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GENERIC COMBINATORIAL IDENTITIES

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Abstract

Some well-known combinatorial identities appear to be special cases of more general identities in integral domains.

Let D be an integral domain (i.e., a commutative ring with 1 but without zero divisors) embedded in its quotient field, so that division makes sense. If $d_1, d_2, ..., d_n$ are nonzero elements of D, $n \geq 1$, and $d_0 = 0$, then the following relations hold:

$$\sum_{i=1}^{n} (-1)^i \frac{(d_n - d_0) \cdots (d_n - d_{i-1})}{d_1 \cdots d_i} = -1. \tag{A}$$

This is a generalization of the familiar combinatorial identity

$$\sum_{i=1}^{n} (-1)^i \binom{n}{i} = -1 \tag{B}$$

to which (A) reduces when $d_i = i$, $i = 1, 2, ..., n$. The generalization was proved in [1], in the context of matrix inversion in integral domains.

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If d_n in (A) is replaced by $-d_n$, then

$$\sum_{i=1}^{n} \frac{(d_n + d_0)\cdots(d_n + d_{i-1})}{d_1 \cdots d_i} = 2\,\frac{(d_n + d_0)\cdots(d_n + d_{n-2})(d_n + d_{n-1})}{d_1 \cdots d_{n-1} d_n} - 1 \quad \text{(C)}$$

which is a generalization of the identity

$$\sum_{i=1}^{n} \binom{n+i-1}{i} = \binom{2n}{n} - 1 \qquad \text{(D)}$$

to which (C) reduces when $d_i = i$, $i = 1, 2, \ldots, n$. This is so because

$$2\,\frac{n(n+1)\cdots(2n-1)}{n!} = 2\,\frac{(n-1)!\,n(n+1)\cdots(2n-1)}{(n-1)!\,n!} = 2\,\frac{(2n-1)!}{(n-1)!\,n!}$$

and

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!} = \frac{2n(2n-1)!}{n!\,n!} = 2\,\frac{(2n-1)!}{(n-1)!\,n!}.$$

See [2, p. 54]. More generally, when $d_i = id$, $d \in D$, $d \neq 0$, $i = 1, 2, \ldots, n$, the same result is obtained.

When $d_i = d$, $d \in D$, $d \neq 0$, $i = 1, 2, \ldots, n$, then (C) becomes a geometric series,

$$\sum_{i=1}^{n} \frac{d(2d)^{i-1}}{(d)^i} = \sum_{i=1}^{n} 2^{i-1} = 2^n - 1 = 2\,\frac{d(2d)^{n-1}}{d^n} - 1 = \frac{2^n d^n}{d^n} - 1.$$

Of particular interest is the case when the d_i form a geometric progression. When $d_i = r^i$, $r \in D$, $r \neq 0$, $i = 1, 2, \ldots, n$, then the terms in (C) become

$$\frac{r^n(r^n + r)\cdots(r^n + r^{i-1})}{r \cdots r^i} = r^n r(r^{n-1} + 1)\cdots r^{i-1}(r^{n-i+1} + 1)/r^{\frac{i(i+1)}{2}}$$

$$= r^{n-i}(r^{n-1} + 1)\cdots(r^{n-i+1} + 1),$$

so that

$$S_n = \sum_{i=1}^{n} r^{n-i}(r^{n-1}+1)\cdots(r^{n-i+1}+1) = 2(r^{n-1}+1)\cdots(r+1)-1$$

$$= 2\prod_{i=1}^{n-1}(r^i+1)-1.$$

Although the initial assumption was $r \neq 0$, if $r = x$, a complex number satisfying $|x| < 1$, then the sequence of functions $S_n = S_n(x)$ converges to some $S(x)$. Recall that if $q(j)$ represents the number of partitions of the integer j into distinct parts, $j \in \mathbb{N}$, then the generating function for the $q(j)$ is given by $\sum_{j=0}^{\infty} q(j)x^j = \prod_{k=1}^{\infty}(1+x^k)$ ([3]). Both the infinite product and the infinite series converge for $|x| < 1$, to some $Q(x) = \sum_{j=0}^{\infty} q(j)x^j$. Thus,

$$S(x) = 2Q(x) - 1.$$

Although $S_n = \sum_{i=1}^{n} \frac{(d_n-d_0)\cdots(d_n-d_{i-1})}{d_1\cdots d_i}$ does reduce to $\sum_{i=1}^{n} 2^n - 1$ when $d_i = i$, $i = 1, 2, ..., n$, in general, S_n does not appear to have a particularly compact expression. To compute S_n, note that

$$\sum_{i=1}^{n} \frac{(d_n-d_0)\cdots(d_n-d_{i-1})}{d_1\cdots d_i}$$

$$= \frac{d_n-d_0}{d_1} + \frac{(d_n-d_0)(d_n-d_1)}{d_1 d_2}$$

$$+ \cdots + \frac{(d_n-d_0)\cdots(d_n-d_{n-2})}{d_1\cdots d_{n-1}} + \frac{(d_n-d_0)\cdots(d_n-d_{n-1})}{d_1\cdots d_n}$$

$$= \frac{d_n}{d_1\cdots d_{n-1}d_n}([d_2\cdots d_n] + [d_3\cdots d_n(d_n-d_1)]$$

$$+ \cdots + [d_i \cdots d_n(d_n - d_1) \cdots (d_n - d_{i-2})]$$

$$+ \cdots + [d_n(d_n - d_1) \cdots (d_n - d_{n-2})] + [(d_n - d_1) \cdots (d_n - d_{n-1})]).$$

With the conventions that $d_0 = d_n - 1$ (so that $d_n - d_0 = 1$) and $d_{n+1} = 1$, then $S_n = \frac{1}{d_1 \cdots d_{n-1}} \sum_{j=2}^{n+1} d_j \cdots d_n(d_n - d_1) \cdots (d_n - d_{j-2})$; i.e., when $j = 2$, the product $d_j \cdots d_n(d_n - d_1) \cdots (d_n - d_{j-2}) = d_2 \cdots d_n$, and when $j = n+1$, that product equals $(d_n - d_1) \cdots (d_n - d_{n-1})$. S_n also can be expressed as $\frac{1}{d_1 \cdots d_{n-1}} \sum_{j=2}^{n+1} \prod_{j \le k \le n,\ 1 \le l \le j-2} d_k(d_n - d_l)$.

The principal hurdle in attempting analogize these generic formulas to classical combinatorics is the lack of symmetry analogous to $\binom{n}{i} = \binom{n}{n-i}$. In general, it is not reasonable to expect that

$$\frac{(d_n - d_0) \cdots (d_n - d_{i-1})}{d_1 \cdots d_i} = \frac{(d_n - d_0) \cdots (d_n - d_{n-i-1})}{d_1 \cdots d_{n-i}}, \quad i = 1, \ldots, n-1.$$

References

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[2] D. Knuth, Fundamental Algorithms, Vol. 1, The Art of Computer Programming, Addison-Wesley, 2nd ed., 1973.

[3] http://en.wikipedia.org/wiki/Partition_(number_theory).