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LINEAR DISCREPANCY OF THE COMPLETE *k*-ARY TREE-POSET

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Abstract

In this paper, we determine, constructively, the linear discrepancy of $T_{k,h}$, the complete k-ary tree-poset of height h. By a construction of a natural labeling on the tree-poset, we give an upper bound of its linear discrepancy, and then we prove that this upper bound is tight. Finally, we establish $ld(T_{k,h}) = \frac{1}{k-1}(k^h - 1) - h$.

1. Introduction

Throughout this paper, we use the notations $[n]$ and $[m, n]$ to denote

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$\{1, ..., n\}$ and $\{m, m+1, ..., n\}$, respectively, for positive integers *m*, *n* with $m \leq n$. A *partially ordered set* (simply called a *poset*) *P* is denoted by $(X, \leq_P)$, where *X* is a finite ground set, and $\leq_P$ is the order relation of *P*. For a given poset $P = (X, \leq_P)$, the notation $x \in P$ simply denotes $x \in X$. For *x* and $y \in P$ if $x \leq_P y$, i.e., $(x, y) \in \leq_P$, then *x* and *y* are said to be *comparable*. If $x, y \in P$, are not comparable, then it is denoted by $x \parallel_P y$. For $x \in P$, the *upset of x*, denoted by $U[x]$, and the *deleted upset of x*, denoted by $U(x)$, are defined as the set of all elements $y \in P$ with $x \leq_P y$, and $U[x] - \{x\}$, respectively. Similar definitions of the *downset* and the *deleted downset* of an element $x \in P$ are as follows: $D[x] = \{y \in P : y \leq_P x\}$, and $D(x) = D[x] - \{x\}$, respectively.

A finite poset of cardinality $n \in \mathbb{N}$ is called an *n-poset*. The *n*-poset $\mathbf{n} = (X, \leq_{\mathbf{n}})$ is called a *chain* of order *n* if either $x \leq_{\mathbf{n}} y$ or $y \leq_{\mathbf{n}} x$ for any $x, y \in X$. The *height* of an *n*-poset *P* is the maximal cardinality of any chain in *P*. For a poset $P = (X, \leq_P)$, if a chain $L(P) = (X, \leq_{L(P)})$ has all relations of *P*, i.e., $\leq_P \subseteq \leq_{L(P)}$, then $L(P)$ is called a *linear extension* of *P*. For a given *n*-poset $P = (X, \leq_P)$, an order preserving bijection $f : P \to [n]$ is called a *natural labeling* on *P*, or simply called a *labeling* on *P*. The *linear discrepancy* of a poset $P = (X, \leq_P)$ is defined as

$$ld(P) = \min_{f \in \mathcal{F}} \max_{x \parallel_P y \in P} | f(x) - f(y) |,$$

where $\mathcal{F}$ is the set of all labelings of *P*. If we define the *tightness* of a labeling *f* on *P* as $T_f(P) = \max_{x \parallel_P y \in P} | f(x) - f(y) |$, then we can simply write $ld(P)$ as $\min_{f \in \mathcal{F}} T_f(P)$. For a given labeling *f* on *P*, if $T_f(P) = ld(P)$, then we call such *f* an *optimal labeling*. For a given optimal labeling *f* and $x, y \in P$ with $x \parallel_P y$, if $ld(P) = | f(x) - f(y) |$, then such pair of *x* and *y* is called an *optimal pair* of *f*.

The problem of determining the linear discrepancy of a poset was known as an NP-complete problem (see Fishburn et al. [3] in 2001), in general. Since then, the linear discrepancies of well-known posets have been separately determined without any general algorithm. In 2001, Tanenbaum et al. [8] determined not only the linear discrepancies of some simple structured posets such as the sum of chains, a standard example S_n, a Boolean lattice B_n, etc., but also that of a Boolean lattice of order *n*. In 2005, Hong et al. [4] gave the linear discrepancy of the Cartesian product of two chains of size *m* and *n*. In 2008, Kim and Cheong [6] gave the linear discrepancy of the Cartesian product of three chains of size 2*t*. Recently, the same authors gave the complete answer for the Cartesian product of three chains of the same size in [2], and some asymptotic results on the general product of chains of the same size in [1].

A tree-poset is the poset whose covering graph (or Hesse diagram) forms a rooted tree as a graph, which is a connected graph having no cycles. In terms of poset theory (see [9]), it can also be defined as a connected poset T such that either $U[x]$ is always a chain for each $x \in T$, called a *down-tree*, or $D[x]$ is always a chain for each $x \in T$, called an *up-tree*. It is obvious that the only tree poset which is a down-tree as well as an up-tree is a chain. In graph theory, the bandwidth can be thought of an analogous parameter to the linear discrepancy in poset theory. While the bandwidth of the complete tree has been determined (see Smithline [7] in 1995), the linear discrepancy of the complete tree-poset has not been seen in the previous studies, yet.

In this paper we deal with the linear discrepancies of complete tree-posets. We first give a construction of a natural labeling of a complete *k*-ary tree-poset of height *h* whose tightness is the cardinality of the tree-poset minus its height so that we have an upper bound of the linear discrepancy of a complete *k*-ary tree poset of height *h*. Next, we prove that this upper bound is also a lower bound, so that we finally obtain the linear discrepancies of the complete tree-posets.

2. Construction of a Labeling of the Complete *k*-ary Tree-poset

In this section, we construct a labeling of a complete tree-poset from which we can obtain an upper bound of the linear discrepancy of a complete tree-poset. We begin with some terminologies and definitions for tree-posets. For the sake of avoiding redundancy, we restrict our definition of a tree-poset to that of a down-tree, i.e., we define a tree-poset T as a connected poset with the property that $U[x]$ is always a chain for each $x \in T$. As in graph theory, for positive integers k and h, we use the notation for the k-ary complete tree-poset of height h tree to be denoted by $T_{k,h}$. Let $T_i = \{x \in T_{k,h} : |U[x]| = h - i + 1\}$ for $i = 1, \ldots, h$. Then $\{T_i\}$ partitions the elements of $T_{k,h}$ into h subsets of the ground set of $T_{k,h}$ in which any two distinct elements in the same subset T_i are incomparable $(1 \le i \le h)$. The element of T_h is called the *root* of $T_{k,h}$, and each element in T_1 is called a *leaf* of $T_{k,h}$. From the definition of down-tree, we directly obtain the following lemma on counting.

Lemma 2.1. *Let k and h be positive integers with $k \ge 2$. For a given complete k-ary down-tree $T_{k,h}$, we have the followings*:

(i) $|T_{k,h}| = |T_1| + \cdots + |T_h| = \dfrac{1}{k-1}(k^h - 1)$.

(ii) $|T_q| = k^{h-q}$ *for* $q = 1, \ldots, h$.

(iii) $|U(x)| = h + 1 - q$ *if* $x \in T_q$ *for* $q = 1, \ldots, h$.

(iv) $|D(x)| = \dfrac{1}{k-1}(k^q - 1)$ *if* $x \in T_q$ *for* $q = 1, \ldots, h$.

Let $|T_{k,h}| = \alpha$. Then we now give a construction of a natural labeling $f : T_{k,h} \to |\alpha|$ which becomes a bijective order preserving map, as follows.

Construction 2.2. Let $T_{k,h} = (X, \le_T)$ be a complete k-ary down-tree of height h, and v be a leaf of $T_{k,h}$, i.e., $v \in T_1$. From (iii) in Lemma 2.1, we note that $U[v]$ is a chain of height h which is a maximal chain in $T_{k,h}$. Now we define some subsets of X for our construction of a natural labeling, as follows:

- Index the elements of $U[v]$ as $x_1, ..., x_h$ such that $x_1 = v$, and $x_i \in T_i$ so that $x_i \leq_T x_{i+1}$ for $i = 1, ..., h-1$. Note that the leaf $v = x_1$ is the minimum of $U[v]$, and the root x_h is the maximum of $U[v]$.

- Pick an element z in $T_{h-1} - U[v]$. Let $z = x_{h+1}$ and $A_1 = U[v] \bigcup \{x_{h+1}\}$. Note that we can always choose such an element z since the root x_h has $k(\geq 2)$ children. Clearly, A_1 is the set of these $h+1$ indexed elements x_i for $i = 1, ..., h+1$.

- Define $A_i = D[x_i] - \bigcup_{j=1}^{i-1} A_j$ for $i = 2, ..., h$. Note that

$$|A_i| = \begin{cases} h+1, & \text{if } i = 1, \\ k^{i-1} - 1, & \text{if } i = 2, ..., h-1, \\ k^{h-1} - 2, & \text{if } i = h \end{cases}$$

from Lemma 2.1.

Let $\alpha = \frac{1}{k-1}(k^h - 1)$ be the cardinality of $T_{k,h}$. With the set of indexed elements A_i's $(i = 1, ..., h)$, we now define a labeling map $f : T_{k,h} \to [\alpha]$ on the elements of $T_{k,h}$, as follows:

(1) For $x = x_i \in A_1$, define $f(x)$ as

$$f(x_1) = 1, \quad f(x_{h+1}) = \alpha - h + 1,$$

$$f(x_i) = \alpha - h + i \text{ if } i = 2, ..., h.$$

(2) For $i = 2, ..., h$, we assign an element x in A_i to $f(x)$ as to satisfy the following conditions:

- $$f(x) \in \begin{cases} \left[\dfrac{k^{i-1} - k}{k-1} - i + 4,\ \dfrac{k^{i} - k}{k-1} - i + 2\right], & \text{if } i = 2, \ldots, h-1, \\ \left[\dfrac{k^{h-1} - k}{k-1} - h + 4,\ \alpha - h\right] & \text{if } i = h, \end{cases}$$

- $f(x) < f(y)$ if $x \leq_T y$,

- $f(x) < f(y)$ if $x \in A_i \cap T_s$, and $y \in A_i \cap T_t$ with $s < t$.

For x and $y \in P$, an incomparable pair (x, y) is *critical* if and only if $D(x) \subseteq D(y)$ and $U(y) \subseteq U(x)$. The following lemma, introduced by Keller and Young [5], is useful for determining the linear discrepancy of a poset.

Lemma 2.3 [5]**.** *For a poset P, and its labeling f, let x and x' be elements in P. If (x, x') is a tight pair, i.e., $f(x') - f(x) = T_f(P)$, then (x, x') is a critical pair in P.*

Proposition 2.4. *For a given complete tree-poset $T_{k,h}$, the bijective map $f : T_{k,h} \to |T_{k,h}|$ given by Construction* 2.2 *is a natural labeling whose tightness is $T_f(T_{k,h}) = \alpha - h = \dfrac{1}{k-1}(k^h - 1) - h$.*

Proof. It is obvious that the constructed map f is a bijection, since the sets A_i $(i = 1, \ldots, h)$ form a partition of $T_{k,h}$, and the closed intervals in part (2) in the construction have the same cardinalities as the A_i's. Furthermore, f preserves the order relation of $T_{k,h}$. For, in Construction 2.2, f is defined as to keep the order relation of $T_{k,h}$ within each A_i $(i = 2, \ldots, h)$, respectively, and any two elements belonging to two distinct sets are incomparable. Furthermore, A_1 consists of one lowest labeled element x_1 and $h - 1$ highest labeled elements from the top $|T_{k,h}|$.

We show that $T_f(T_{k,h}) = |T_{k,h}| - h,$ that is, for all incomparable pairs x and y in $T_{k,h},$

$$|f(y) - f(x)| \leq \frac{1}{k-1}(k^h - 1) - h = f(x_{h+1}) - f(x_1). \qquad (1)$$

From Lemma 2.3, it is sufficient to evaluate the differences of labels of critical pairs for determining the linear discrepancy. Let (x, y) be a critical pair in $T_{k,h}$. Then x is a leaf of $T_{k,h}$. For a leaf x, there is $y \in T_{h-1}$ such that $f(x) < f(y)$ and $f(y) - f(x)$ is maximal for a given x from Construction 2.2. Hence, we only check two cases: (i) $x \in A_h,$ and (ii) $x \notin A_h$.

Suppose that $x \in A_h$. Then $f(x) \geq \frac{k^{h-1} - k}{k-1} - h + 4,$ and $f(y) \leq \frac{k^h - 1}{k-1} - 1.$ Hence, we have

$$f(y) - f(x) \leq (\alpha - 1) - \left(\frac{k^{h-1} - k}{k-1} - h + 4\right) < \alpha - h$$

since $h < \frac{k^{h-1} - k}{k-1} - h + 5.$

Now, suppose that $x \notin A_h$. Then $f(x) \geq 1,$ and $f(y) \leq \alpha - h + 1.$ Hence, we have

$$f(y) - f(x) \leq \alpha - h + 1 - 1 = \alpha - h.$$

In fact, $f(x_{h+1}) - f(x_1) = \alpha - h.$ Therefore, we conclude that the tightness of f is $f(x_{h+1}) - f(x_1),$ which is $\frac{1}{k-1}(k^h - 1) - h.$ $\square$

Example 2.5. From Construction 2.2, we give an order-preserving bijection (a natural labeling) on $T_{3,4}$. In this example, $k = 3,$ $h = 4,$ and $|T_{3,4}| = 40.$ The sets $A_i,$ for $i = 1, 2, 3, 4,$ and the values for the labels of the elements can be obtained from the definitions. Furthermore, the labels satisfy the following values, so that a possible assignment to the elements of $T_{3,4}$ is illustrated in Figure 1.

$$f(A_1) = [37, 40], \; f(A_2) = [2, 3], \; f(A_3) = [4, 11], \; f(A_4) = [12, 36]$$

where $f(x_1) = 1$, $f(x_2) = 38$, $f(x_3) = 39$, $f(x_4) = 40$, and $f(x_5) = 37$. From Proposition 2.4, we have $T_f(T_{3,4}) = 36$.

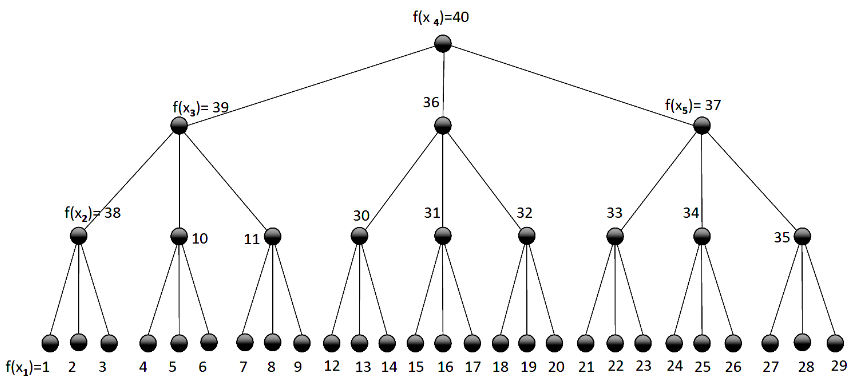

Figure 1. The labeling f of $T_{3,4}$ obtained from Construction 2.2.

3. $ld(T_{k,h})$ from the Tight Upper Bound

In this section, we give an upper bound of the linear discrepancy of $T_{k,h}$, from Construction 2.2, and prove that the upper bound is also its lower bound, that means the upper bound is tight.

From Proposition 2.4, we have an upper bound of the linear discrepancy of $T_{k,h}$ as follows.

Proposition 3.1. *For a k-ary complete tree-poset $T_{k,h}$ of height h, we have*

$$ld(T_{k,h}) \leq \frac{1}{k-1}(k^h - 1) - h = |T_{k,h}| - h.$$

The following proposition shows that the upper bound in Proposition 3.1 is also a lower bound, i.e., the constructive lower bound is tight.

Proposition 3.2. *For a given complete tree-poset* $T_{k,h}$, *we have*

$$ld(T_{k,h}) \geq \frac{1}{k-1}(k^h - 1) - h = |T_{k,h}| - h.$$

Proof. Let $|T_{k,h}| = \alpha$. Assume that $ld(T_{k,h}) \leq \alpha - h - 1$, and let f be an optimal labeling. Then there is an element x_1 in T_1 such that $f(x_1) = 1$. Let y be an element in $T_{k,h}$ such that $x_1 \|_T y$. Then $f(y) \leq \alpha - h$ since $ld(T_{k,h}) \leq \alpha - h - 1$. Hence, every element $z \in T_{k,h}$ such that $f(z) > \alpha - h$ is comparable to x_1, i.e., $z \in U(x_1)$. Note that the set consisting of incomparable elements to x_1 is $T_{k,h} - U[x_1]$. Hence, for $y \in T_{k,h} - U[x_1]$, we have $2 \leq f(y) \leq \alpha - h$, i.e., $|T_{k,h} - U[x_1]| \leq \alpha - h - 1$. Since $|U(x_1)| = h$, we have

$$\begin{aligned} |T_{k,h}| &= |T_{k,h} - U[x_1]| + |U[x_1]| \\ &\leq \alpha - h - 1 + h \\ &= \alpha - 1 < \alpha = |T_{k,h}|. \end{aligned}$$

This is a contradiction. Therefore, we conclude $ld(T_{k,h}) \geq \alpha - h.$ □

From Propositions 3.1 and 3.2, we finally determine the linear discrepancy of the complete k-ary tree poset $T_{k,h}$ of height h as follows:

Theorem 3.3. $ld(T_{k,h}) = \frac{1}{k-1}(k^h - 1) - h = |T_{k,h}| - h.$

Example 3.4. We note that the tightness of the natural labeling f on $T_{3,4}$ in Example 2.5 is 36. From Theorem 3.3, we finally have $ld(T_{3,4}) = \alpha - h = 40 - 4 = 36$ which coincides with the tightness of f constructed by Construction 2.2.

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