# A GENERALIZED NONLINEAR INTEGRABLE EQUATION AND SOME PROPERTIES 

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#### Abstract

We present a nonlinear evolution equation with variable coefficients. With the help of Bell Polynomials, we obtain its Lax pair, Bäcklund transformation and infinite conserved laws.


## 1. Introduction

Recently, Gilson et al. [1], Lambert and Springael [2] and Bell [3] proposed an efficient method for constructing bilinear Bäcklund transformations of nonlinear integrable equations by using the Bell polynomials. Fan [4] and Fan and Hon [5] further generalized the method for generating bilinear Bäcklund transformations, Lax pairs, infinite conserved laws of variable-coefficient integrable equations. Based on the approaches, we want to consider the following nonlinear evolution equation with variable coefficients:

$$
\begin{equation*}
u_{t t}=u_{t x}+4 u+h_{1}(t) u u_{x}+h_{2}(t) x u_{x}+h_{3}(t) u_{3 x}+h_{4}(t) u_{x} \tag{1}
\end{equation*}
$$

so that its bilinear form, Bäcklund transformation, Lax pair and infinite
conserved laws are worked out, respectively. To proceed this, we first present some necessary acknowledges on Bell polynomials [1-3]. Set $f=f\left(x_{1}, \ldots, x_{l}\right)$ to be a $C^{\infty}$ multi-variable function, the following polynomials:

$$
\begin{equation*}
Y_{n_{1} x_{1}}, \ldots, n_{l} x_{l}(f)=\exp (-f) \partial_{x_{1}}^{n_{1}} \ldots \partial_{x_{l}}^{n_{1}} \exp (f) \tag{2}
\end{equation*}
$$

are known as multi-dimensional Bell polynomials. Denote by [4] $f_{r_{1} x_{1}, \ldots, r_{l} x_{l}}=\partial_{x_{1}}^{r_{1}} \ldots \partial_{x_{l}}^{\eta_{l}} f\left(r_{k}=0, \ldots, n_{k} ; k=1, \ldots, l\right)$ which represents all partial derivatives of $f$. When $f=f(x, t)$, we can get that

$$
Y_{x}(f)=f_{x}, Y_{2 x}(f)=f_{2 x}+f_{x}^{2}, \ldots .
$$

The multi-dimensional binary Bell polynomials are defined by

$$
\begin{align*}
& \Upsilon_{1 x_{1}}, \ldots, n_{l} x_{l} \\
&(v, w)
\end{aligned}=Y_{n_{1} x_{1}, \ldots, n_{l} x_{l}}(f)=f\left(r_{1} x_{1}, \ldots, r_{l} x_{l}\right), \begin{aligned}
& v_{r_{1} x_{1}, \ldots, r_{l} x_{l}}, r_{1}+\cdots+r_{l}=2 k-1,  \tag{3}\\
& w_{r_{1} x_{1}}, \ldots, r_{l} x_{l}, r_{1}+\cdots+r_{l}=2 k .
\end{align*} .
$$

When take $v=\ln F / G, w=\ln F G$, we can get [1]:

$$
\begin{equation*}
\Upsilon_{n_{1} x_{1}, \ldots, n_{l} x_{l}}(v, w)=(F G)^{-1} D_{x_{1}}^{n_{1}} \ldots D_{x_{l}}^{n_{l}} F \cdot G . \tag{4}
\end{equation*}
$$

Specially, set $F=G$, we have that

$$
G^{-2} D_{x_{1}}^{n_{1}} \ldots D_{x_{l}}^{n_{l}} F \cdot G=\left\{\begin{array}{l}
0, n_{1}+\cdots+n_{l}=2 k-1,  \tag{5}\\
P_{n_{1} x_{x}}, \ldots, n_{l} x_{l} \\
(q), n_{1}+\cdots+n_{l}=2 k, k=1,2, \ldots .
\end{array}\right.
$$

If again set $v=\ln \psi$, one infers that

$$
\begin{equation*}
Y_{n_{1} x_{1}}, \ldots, n_{l} x_{l}(v)=\psi_{n_{1} x_{1}}, \ldots, n_{l} x_{l} / \psi, \tag{6}
\end{equation*}
$$

which is an efficient tool for producing Lax pairs of integrable equations.

## 2. Integrability of Equation (1)

Set $u=q_{2 x}$, and integrate once, Eq. (1) becomes

$$
\begin{align*}
E(q)= & q_{2 t, x}-q_{2 x, t}-4 q_{x}-\frac{1}{2} h_{1}(t) q_{2 x}^{2}-h_{2}(t)\left(x q_{2 x}-q_{x}\right) \\
& -h_{3}(t) q_{4 x}-h_{4}(t) q_{2 x}=0 . \tag{7}
\end{align*}
$$

If set $h_{1}(t)=6 h_{3}(t)$, then we get

$$
\begin{align*}
E(q)= & \partial_{x}\left(P_{2 t}(q)-P_{x t}(q)\right)-h_{3}(t) P_{4 x}(q)-h_{2}(t) x P_{2 x}(q) \\
& -h_{4}(t) q_{2 x}+\left(h_{2}(t)-4\right) q_{x}=0 . \tag{8}
\end{align*}
$$

Set $q=2 \ln G \Leftrightarrow u=q_{2 x}=2(\ln G)_{2 x}$, one yields that

$$
\begin{align*}
E(q)= & {\left[\partial_{x}\left(D_{t}^{2}-D_{x} D_{t}\right)-h_{3}(t) D_{x}^{4}-h_{2}(t) x D_{2 x}-h_{4}(t) D_{2 x}\right.} \\
& \left.+\left(h_{2}(t)-4\right) \partial_{x}\right] G \cdot G=0, \tag{9}
\end{align*}
$$

which is the bilinear form of Eq. (1).
Let $q, \tilde{q}$ be two different solutions of Eq. (9). Then the following twofield condition:

$$
\begin{equation*}
E(\tilde{q})-E(q)=0 \tag{10}
\end{equation*}
$$

can be thought as an ansatz for a bilinear Bäcklund transformation. Eq. (9) implies that

$$
\begin{align*}
& (\tilde{q}-q)_{2 t, x}-(\tilde{q}-q)_{2 x, t}-4(\tilde{q}-q)_{x} \\
& -h_{3}(t)\left[(\tilde{q}-q)_{4 x}+3(\tilde{q}-q)_{2 x}(\tilde{q}+q)_{2 x}\right] \\
& -h_{2}(t)\left[x(\tilde{q}-q)_{2 x}-(\tilde{q}-q)_{x}\right]-h_{4}(t)(\tilde{q}-q)_{2 x}=0 . \tag{11}
\end{align*}
$$

Set $\tilde{q}-q=2 v, \tilde{q}+q=2 w$, then Eq. (11) turns into

$$
\begin{align*}
E(\tilde{q})-E(q)= & v_{2 t, x}-v_{2 x, t}-4 v_{x}-h_{3}(t)\left(v_{4 x}+6 v_{2 x} w_{2 x}\right) \\
& -h_{2}(t)\left(x v_{2 x}-v_{x}\right)-h_{4}(t) v_{2 x}=0 . \tag{12}
\end{align*}
$$

Eq. (12) can be cast into

$$
\begin{align*}
E(\widetilde{q})-E(q) & \left.=\partial_{x} \partial_{t}\left(\Upsilon_{t}(v)-\Upsilon_{x}(v)\right)-h_{3}(t) \partial_{x} \Upsilon_{3 x}(v, w)+R(v, w)\right] \\
& =0, \tag{13}
\end{align*}
$$

where

$$
\begin{align*}
R(v, w)= & -3 h_{3}(t) \text { Wronskian }\left[\Upsilon_{2 x}(v, w), \Upsilon_{x}(v)\right]-h_{2}(t) x v_{2 x}-h_{4}(t) v_{2 x} \\
& +\left(h_{2}(t)-4\right) v_{x} . \tag{14}
\end{align*}
$$

Set

$$
\begin{equation*}
\Upsilon_{2 x}(v, w)=\lambda, h_{2}(t)=2 \tag{15}
\end{equation*}
$$

Eq. (14) becomes

$$
\begin{equation*}
R(v, w)=-\partial_{x}\left[2 \lambda h_{3}(t) \Upsilon_{x}(v)+h_{2}(t) x \Upsilon_{x}(v)+h_{4}(t) \Upsilon_{x}(v)\right] . \tag{16}
\end{equation*}
$$

Thus, Eq. (13) presents that

$$
\begin{align*}
E(\tilde{q})-E(q)= & \partial_{x}\left[\partial_{t}\left(\Upsilon_{t}(v)-\Upsilon_{x}(v)\right)-h_{3}(t) \Upsilon_{3 x}(v, w)\right. \\
& \left.-\left(3 \lambda h_{3}(t)+h_{4}(t) \Upsilon_{x}(v)+2 x \Upsilon_{x}(v)\right)\right]=0 . \tag{17}
\end{align*}
$$

Therefore, we obtain the bilinear Bäcklund transformation of Eq. (1):

$$
\left\{\begin{array}{l}
\left(D_{x}^{2}-\lambda\right) F \cdot G=0,  \tag{18}\\
{\left[\partial_{t}\left(D_{t}-D_{x}\right)-h_{3}(t) D_{x}^{3}-\left(3 \lambda h_{3}(t)+h_{4}(t)+2 x\right) D_{x}+\beta\right] F \cdot G=0 .}
\end{array}\right.
$$

Let $v=\ln \psi, w=v+q$. Then one infers

$$
\begin{align*}
& \Upsilon_{x}(v)=v_{x}, \quad \Upsilon_{2 x}(v, w)=q_{2 x}+\psi_{2 x} / \psi, \\
& \Upsilon_{3 x}(v, w)=3 q_{2 x} \psi_{x} / \psi+\psi_{3 x} / \psi, \ldots \tag{19}
\end{align*}
$$

Substituting Eq. (19) into Eq. (18) yields the Lax pair of Eq. (1)

$$
\begin{align*}
& \psi_{2 x}+(u-\lambda) \psi=0,  \tag{20}\\
& \psi_{2 t}-\psi_{x t}-h_{3}(t)\left(3 u \psi_{x}+\psi_{3 x}\right)-\left(3 \lambda h_{3}(t)+h_{4}(t)\right) \psi_{x}+2 x \psi_{x}=0 . \tag{21}
\end{align*}
$$

Next, we search for the infinite conserved laws of Eq. (1). Set $\eta=v_{x}$, $w_{x}=q_{X}+\eta$, we have

$$
\begin{equation*}
\eta_{x}+\eta^{2}+q_{2 x}=\lambda=\varepsilon^{2} . \tag{22}
\end{equation*}
$$

Thus, we obtain

$$
\begin{align*}
& \partial_{t}\left(\eta_{t}-\eta_{x}\right)+\partial_{x}\left[h_{3}(t)\left(\eta_{2 x}+3 q_{2 x} \eta+3 \eta \eta_{x}+\eta^{3}\right)\right. \\
& \left.+\left(3 \lambda h_{3}(t)+h_{4}(t)\right) \eta+2 x \eta\right]=0 . \tag{23}
\end{align*}
$$

Set $\eta=\varepsilon+\sum_{n=1}^{\infty} I_{n}\left(q, q_{X}, q_{2 x}, \ldots\right) \varepsilon^{-n}$ and substituting into (23), we have the infinite conserved densities

$$
I_{1}=-\frac{1}{2} u, I_{2}=\frac{1}{4} u_{x}, \ldots, I_{n+1}=-\frac{1}{2}\left(I_{n, x}+\sum_{i+j=n} I_{i} I_{j}\right), n=3,4, \ldots .
$$

Inserting $\eta$ into Eq. (23), we get

$$
\begin{align*}
& \sum_{n=1}^{\infty} I_{n, 2} \varepsilon^{-n}-\sum_{n=1}^{\infty} I_{n, x t} \varepsilon^{-n}+\partial_{x}\left[h _ { 3 } ( t ) \left(\sum_{n=1}^{\infty} I_{n, 2 x} \varepsilon^{-n}+3 u \sum_{n=1}^{\infty} J_{n} \varepsilon^{-n}\right.\right. \\
& \left.+3 \sum_{n=1}^{\infty} I_{n} \varepsilon^{-n} \sum_{k=1}^{\infty} I_{k, x} \varepsilon^{-k}+\sum_{n=1}^{\infty} I_{n} \varepsilon^{-n} \sum_{k=1}^{\infty} I_{k} \varepsilon^{-k} \sum_{j=1}^{\infty} I_{j} \varepsilon^{-j}\right) \\
& \left.+3 h_{3}(t) \sum_{n=1}^{\infty} I_{n} \varepsilon^{-n+2}+h_{4}(t) \sum_{n=1}^{\infty} I_{n} \varepsilon^{-n}+2 x \sum_{n=1}^{\infty} I_{n} \varepsilon^{-n}\right]=0 . \tag{24}
\end{align*}
$$

It is easy to get the corresponding fluxes that

$$
\begin{aligned}
F_{1}= & -\frac{1}{2} h_{3}(t)\left(u_{x}+3 u^{2}+3 u\right)-\frac{3}{2} u_{x}-\frac{3}{2} u-\frac{1}{2} h_{4}(t) u-x u, \\
F_{n}= & h_{3}(t)\left(I_{n, 2 x}+3 u I_{n}+3 \sum_{i+j=n} I_{i} I_{j, x}+\sum_{i+j+k=n} I_{i} I_{j} I_{k}\right) \\
& +3 h_{3}(t) I_{n+2}+\left(h_{4}(t)+2 x\right) I_{n}, \quad n=2,3, \ldots .
\end{aligned}
$$

## References

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