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FLAT ALMOST NORDEN METRICS WITH NONINTEGRABLE ALMOST COMPLEX STRUCTURES IN DIMENSION FOUR

A. BONOME
Departamento de Xeometría e Topoloxía, Facultade de Matematicas
Universidade de Santiago de Compostela
15782 *Santiago de Compostela, Spain*
e-mail: bonome@zmat.usc.es

R. CASTRO
Departamento de Xeometría e Topoloxía, Facultade de Matematicas
Universidade de Santiago de Compostela
15782 *Santiago de Compostela, Spain*
e-mail: regina@zmat.usc.es

L. M. HERVELLA
Departamento de Xeometría e Topoloxía, Facultade de Matematicas
Universidade de Santiago de Compostela
15782 *Santiago de Compostela, Spain*
e-mail: hervella@zmat.usc.es

Y. MATSUSHITA
School of Engineering, University of Shiga Prefecture
Hikone 522-8533, *Japan*
e-mail: yasuo@mech.usp.ac.jp

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Abstract

We shall exhibit examples of almost Norden structures in dimension four, which consist of a flat metric and of a nonintegrable almost complex structure.

1. Introduction and Preliminaries

The purpose of the present paper is to exhibit examples of flat almost Norden 4-manifolds whose almost complex structures are not integrable. Examples of flat almost Hermitian manifolds with nonintegrable almost complex structures in dimensions four and six are reported by Tricerri and Vanhecke [10, 11]. Our examples of flat Norden metrics are constructed on a Walker 4-manifold, according to the way of construction of Norden metrics on a neutral 4-manifold with two kinds of almost complex structures [1]. Since four-dimensional Norden metric is necessarily of neutral signature $(++--)$, we must recall the following basic fact for the neutral geometry.

Fact 1 ([4], [5], [6], [7], [9]). The existence conditions of the following three geometric objects on an oriented 4-manifold M are equivalent to each other:

(i) a metric of signature $(++--)$ with $G = SO_0(2, 2)$,

(ii) a pair (J, J') of two kinds of an almost complex structure J and an opposite almost complex structure J' which commute with each other,

(iii) a field of oriented tangent 2-planes.

It is known [5] that we can always choose g and (J, J') so that g is invariant by both structures J and J', and moreover that J and J' commute with each other:

$$J^2 = J'^2 = -1, \quad JJ' = J'J, \tag{1}$$

$$g(JX, JY) = g(X, Y), \quad g(J'X, J'Y) = g(X, Y) \quad \forall X, Y \in \mathfrak{X}(M), \tag{2}$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M, see also [8].

It is convenient to express a neutral metric g and the pair (J, J') in terms of an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ of vectors, and its dual frame $\{e^1, e^2, e^3, e^4\}$ of 1-forms, with $e^i(e_j) = \delta_{ij}$. In fact, the metric g can be written as

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4. \tag{3}$$

The almost complex structure J and an opposite almost complex structure J' can be written as

$$J = e_2 \otimes e^1 - e_1 \otimes e^2 + e_4 \otimes e^3 - e_3 \otimes e^4, \tag{4}$$

$$J' = e_2 \otimes e^1 - e_1 \otimes e^2 - e_4 \otimes e^3 + e_3 \otimes e^4. \tag{5}$$

In the present paper, by a Norden metric on a neutral 4-manifold we mean a neutral metric g^{N+} which is J-skew invariant, i.e., $g^{N+}(JX, JY) = -g^{N+}(X, Y)$. Similarly, by an opposite Norden metric on a neutral 4-manifold we mean a neutral metric g^{N-} which is J'-skew invariant, i.e., $g^{N-}(J'X, J'Y) = -g^{N-}(X, Y)$. If a neutral metric is skew invariant by both J and J', then the metric is called *double Norden*, and is denoted by $g^{N\pm}$. The generic forms of these three kinds of Norden metrics are obtained in the authors' previous paper [1, Theorem 4 and Corollary 5]. In [1], typical examples of these Norden metrics are also presented.

There are two simple forms of Norden metrics [1, (21), (22)]:

$$g^{N+} = e^1 \otimes e^4 + e^4 \otimes e^1 + e^2 \otimes e^3 + e^3 \otimes e^2, \tag{6}$$

or

$$g^{N+} = e^1 \otimes e^3 + e^3 \otimes e^1 - e^2 \otimes e^4 - e^4 \otimes e^2. \tag{7}$$

Similarly, there are two simple forms of opposite Norden metrics [1, (23), (24)]:

$$g^{N-} = e^1 \otimes e^4 + e^4 \otimes e^1 - e^2 \otimes e^3 - e^3 \otimes e^2, \tag{8}$$

or

$$g^{N-} = e^1 \otimes e^3 + e^3 \otimes e^1 + e^2 \otimes e^4 + e^4 \otimes e^2. \tag{9}$$

We shall exhibit in the present paper a flat opposite Norden metric g^{N-} of type (8), with a nonintegrable opposite almost complex structure (5), on a Walker 4-manifold as a neutral 4-manifold.

2. A Walker 4-manifold as a Neutral 4-manifold

A Walker 4-manifold is a triple (M, g, D) consisting of a 4-manifold M, together with an indefinite metric g and a nonsingular field of 2-dimensional planes D (or distribution) such that D is parallel and null with respect to g. From Walker's theorem [12, Theorem 1 and Section 6, Case 1], there is a system of coordinates (x^1, x^2, x^3, x^4) with respect to which g takes the canonical form

$$g = [g_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x^1, x^2, x^3, x^4) & c(x^1, x^2, x^3, x^4) \\ 0 & 1 & c(x^1, x^2, x^3, x^4) & b(x^1, x^2, x^3, x^4) \end{bmatrix}, \tag{10}$$

where a, b and c are functions of the coordinates (x^1, x^2, x^3, x^4). The metric (10) is the most generic form of Walker metrics. We see that g is of signature $(+ + - -)$ (or neutral). The parallel null 2-plane D is spanned locally by $\{\partial_1, \partial_2\}$, where ∂_i are the abbreviated forms of $\frac{\partial}{\partial x^i}$, $(i = 1, \ldots, 4)$. The components R_{ijkl} of the pseudo-Riemann curvature tensor are given in [7, Appendix A] (note that p_i mean $\partial p / \partial x^i$). From the components R_{ijkl}, it should be recognized that the metric (10) is, of course, not flat.

One of local orthonormal frames for (10), which we will apply in the present analysis, is

$$e_1 = \frac{1}{2}(1 - a)\partial_1 + \partial_3, \qquad e_2 = -c\partial_1 + \frac{1}{2}(1 - b)\partial_2 + \partial_4,$$

$$e_3 = -\frac{1}{2}(1 + a)\partial_1 + \partial_3, \quad e_4 = -c\partial_1 - \frac{1}{2}(1 + b)\partial_2 + \partial_4, \tag{11}$$

with respect to which the Walker metric g can be diagonalized as in (3). Its dual basis of 1-forms is given by

$$e^1 = dx^1 + \frac{1}{2}(1+a)dx^3 + cdx^4, \qquad e^2 = dx^2 + \frac{1}{2}(1+b)dx^4,$$

$$e^3 = -dx^1 + \frac{1}{2}(1-a)dx^3 - cdx^4, \quad e^4 = -dx^2 + \frac{1}{2}(1-b)dx^4. \tag{12}$$

In terms of such an orthonormal frame $\{e_i\}$ and its dual basis $\{e^i\}$, we can easily construct Norden metrics (6), (7) and opposite Norden metrics (8), (9) on a Walker 4-manifold.

In the present paper, we shall turn our attention to a restricted form, as treated in a recent paper [6], rather than the generic metric (10). This restricted Walker metric is the metric (10) with $c = 0$, i.e.,

$$g = [g_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x^1, x^2, x^3, x^4) & 0 \\ 0 & 1 & 0 & b(x^1, x^2, x^3, x^4) \end{bmatrix}. \tag{13}$$

The nonzero components R_{ijkl} of the curvature tensor of the metric (13) are given by

$$R_{1313} = -\frac{1}{2}a_{11}, \quad R_{1323} = -\frac{1}{2}a_{12}, \quad R_{1334} = \frac{1}{2}a_{14} - \frac{1}{4}a_2b_1,$$

$$R_{1414} = -\frac{1}{2}b_{11}, \quad R_{1424} = -\frac{1}{2}b_{12}, \quad R_{1434} = -\frac{1}{2}b_{13} + \frac{1}{4}a_1b_1,$$

$$R_{2323} = -\frac{1}{2}a_{22}, \quad R_{2334} = \frac{1}{2}a_{24} - \frac{1}{4}a_2b_2,$$

$$R_{2424} = -\frac{1}{2}b_{22}, \quad R_{2434} = -\frac{1}{2}b_{23} + \frac{1}{4}a_2b_1,$$

$$R_{3434} = -\frac{1}{2}a_{44} - \frac{1}{2}b_{33} + \frac{1}{4}aa_1b_1 - \frac{1}{4}a_1b_3$$

$$+ \frac{1}{4}ba_2b_2 + \frac{1}{4}a_2b_4 + \frac{1}{4}a_3b_1 - \frac{1}{4}a_4b_2. \tag{14}$$

3. Opposite Norden Metrics

We study an opposite Norden metric g^{N-} of type (8) as a candidate for a flat Norden metric.

Proposition 2. *If* $c = 0$, *then an opposite Norden metric* g^{N-} *of type* (8) *takes the form*

$$g^{N-} = e^1 \otimes e^4 + e^4 \otimes e^1 - e^2 \otimes e^3 - e^3 \otimes e^2$$

$$= 2dx^1 \otimes dx^4 - 2dx^2 \otimes dx^3 + (a - b)dx^3 \otimes dx^4, \tag{15}$$

where $a = a(x^1, x^2, x^3, x^4)$ *and* $b = b(x^1, x^2, x^3, x^4)$.

Thus, g^{N-} depends only on the difference $a - b$. If we put

$$a(x^1, x^2, x^3, x^4) - b(x^1, x^2, x^3, x^4) = 2f(x^1, x^2, x^3, x^4), \tag{16}$$

then the nonzero components R^{N-}_{ijkl} of the Riemann curvature tensor are given by

$$R^{N-}_{1314} = -\frac{1}{2} f_{11}, \quad R^{N-}_{1324} = -\frac{1}{2} f_{12},$$

$$R^{N-}_{1334} = -\frac{1}{2} f_{13} + \frac{1}{4} f_1^2, \quad R^{N-}_{1423} = -\frac{1}{2} f_{12},$$

$$R^{N-}_{1434} = \frac{1}{2} f_{14} + \frac{1}{4} f_1 f_2, \quad R^{N-}_{2324} = -\frac{1}{2} f_{22},$$

$$R^{N-}_{2334} = -\frac{1}{2} f_{23} + \frac{1}{4} f_1 f_2, \quad R^{N-}_{2434} = \frac{1}{2} f_{24} + \frac{1}{4} f_2^2,$$

$$R^{N-}_{3434} = f_{34} + \frac{1}{2} f f_1 f_2. \tag{17}$$

We see that there are many partial derivatives of f with respect to x^1 and x^2. Since we are seeking a flat metric, we now suppose that

$$a(x^1, x^2, x^3, x^4) - b(x^1, x^2, x^3, x^4) = 2f(x^3, x^4). \tag{18}$$

Then, the only nonzero component of the curvature tensor, is

$$R^{N-}_{3434} = f_{34}, \tag{19}$$

which implies that if $f = f(x^3, x^4)$, then g^{N-} is still not flat.

Proposition 3. *If* $f = \phi(x^3) + \psi(x^4)$, *that is,*

$$a(x^1, x^2, x^3, x^4) - b(x^1, x^2, x^3, x^4) = 2\phi(x^3) + 2\psi(x^4), \tag{20}$$

then the opposite Norden metric

$$g^{N-} = 2dx^1 \otimes dx^4 - 2dx^2 \otimes dx^3 + 2(\phi(x^3) + \psi(x^4))dx^3 \otimes dx^4 \tag{21}$$

is flat.

4. Integrability of Opposite Almost Complex Structure J'

The flat opposite Norden metric g^{N-} in (21) is skew invariant by the following opposite almost complex structure

$$\begin{aligned} J' &= e_2 \otimes e^1 - e_1 \otimes e^2 - e_4 \otimes e^3 + e_3 \otimes e^4 \\ &= (-b\partial_2 + 2\partial_4) \otimes dx^1 + (a\partial_1 - 2\partial_3) \otimes dx^2 \\ &\quad + \left\{-\frac{1}{2}(ab-1)\partial_2 + a\partial_4\right\} \otimes dx^3 + \left\{\frac{1}{2}(ab-1)\partial_1 - b\partial_3\right\} \otimes dx^4. \end{aligned} \tag{22}$$

We now analyze if such an opposite almost complex structure J' is integrable or not.

If we write $J'\partial_i = \sum_{j=1}^{4} J'^{j}_{i}\partial_j$, then the components J'^{j}_{i} of J' are given as follows:

$$\begin{aligned} &J'^{2}_{1} = -b, \quad J'^{4}_{1} = 2, \quad J'^{1}_{2} = a, \quad J'^{3}_{2} = -2, \\ &J'^{2}_{3} = -\frac{1}{2}(ab-1), \quad J'^{4}_{3} = a, \quad J'^{1}_{4} = \frac{1}{2}(ab-1), \quad J'^{3}_{4} = -b. \end{aligned} \tag{23}$$

The components N^i_{jk} of the Nijenhuis tensor or torsion of J' are defined by

$$N^i_{jk}[J'] = 2\sum_{h=1}^{4}\left(J'^h_j \frac{\partial J'^i_k}{\partial x^h} - J'^h_k \frac{\partial J'^i_j}{\partial x^h} - J'^i_h \frac{\partial J'^h_k}{\partial x^j} + J'^i_h \frac{\partial J'^h_j}{\partial x^k}\right). \tag{24}$$

It is well known [2, p. 124] that J' is integrable if and only if all the components $N^i_{jk}[J']$ vanish.

Proposition 4. *If a and b satisfy*

$$a(x^1, x^2, x^3, x^4) - b(x^1, x^2, x^3, x^4) = 2\phi(x^3) + 2\psi(x^4), \tag{25}$$

then J' is in general not integrable.

Proof. From a straightforward calculation, we have the components of the Nijenhuis tensor explicitly as follows:

$$N^1_{12} = -4\{(a - \phi - \psi)a_2 - a_4\}$$

$$N^1_{13} = N^1_{24} = (a^2 + 1)a_1 - 2a(a_3 - 2\phi_3)$$

$$N^1_{14} = -2(a - 2\phi - 2\psi)\{(a - \phi - \psi)a_2 - a_4\}$$

$$N^1_{23} = 2a\{(a - \phi - \psi)a_2 - a_4\}$$

$$N^1_{34} = -(a^2 - 2a\phi - 2a\psi - 1)\{(a - \phi - \psi)a_2 - a_4\}$$

$$N^2_{12} = 4(a - \phi - \psi)a_1 - 4a_3 + 8\phi_3$$

$$N^2_{13} = N^2_{24} = (a^2 - 4a\phi - 4a\psi + 4\phi^2 + 8\phi\psi + 4\psi^2 + 1)a_2 - (a - 2\phi - 2\psi)a_4$$

$$N^2_{14} = 2(a - 2\phi - 2\psi)\{(a - \phi - \psi)a_1 - a_3 + 2\phi_3\}$$

$$N^2_{23} = -2a\{(a - \phi - \psi)a_1 - a_3 + 2\phi_3\}$$

$$N_{34}^{2} = (a^2 - 2a\phi - 2a\psi - 1)\{(a - \phi - \psi)a_1 - a_3 + 2\phi_3\}$$

$$N_{12}^{3} = 4a_2$$

$$N_{13}^{3} = N_{24}^{3} = -2aa_1 + 4a_3 - 8\phi_3$$

$$N_{14}^{3} = 2(a - 2\phi - 2\psi)a_2$$

$$N_{23}^{3} = -2aa_2$$

$$N_{34}^{3} = (a^2 - 2a\phi - 2a\psi - 1)a_2$$

$$N_{12}^{4} = -4a_1$$

$$N_{13}^{4} = N_{24}^{4} = -2(a - 2\phi - 2\psi)a_2 + 4a_4$$

$$N_{14}^{4} = -2(a - 2\phi - 2\psi)a_1$$

$$N_{23}^{4} = 2aa_1$$

$$N_{34}^{4} = -(a^2 - 2a\phi - 2a\psi - 1)a_1.$$

Since these components cannot vanish in general, we see that J' is not integrable.

In fact, $N_{jk}^{i}[J'] = 0$ if and only if $a_1 = a_2 = a_4 = 0$, $a_3 = 2\phi_3$.

5. Flat Norden Metrics with Nonintegrable Almost Complex Structures

In this last section, we shall state our main results. We now summarize the preceding argument as the main theorem.

Theorem 5. *Let* (M, g) *be a Walker* 4-*manifold*, *endowed with the metric as in* (13), *i.e.*,

$$g = 2dx^1 \otimes dx^3 + 2dx^2 \otimes dx^4 + a(x^1, x^2, x^3, x^4)dx^3 \otimes dx^3$$
$$+ b(x^1, x^2, x^3, x^4)dx^4 \otimes dx^4. \qquad (26)$$

Then, the opposite almost complex structure, given in (22),

$$J' = (-b\partial_2 + 2\partial_4) \otimes dx^1 + (a\partial_1 - 2\partial_3) \otimes dx^2 + \left\{-\frac{1}{2}(ab-1)\partial_2 + a\partial_4\right\} \otimes dx^3 + \left\{\frac{1}{2}(ab-1)\partial_1 - b\partial_3\right\} \otimes dx^4 \quad (27)$$

is in general not integrable, and constitutes, together with g above, an opposite almost pseudo-Hermitian structure (g, J') satisfying the J'-invariance:

$$g(J'X, J'Y) = g(X, Y). \quad (28)$$

If $a(x^1, x^2, x^3, x^4) - b(x^1, x^2, x^3, x^4) = 2\phi(x^3) + 2\psi(x^4)$ *as in* (25), *then the Walker* 4-*manifold M admits a flat opposite Norden metric g^{N-} as in* (21), *i.e.*,

$$g^{N-} = 2dx^1 \otimes dx^4 - 2dx^2 \otimes dx^3 + 2(\phi(x^3) + \psi(x^4))dx^3 \otimes dx^4, \quad (29)$$

which is J'-skew invariant:

$$g^{N-}(J'X, J'Y) = -g^{N-}(X, Y). \quad (30)$$

We shall analyze in detail the family of Walker 4-manifolds, which are characterized in the main theorem.

Case I. $a = a(x^1, x^2, x^3, x^4)$ and $b = a(x^1, x^2, x^3, x^4) - 2\phi(x^3) - 2\psi(x^4)$.

The Walker 4-manifold described in this theorem admits a nonflat opposite almost pseudo-Hermitian structure (g, J'), but admits a flat opposite Norden structure (g^{N-}, J'), with a nonintegrable almost complex structure J'.

If we restrict our attention to the case

$$a = 0, \quad b = -2\phi(x^3) - 2\psi(x^4), \quad (31)$$

then there is still one nonzero component of the curvature as follows:

$$R_{3434} = \phi_{33}(x^3). \quad (32)$$

Therefore, a Walker 4-manifold M with a metric

$$g = 2dx^1 \otimes dx^3 + 2dx^2 \otimes dx^4 + 2(\phi(x^3) + \psi(x^4))dx^4 \otimes dx^4 \tag{33}$$

has a nonzero curvature component $R_{3434} = \phi_{33}(x^3)$, and admits a *flat* opposite Norden metric (29). In this restricted case, the Nijenhuis tensor of J' has still nonzero components as follows:

$$\begin{aligned} N_{12}^2 &= -8N_{34}^2 = -N_{13}^3 = -N_{24}^3 = 8\phi_3(x^3), \\ N_{14}^2 &= -8(\phi(x^3) + \psi(x^4))\phi_3(x^3). \end{aligned} \tag{34}$$

Case II. $a = 0$ and $b = -2kx^3 - 2\psi(x^4)$ (k : constant).

This is the case that ϕ is linear in x^3, and therefore $R_{3434} = 0$. That is, g is a flat metric. However, the components of the Nijenhuis tensor in (34) are still not all zero:

$$N_{12}^2 = -8N_{34}^2 = -N_{13}^3 = -N_{24}^3 = 8k, \quad N_{14}^2 = -8(kx^3 + \psi(x^4))k. \tag{35}$$

Thus, in this Case II, we have a Walker 4-manifold M, which admits a *flat* metric

$$g = 2dx^1 \otimes dx^3 + 2dx^2 \otimes dx^4 + 2(kx^3 + \psi(x^4))dx^4 \otimes dx^4, \tag{36}$$

and also a *flat* opposite Norden metric

$$g^{N-} = 2dx^1 \otimes dx^4 - 2dx^2 \otimes dx^3 + 2(kx^3 + \psi(x^4))dx^3 \otimes dx^4, \tag{37}$$

with a nonintegrable opposite almost complex structure

$$\begin{aligned} J' &= 2(kx^3 + \psi(x^4))(\partial_2 \otimes dx^1 - \partial_3 \otimes dx^4) \\ &\quad + 2\partial_4 \otimes dx^1 - 2\partial_3 \otimes dx^2 + \frac{1}{2}\partial_2 \otimes dx^3 - \frac{1}{2}\partial_1 \otimes dx^4. \end{aligned} \tag{38}$$

Case III. $a = 0$ and $b = -2\psi(x^4)$.

If b is further independent of $x^3 (k = 0)$, then all the components of Nijenhuis tensor in (35) vanish. Thus, in this case we have a Walker

4-manifold which admits two kinds of flat metrics

$$g = 2dx^1 \otimes dx^3 + 2dx^2 \otimes dx^4 + 2\psi(x^4)dx^4 \otimes dx^4, \tag{39}$$

$$g^{N-} = 2dx^1 \otimes dx^4 - 2dx^2 \otimes dx^3 + 2\psi(x^4)dx^3 \otimes dx^4, \tag{40}$$

with an *integrable* opposite almost complex structure

$$J' = 2\psi(x^4)(\partial_2 \otimes dx^1 - \partial_3 \otimes dx^4)$$
$$+ 2\partial_4 \otimes dx^1 - 2\partial_3 \otimes dx^2 + \frac{1}{2}\partial_2 \otimes dx^3 - \frac{1}{2}\partial_1 \otimes dx^4. \tag{41}$$

We have thus exhibit examples of flat opposite Norden metrics g^{N-} with nonintegrable almost complex structures J'. As a by-product, we also have a flat almost Hermitian metric g as in (36). It may be also interesting to see if a Walker 4-manifold admits a *flat Norden metric* g^{N+} with respect to an almost complex structure J. In relation to the present issue, a recent work of Manev and Sekigawa [3] asserts that a pseudo-hyper-Kähler $4n$-manifold $(4n \geq 4)$ is a *flat neutral* pseudo-Riemannian manifold.

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