, $\lfloor x \rfloor$

denotes the largest integer less than or equal to x, and let also $k_{t,\tau} = \left\lfloor \frac{t}{\tau} \right\rfloor$. Then, for any given real number $t \geq 0$, we can write $t = k_{t,\tau}\tau + t_r$, with $0 \leq t_r < \tau$. Let $\theta = \frac{1}{\tau}\int_0^\tau \lambda(s)ds$, that is the global intensity of the cyclic Poisson process $\{N(t), t \geq 0\}$. We assume that

$$\theta > 0. \tag{1.3}$$

Then, for any given $t \geq 0$, we have

$$\Lambda(t) = k_{t,\tau}\tau\theta + \Lambda(t_r),$$

which implies

$$\psi(t) = (k_{t,\tau}\tau\theta + \Lambda(t_r))\mu.$$

In [7], an estimator for the mean function $\psi(t)$ of the process $\{Y(t), t \geq 0\}$ using the observed realization was constructed and this estimator has been proved to be weakly and strongly consistent. Our goal in this paper is to compute asymptotic bias and variance of a (slightly) modified estimator for the mean function $\psi(t)$ of the process $\{Y(t), t \geq 0\}$.

The rest of this paper is organized as follows. The estimator and main results are presented in Section 2, in Section 3 we present two technical lemmas, which are needed in the proofs of our theorems, and proofs of the main results are given in Section 4.

2. The Estimator and Results

Let $k_{n,\tau} = \left\lfloor \frac{n}{\tau} \right\rfloor$. An estimator of the mean function $\psi(t)$ using the available data set at hand is given by

$$\hat{\psi}_n(t) = (k_{t,\tau}\tau\hat{\theta}_n + \hat{\Lambda}_n(t_r))\hat{\mu}_n, \tag{2.1}$$

where

$$\hat{\theta}_n = \frac{1}{k_{n,\tau}\tau} \sum_{k=0}^{k_{n,\tau}-1} N([k\tau, k\tau + \tau]),$$

$$\hat{\Lambda}_n(t_r) = \frac{1}{k_{n,\tau}} \sum_{k=0}^{k_{n,\tau}-1} N([k\tau, k\tau + t_r])$$

and

$$\hat{\mu}_n = \frac{1}{N([0, n])} \sum_{i=1}^{N([0,n])} X_i,$$

with the understanding that $\hat{\mu}_n = 0$ when $N([0, n]) = 0$. Thus, $\hat{\psi}_n(t) = 0$ when $N([0, n]) = 0$. Note that, since in this paper we want to have more precise results compared to the ones in [7], our estimators $\hat{\theta}_n$ and $\hat{\Lambda}_n(t_r)$ of respectively θ and $\Lambda(t_r)$ in this paper have to be slightly modified from the ones in [7].

Our main results are presented in the following two theorems. The first theorem is about asymptotic approximation to the bias of $\hat{\psi}_n(t)$ and the second theorem is about asymptotic approximation to the variance of $\hat{\psi}_n(t)$.

Theorem 1 (Asymptotic approximation to the bias)**.** *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable. If, in addition,* $Y(t)$ *satisfies condition* (1.2), *then*

$$\mathbf{E}\hat{\psi}_n(t) = \psi(t) - \frac{\psi(t)}{e^{\theta n}} + o(e^{-n}) \tag{2.2}$$

as $n \to \infty$.

Theorem 2 (Asymptotic approximation to the variance)**.** *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable. If, in addition,*

$Y(t)$ *satisfies condition* (1.2), *then*

$$Var(\hat{\psi}_n(t)) = \frac{\mu^2\tau}{n}(k_{t,\tau}^2\theta\tau + \Lambda(t_r)(1 + 2k_{t,\tau}))$$

$$+ \frac{\sigma^2}{\theta n}(k_{t,\tau}\theta\tau + \Lambda(t_r))^2 + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{2.3}$$

as $n \to \infty$.

We note that, since the bias of $\hat{\psi}_n(t)$ is of smaller order than $\mathcal{O}(n^{-2})$, then the asymptotic approximation to the MSE (mean-squared-error) of $\hat{\psi}_n(t)$ is also given by the r.h.s. of (2.3). Hence, the rate of decrease of $MSE(\hat{\psi}_n(t))$ is of order $\mathcal{O}(n^{-1})$ as $n \to \infty$.

3. Some Technical Lemmas

In this section, we present two lemmas which are needed in the proofs of our theorems.

Lemma 1. *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable*, *then*

$$\mathbf{E}(\hat{\theta}_n) = \theta \ \textit{and} \ Var(\hat{\theta}_n) = \frac{\theta}{k_{n,\tau}\tau}.$$

Hence, $\hat{\theta}_n$ *is unbiased estimator of* θ *and* $Var(\hat{\theta}_n)$ *is of order* $\mathcal{O}(n^{-1})$ *as* $n \to \infty$.

Proof. The expected value of $\hat{\theta}_n$ can be computed as follows:

$$\mathbf{E}(\hat{\theta}_n) = \frac{1}{k_{n,\tau}\tau}\sum_{k=0}^{k_{n,\tau}-1}\int_{k\tau}^{(k+1)\tau}\lambda(s)ds$$

$$= \frac{1}{k_{n,\tau}}\sum_{k=0}^{k_{n,\tau}-1}\frac{1}{\tau}\int_0^{\tau}\lambda(s)ds = \theta.$$

The variance of $\hat{\theta}_n$ can be computed as follows:

$$Var(\hat{\theta}_n) = \frac{1}{k_{n,\tau}^2 \tau^2} \sum_{k=0}^{k_{n,\tau}-1} Var(N([k\tau, k\tau + \tau]))$$

$$= \frac{1}{k_{n,\tau}^2 \tau^2} \sum_{k=0}^{k_{n,\tau}-1} \mathbf{E}(N([k\tau, k\tau + \tau]))$$

$$= \frac{1}{k_{n,\tau}^2 \tau^2} \sum_{k=0}^{k_{n,\tau}-1} \int_{k\tau}^{(k+1)\tau} \lambda(s) ds$$

$$= \frac{\theta}{k_{n,\tau} \tau}.$$

This completes the proof of Lemma 1.

Lemma 2. *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable, then*

$$\mathbf{E}\hat{\Lambda}_n(t_r) = \Lambda(t_r) \text{ and } Var(\hat{\Lambda}_n(t_r)) = \frac{\Lambda(t_r)}{k_{n,\tau}}.$$

Hence, $\hat{\Lambda}_n(t_r)$ *is unbiased estimator of* $\Lambda(t_r)$ *and* $Var(\hat{\Lambda}_n(t_r))$ *is of order* $\mathcal{O}(n^{-1})$ *as* $n \to \infty$.

Proof. The expected value of $\hat{\Lambda}_n(t_r)$ can be computed as follows:

$$\mathbf{E}(\hat{\Lambda}_n(t_r)) = \frac{1}{k_{n\,\tau}} \sum_{k=0}^{k_{n,\tau}-1} \int_{k\tau}^{k\tau + t_r} \lambda(s) ds$$

$$= \frac{1}{k_{n,\tau}} \sum_{k=0}^{k_{n,\tau}-1} \int_0^{t_r} \lambda(s) ds$$

$$= \Lambda(t_r).$$

The variance of $\hat{\Lambda}_n(t_r)$ can be computed as follows:

$$Var(\hat{\Lambda}_n(t_r)) = \frac{1}{k_{n,\tau}^2} \sum_{k=0}^{k_{n,\tau}-1} Var(N([k\tau, k\tau + t_r]))$$

$$= \frac{1}{k_{n,\tau}^2} \sum_{k=0}^{k_{n,\tau}-1} \mathbf{E}(N([k\tau, k\tau + t_r]))$$

$$= \frac{1}{k_{n,\tau}^2} \sum_{k=0}^{k_{n,\tau}-1} \int_{k\tau}^{k\tau+t_r} \lambda(s)ds$$

$$= \frac{\Lambda(t_r)}{k_{n,\tau}}.$$

This completes the proof of Lemma 2.

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. The expected value of $\hat{\psi}_n(t)$ can be computed as follows:

$$\mathbf{E}(\hat{\psi}_n(t)) = \mathbf{E}(\mathbf{E}(\hat{\psi}_n(t) | N([0, n])))$$

$$= \sum_{m=0}^{\infty} \mathbf{E}(\hat{\psi}_n(t) | N([0, n]) = m)\mathbf{P}(N([0, n]) = m)$$

$$= \sum_{m=1}^{\infty} (k_{t,\tau}\tau\mathbf{E}\hat{\theta}_n + \mathbf{E}\hat{\Lambda}_n(t_r))\mathbf{E}\left(\frac{1}{m}\sum_{i=1}^{m} X_i\right)\mathbf{P}(N([0, n]) = m)$$

$$= \sum_{m=1}^{\infty} (k_{t,\tau}\tau\theta + \Lambda(t_r))\mu\mathbf{P}(N([0, n]) = m)$$

$$= (k_{t,\tau}\tau\theta + \Lambda(t_r))\mu\sum_{m=1}^{\infty} \mathbf{P}(N([0, n]) = m)$$

$$= \psi(t)(1 - \mathbf{P}(N([0, n]) = 0))$$

$$= \psi(t)(1 - e^{-\Lambda(n)}). \tag{4.1}$$

A simple calculation shows that

$$\Lambda(n) = \int_0^n \lambda(s)ds = \theta n + \mathcal{O}(1) \tag{4.2}$$

as $n \to \infty$. Substituting the r.h.s. of (4.2) into the r.h.s. of (4.1), we then obtain the r.h.s. of (2.2). This completes the proof of Theorem 1.

Proof of Theorem 2. To prove Theorem 2, first we compute $\mathbf{E}(\hat{\psi}_n^2(t))$ as follows:

$$\mathbf{E}(\hat{\psi}_n^2(t)) = \mathbf{E}(\mathbf{E}(\hat{\psi}_n^2(t) \mid N([0, n])))$$

$$= \sum_{m=0}^{\infty} \mathbf{E}(\hat{\psi}_n^2(t) \mid N([0, n]) = m)\mathbf{P}(N([0, n]) = m)$$

$$= \sum_{m=1}^{\infty} \mathbf{E}(k_{t,\tau}\tau\hat{\theta}_n + \hat{\Lambda}_n(t_r))^2 \mathbf{E}\left(\frac{1}{m}\sum_{i=1}^{m} X_i\right)^2 \mathbf{P}(N([0, n]) = m). \tag{4.3}$$

Since $\{X_i, i \geq 1\}$ is a sequence of independent and identically distributed random variables with mean μ and variance σ^2, a simple calculation shows that

$$\mathbf{E}\left(\frac{1}{m}\sum_{i=1}^{m} X_i\right)^2 = \mu^2 + \frac{\sigma^2}{m}. \tag{4.4}$$

The other expectation on the r.h.s. of (4.3) can be computed as follows:

$$\mathbf{E}(k_{t,\tau}\tau\hat{\theta}_n + \hat{\Lambda}_n(t_r))^2 = k_{t,\tau}^2\tau^2\mathbf{E}(\hat{\theta}_n)^2 + \mathbf{E}(\hat{\Lambda}_n(t_r))^2 + 2k_{t,\tau}\tau\mathbf{E}(\hat{\theta}_n\hat{\Lambda}_n(t_r)). \tag{4.5}$$

Now note that, by Lemma 1 we have

$$\mathbf{E}(\hat{\theta}_n)^2 = \theta^2 + \frac{\theta}{k_{n,\tau}\tau}, \tag{4.6}$$

and by Lemma 2 we see that

$$\mathbf{E}(\hat{\Lambda}_n(t_r))^2 = \Lambda^2(t_r) + \frac{\Lambda(t_r)}{k_{n,\tau}}. \tag{4.7}$$

To compute $\mathbf{E}(\hat{\theta}_n\hat{\Lambda}_n(t_r))$ we argue as follows. Let

$$\hat{\Lambda}_n^c(t_r) = \frac{1}{k_{n,\tau}} \sum_{k=0}^{k_{n,\tau}-1} N([k\tau + t_r, k\tau + \tau]).$$

Then $\hat{\theta}_n$ can be written as

$$\hat{\theta}_n = \frac{1}{\tau}(\hat{\Lambda}_n(t_r) + \hat{\Lambda}_n^c(t_r)).$$

Note also that $\hat{\Lambda}_n(t_r)$ and $\hat{\Lambda}_n^c(t_r)$ are independent. Then, a simple calculation shows that

$$\mathbf{E}(\hat{\theta}_n\hat{\Lambda}_n(t_r)) = \Lambda(t_r)\theta + \frac{\Lambda(t_r)}{k_{n,\tau}\tau}. \tag{4.8}$$

Substituting (4.6), (4.7) and (4.8) into the r.h.s. of (4.5), we then have

$$\mathbf{E}(k_{t,\tau}\tau\hat{\theta}_n + \hat{\Lambda}_n(t_r))^2$$

$$= (k_{t,\tau}\tau\theta + \Lambda(t_r))^2 + \frac{1}{k_{n,\tau}}(k_{t,\tau}^2\tau\theta + \Lambda(t_r)(1 + 2k_{t,\tau})). \tag{4.9}$$

Substituting (4.4) and (4.9) into the r.h.s. of (4.3), we then obtain

$$\mathbf{E}(\hat{\psi}_n^2(t)) = \left((k_{t,\tau}\tau\theta + \Lambda(t_r))^2 + \frac{1}{k_{n,\tau}}(k_{t,\tau}^2\tau\theta + \Lambda(t_r)(1 + 2k_{t,\tau}))\right)\mu^2$$

$$\cdot \sum_{m=1}^{\infty} \mathbf{P}(N([0, n]) = m)$$

$$+\left((k_{t,\tau}\tau\theta + \Lambda(t_r))^2 + \frac{1}{k_{n,\tau}}(k_{t,\tau}^2\tau\theta + \Lambda(t_r)(1 + 2k_{t,\tau}))\right)\sigma^2$$

$$\cdot \sum_{m=1}^{\infty}\frac{1}{m}\mathbf{P}(N([0, n]) = m). \tag{4.10}$$

The first term on the r.h.s. of (4.10) is equal to

$$\psi^2(t) + \frac{\mu^2}{k_{n,\tau}}(k_{t,\tau}^2\tau\theta + \Lambda(t_r)(1 + 2k_{t,\tau})) + \mathcal{O}(e^{-n}) \tag{4.11}$$

as $n \to \infty$, while its second term can be simplified as

$$\left((k_{t,\tau}\tau\theta + \Lambda(t_r))^2 + \frac{1}{k_{n,\tau}}(k_{t,\tau}^2\tau\theta + \Lambda(t_r)(1 + 2k_{t,\tau}))\right)\sigma^2\left(\frac{1}{\theta n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)$$

$$= \frac{\sigma^2}{\theta n}(k_{t,\tau}\tau\theta + \Lambda(t_r))^2 + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{4.12}$$

as $n \to \infty$. A simple argument shows that

$$\frac{1}{k_{n,\tau}} = \frac{\tau}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{4.13}$$

as $n \to \infty$. Substituting (4.11), (4.12) and (4.13) into the r.h.s. of (4.10), then we have

$$\mathbf{E}(\hat{\psi}_n^2(t)) = \psi^2(t) + \frac{\mu^2\tau}{n}(k_{t,\tau}^2\tau\theta + \Lambda(t_r)(1 + 2k_{t,\tau}))$$

$$+ \frac{\sigma^2}{\theta n}(k_{t,\tau}\tau\theta + \Lambda(t_r))^2 + \mathcal{O}\left(\frac{1}{n^2}\right) \tag{4.14}$$

as $n \to \infty$. By (4.14) and (2.2) we obtain (2.3). This completes the proof of Theorem 2.

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