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BUFFON TYPE PROBLEMS IN ARCHIMEDEAN TILINGS I

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Abstract

In this paper, we consider the elongated triangular tiling of the plane ($(3^3, 4^2)$ Archimedean tiling) and compute the probability that a random circle or a random segment intersects a side of the lattice.

1. Introduction

An *Archimedean tessellation* (semi-regular or uniform tessellation) is a tessellation of the plane made from more than one type of regular polygon so that the same polygons surround each vertex; we can list the types of polygons as they come together at the vertex [10]. The *elongated triangular tiling* is a tiling such that three triangles and two squares come together in any vertex so it can be called a $(3^3, 4^2)$ *Archimedean tiling* (see Figure 1(a)). Many authors studied Buffon type problems for different lattices of figures or tilings and different test bodies: see for example [1-9, 13-15].

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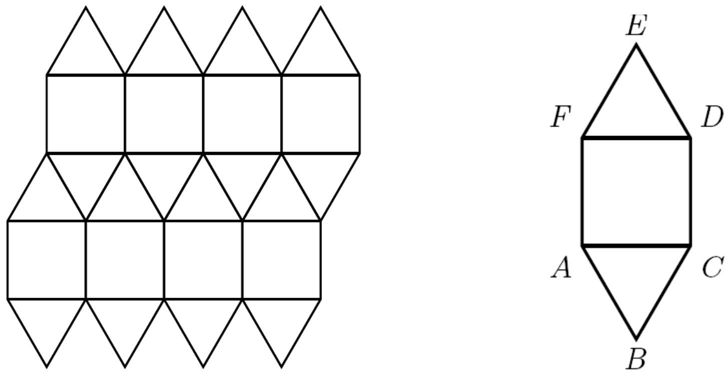


(a) Elongated triangular tiling (b) The fundamental cell T_0

Figure 1. The tiling $\mathcal{R}$.

We will study Buffon type problems for the elongated triangular tiling and two special test bodies: a circle of constant radius and a line segment of length *l*.

Let E_2 be the Euclidean plane and let $\mathcal{R}$ be the elongated triangular tiling of E_2 given in Figure 1(a). We denote by T_0 the *fundamental tile* (or cell) of $\mathcal{R}$ (see Figure 1(b)) and by T_n one of congruent copies of T_0 such that:

(1) $\bigcup_{n\in\mathbb{N}} T_n = E_2$,

(2) $\text{Int}(T_i) \cap \text{Int}(T_j) = \varnothing$, $\forall i, \ j \in \mathbb{N}$ and $i \neq j$,

(3) $T_n = \gamma_n(T_0)$, $\forall n \in \mathbb{N}$, where γ_n are the elements of a discrete subgroup of the group of motions in E_2 that leaves invariant the tiling $\mathcal{R}$.

The body T_0 can be expressed as the union of a square of side *a* and two equilateral triangles of the same side *a*.

Let us denote by *K* a convex body (which means here a compact convex set) which we shall call test body. A general problem of Buffon type can be stated as follows: “Which is the probability $p_{K,\mathcal{R}}$ that the random convex body K, or more precisely, a random congruent copy of K, meets some of the boundary points of at least one of the domains T_n ?".

We will study also the problem of the independence of the two events "The body K meets some of the boundary points of the triangles of the tile $\mathcal{R}$" and "The body K meets some of the boundary points of the squares of the tiling $\mathcal{R}$".

If we denote by $\mathcal{M}$ the set of all test bodies K whose centroid is in the interior of T_0 and by $\mathcal{N}$ the set of all test bodies K that are completely contained in the triangle ABC or in the triangle DEF or in the square $ACDF$, we have

$$p_{K,\mathcal{R}} = 1 - \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})}, \tag{1}$$

where μ is the Lebesgue measure in the plane E_2.

2. The Test Body is a Circle

Let us suppose that the test body K is a circle of diameter D. Easy geometrical considerations will lead us to distinguish between the cases $D < \frac{a}{\sqrt{3}}$, $\frac{a}{\sqrt{3}} \leq D < a$ and $D \geq a$. It is obvious that if $D \geq a$, then the circle always meets the boundary of one of the bodies T_n, so we have to study the other two cases.

Proposition 1. *The probability that the circle K of diameter D intersects the tiling $\mathcal{R}$ is given by*

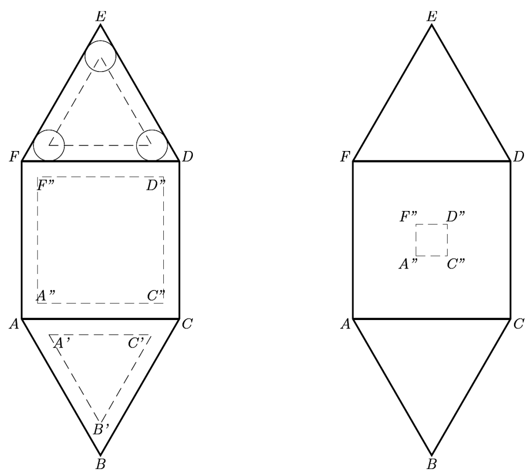
$$p_{K,\mathcal{R}} = \begin{cases} \dfrac{D\left[10a - (2 + 3\sqrt{3})D\right]}{(2 + \sqrt{3})a^2}, & \text{if } D < \dfrac{a}{\sqrt{3}}, \\[2ex] \dfrac{\sqrt{3}a^2 + 4aD - 2D^2}{(2 + \sqrt{3})a^2}, & \text{if } \dfrac{a}{\sqrt{3}} \leq D < a. \end{cases} \tag{2}$$

Proof. We compute the measures $\mu(\mathcal{M})$ and $\mu(\mathcal{N})$ with help of the elementary kinematic measure $dK = dx \wedge dy \wedge d\phi$ of E_2 (see [11, 12]), where x and y are the coordinates of the center of K (or the components of a translation), and ϕ is the angle of rotation. We have

$$\mu(\mathcal{M}) = \int_0^{\pi} d\phi \iint_{(x,y)\in T_0} dxdy = \pi \cdot \text{area}(T_0) = \pi a^2\left(1 + \frac{\sqrt{3}}{2}\right).$$

Let $\mathcal{N}_1$ be the set of circles of diameter D that are contained in the triangle ABC and $\mathcal{N}_2$ be the set of circles of diameter D that are contained in the square $ACDF$. From (1) we obtain

$$p_{K,\mathcal{R}} = 1 - \frac{2\mu(\mathcal{N}_1) + \mu(\mathcal{N}_2)}{\pi a^2\left(1 + \frac{\sqrt{3}}{2}\right)}. \tag{3}$$



(a) The case $D < \frac{a}{\sqrt{3}}$ (b) The case a $\frac{a}{\sqrt{3}} \leq D < a$

Figure 2. The case $K = \text{circle}$.

From Figure 2(a) it is easy to see that $\mu(\mathcal{N}_1)$ is π times the area of the triangle *A'B'C'* whose sides are parallel to the sides of the triangle *ABC* at distance $D/2$ from them (*A'* is the center of a disk interior to the triangle *ABC* and tangent to the sides *AB* and *AC* and so on). Since the side of the triangle is $a - D\sqrt{3}$, we have:

$$\mu(\mathcal{N}_1) = \frac{\pi\sqrt{3}}{4}(a - \sqrt{3}D)^2.$$

In the same way we obtain that

$$\mu(\mathcal{N}_2) = \pi(a - D)^2.$$

Then we have for the case $D < \frac{a}{\sqrt{3}}$

$$p_{K,\mathcal{R}} = \frac{D[10a - (2 + 3\sqrt{3})D]}{(2 + \sqrt{3})a^2}.$$

Let $\frac{a}{\sqrt{3}} \leq D < a$ (see Figure 2(b)). If the center of the circle K is in the triangle ABC, then the circle always intersects one of the sides of the triangle so that

$$\mu(\mathcal{N}_1) = \frac{\pi\sqrt{3}}{4}a^2.$$

If the center of the circle is in the square $ACDF$, then the circle does not intersect the side of the square if its center is in the square $A''C''D''F''$; since the side of this square is $a - D$, we have

$$\mu(\mathcal{N}_2) = \pi(a - D)^2,$$

and so in this case

$$p_{K,\mathcal{R}} = \frac{\sqrt{3}a^2 + 4aD - 2D^2}{(2 + \sqrt{3})a^2}.$$

Let us consider now the problem of the independence of the two events $I_1 =$ “The body K meets some of the boundary points of the triangles of the tile $\mathcal{R}$” and $I_2 =$ “The body K meets some of the boundary points of the squares of the tile $\mathcal{R}$”.

In the same way as above we compute the measure of the circles K whose center is in T_0 and intersect or a triangle or a square of $\mathcal{R}$ (or both).

Proposition 2. *The events* I_1 *and* I_2 *are dependent for any* $D \in [0, a]$.

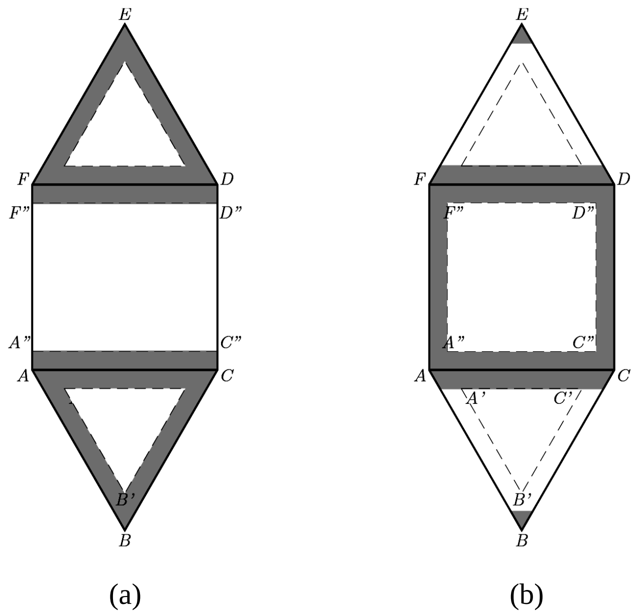


Figure 3

Proof. We have to consider three cases:

(i) Let $D < a/\sqrt{3}$. The circle K intersects the boundary of a triangle of the tiling if its center is in the gray region in Figure 3(a) whose area is

$$2\left[\frac{\sqrt{3}}{4}(a^2 - (a - D\sqrt{3})^2) + \frac{1}{2}aD\right] = 4aD - \frac{3\sqrt{3}}{2}D^2$$

so that

$$p(I_1) = \frac{8aD - 3\sqrt{3}D^2}{a^2(2+\sqrt{3})}.$$

The circle K intersects the boundary of a square of the tiling if its center is in the gray region in Figure 3(b) whose area is

$$a^2 - (a - D)^2 + 2\left[\frac{\left(2a - \frac{D}{\sqrt{3}}\right)\frac{D}{2}}{2} + \frac{D^2}{4\sqrt{3}}\right] = 3aD - D^2$$

so that

$$p(I_2) = \frac{2D(3a - D)}{a^2(2 + \sqrt{3})}.$$

In the same way we obtain:

$$p(I_1 \cap I_2) = \frac{4D}{a(2 + \sqrt{3})}.$$

In order to evaluate the independence of the two events I_1 and I_2 we have to study the equation:

$$p(I_1 \cap I_2) = p(I_1)\, p(I_2),$$

i.e.,

$$2(\sqrt{3} + 2)a^3 = D(3a - D)(8a - 3\sqrt{3}D). \tag{4}$$

Let $D = a \cdot t$ be, with $0 < t < 1/\sqrt{3}$: the equation (4) is satisfied if in such interval there exists a zero of the function

$$f(t) = 3\sqrt{3}t^3 - (9\sqrt{3} + 8)t^2 + 24t - 2\sqrt{3} - 4.$$

Since for $0 < t < 1/\sqrt{3}$ the derivative $f'(t)$ is positive, we have $f(t) < f(1/\sqrt{3}) = 3\sqrt{3} - \frac{17}{3} < 0$ the equation (4) does not have solution and the events I_1 and I_2 are dependent.

(ii) Let now $a/\sqrt{3} \le D < a\sqrt{3}/2$. In this case if the circle K has the center in the triangle, then it always intersects the triangle, moreover it intersects the triangle if its center is in the square at distance less than $D/2$ from the upper or lower side, so:

$$p(I_1) = \frac{a\sqrt{3} + 2D}{a(2 + \sqrt{3})}.$$

The probability that the circle intersects the square and the circle intersects both the triangle and the square is the same as in case (i) and so

$$p(I_2) = \frac{2D(3a - D)}{a^2(2 + \sqrt{3})}$$

and

$$p(I_1 \cap I_2) = \frac{4D}{a(2 + \sqrt{3})}.$$

As above we study the equation:

$$p(I_1 \cap I_2) = p(I_1)\, p(I_2),$$

i.e.,

$$(3a - D)(2D + a\sqrt{3}) = 2a^2(2 + \sqrt{3})$$

who has the solutions $D = a$ and $D = \frac{2 - \sqrt{3}}{2}a$ that are both outside the interval $[a/\sqrt{3},\ a\sqrt{3}/2]$ and so also in this case the events are dependent.

(iii) Finally we consider the case $a\sqrt{3}/2 \le D < a$. In this case if the center of the circle K is in the triangle, then the circle always intersects both a triangle and a square. Moreover it intersects the triangle if its center is in the square at distance less than $D/2$ from the upper or lower side; since in this case the circle intersects also the square, we have

$$p(I_1) = p(I_1 \cap I_2) = \frac{a\sqrt{3} + 2D}{a(2 + \sqrt{3})}$$

and the events are obviously dependent, since $p(I_2) < 1$.

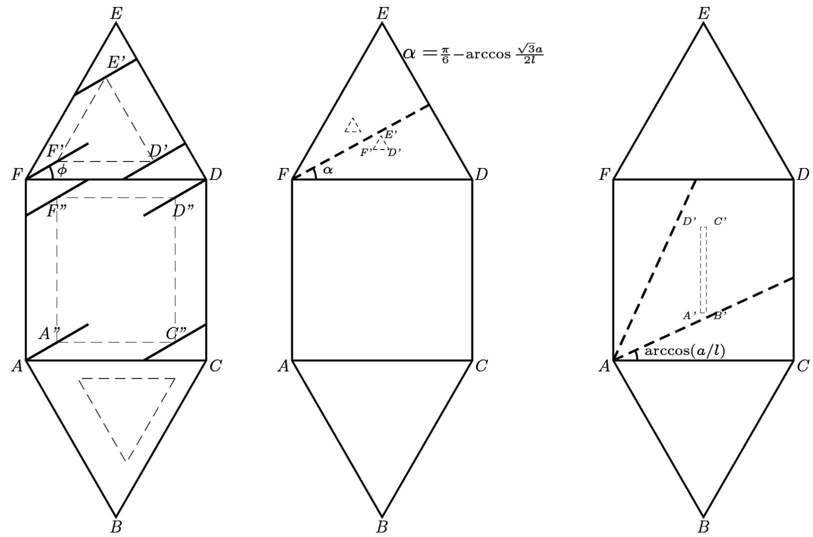
3. The Test Body is a Line Segment

Let us consider now the case K is a line segment of length l. Also in this case easy geometrical considerations give us four cases: $l < \frac{a\sqrt{3}}{2}$, $\frac{a\sqrt{3}}{2} \leq l < a$, $a \leq l < a\sqrt{2}$ and $l \geq a\sqrt{2}$. In the last case the segment always intersects the boundary of one of the bodies T_n, so we have to study the other cases. We have

Proposition 3. *The probability that the line segment K of length l intersects the tiling $\mathcal{R}$ is given by*

$$p_{K,\mathcal{R}} = \begin{cases} \dfrac{l[60a - (15 + 2\pi\sqrt{3})l]}{3\pi(2+\sqrt{3})a^2} & \text{if } l < \dfrac{a\sqrt{3}}{2} \\[2ex] \dfrac{60al - 27a\sqrt{4l^2 - 3a^2} - 15l^2 - 2\sqrt{3}\pi l^2 + 6(3a^2 + 2l^2)\sqrt{3}\arccos\dfrac{a\sqrt{3}}{2l}}{3(2+\sqrt{3})\pi a^2} & \text{if } \dfrac{a\sqrt{3}}{2} \leq D < a \\[2ex] \dfrac{4a^2 - 8a\sqrt{l^2 - a^2} + 2l^2 + \sqrt{3}\pi a^2 + 8a^2\arccos(a/l)}{(2+\sqrt{3})\pi a^2} & \text{if } a \leq D < a\sqrt{2}. \end{cases} \quad (5)$$

Proof. (i) Let us consider the case $l < \frac{a\sqrt{3}}{2}$. We compute first the measure $\mu(\mathcal{N}_1)$ of the set $\mathcal{N}_1$ of all line segments of length l contained in the triangle DEF. For a fixed angle $\phi \in \left[0, \frac{\pi}{3}\right[$ we denote by (see Figure 4(a))



(a) $l < \frac{a\sqrt{3}}{2}$ (b) $\frac{a}{\sqrt{3}} \leq l < a$ (c) $a \leq l < a\sqrt{2}$

Figure 4. The case K = line segment.

- F' the midpoint (in DEF) of the line segment of length l with one endpoint in F that makes an angle ϕ with FD.

- D' the midpoint of the line segment of length l with endpoints on DF and DE that makes an angle ϕ with FD.

- E' the midpoint of the line segment of length l with endpoints on EF and DE that makes an angle ϕ with the direction of FD.

We compute

$$\mathrm{area}(D'E'F') = \frac{\sqrt{3}}{4}\left[a - \frac{2l}{\sqrt{3}}\sin\left(\frac{2}{3}\pi - \phi\right)\right]^2$$

and, by symmetry, we obtain

$$\mu(\mathcal{N}_1) = 3\int_0^{\pi/3} \mathrm{area}(D'E'F')\,d\phi = \int_0^{\pi/3} \frac{\sqrt{3}}{4}\left[a - \frac{2l}{\sqrt{3}}\sin\left(\frac{2}{3}\pi - \phi\right)\right]^2 d\phi$$

$$= \frac{3\sqrt{3}\pi a^2 - 36al + (9 + 2\sqrt{3}\pi)l^2}{12}. \qquad (6)$$

In the same way, if $\psi \in \left[0, \frac{\pi}{2}\right[$, then we obtain for the set $\mathcal{N}_2$ of the line segments contained in the square *ACDF*

$$\text{area}(A' C' D' F') = (a - l \sin \psi)(a - l \cos \psi)$$

and so, by symmetry, we have

$$\mu(\mathcal{N}_2) = 2\int_0^{\pi/2} \text{area}(A' C' D' F') d\psi = \int_0^{\pi/2} (a - l \cos \psi)(a - l \sin \psi) d\psi$$

$$= \pi a^2 - 4al + l^2$$

and so

$$\mu(\mathcal{N}) = \left(1 + \frac{\sqrt{3}}{2}\right)\pi a^2 - 10al + \left(\frac{5}{2} + \frac{\pi}{\sqrt{3}}\right) l^2.$$

Hence we have if $l < \frac{a\sqrt{3}}{2}$

$$p_{K,\mathcal{R}} = \frac{l[60a - (15 + 2\pi\sqrt{3})l]}{3\pi(2 + \sqrt{3})a^2}. \tag{7}$$

(ii) Let now $\frac{a\sqrt{3}}{2} \leq l < a$. With reference to Figure 4(b) it is easy to see that the line segment can be contained in the triangle *EFD* only if the angle $\phi \in [0, \pi/3[$ between the line segment and the side *FD* satisfies

$$0 \leq \phi < \frac{\pi}{6} - \arccos\frac{\sqrt{3}a}{2l} \quad \text{or} \quad \frac{\pi}{6} + \arccos\frac{\sqrt{3}a}{2l} < \phi < \frac{\pi}{3}.$$

So the measure of the line segments completely contained in the triangle *EFD* is, by symmetry,

$$\mu(\mathcal{N}_1) = 6\int_0^{\frac{\pi}{6} - \arccos\frac{\sqrt{3}a}{2l}} \frac{\sqrt{3}}{4}\left[a - \frac{2l}{\sqrt{3}} \sin\left(\frac{2}{3}\pi - \phi\right)\right]^2 d\phi$$

$$= \frac{9a(3\sqrt{4l^2 - 3a^2} - 4l) + 3\sqrt{3}\pi a^2 + (9 + 2\sqrt{3}\pi)l^2 - 6\sqrt{3}(3a^2 + 2l^2)\arccos\frac{a\sqrt{3}}{2l}}{12}.$$

The measure of the line segment completely contained in the square *ACDF* is the same as in the case above:

$$\mu(\mathcal{N}_2) = \pi a^2 - 4al + l^2.$$

Hence we have if $\frac{a\sqrt{3}}{2} \leq l < a$

$$p_{K,\mathcal{R}} = \frac{\begin{array}{c}60al - 27a\sqrt{4l^2 - 3a^2} - 15l^2 - 2\sqrt{3}\pi l^2 \\ + 6(3a^2 + 2l^2)\sqrt{3}\arccos\frac{a\sqrt{3}}{2l}\end{array}}{3(2+\sqrt{3})\pi a^2}. \qquad (8)$$

(iii) Let now $a \leq l < a\sqrt{2}$. With reference to Figure 4(c) it is easy to see that if the centroid of the line segment is in the triangle *EFD* the line segment always meets one of the side of the triangle and can be contained in the square *ACDF* only if the angle $\phi \in [0, \pi/2[$ between the line segment and the side *AC* satisfies $\arccos\frac{a}{l} < \phi \leq \frac{\pi}{2} - \arccos\frac{a}{l}$.

So the measure of the line segments completely contained in the square *ACDF* is given by:

$$\mu(\mathcal{N}_2) = 2\int_{\arccos(a/l)}^{\frac{\pi}{2}-\arccos\frac{a}{l}} (a - l\cos\phi)(a - l\sin\phi)d\phi$$

$$= 4a\sqrt{l^2 - a^2} - l^2 + (\pi - 2)a^2 - 4a^2\arccos(a/l).$$

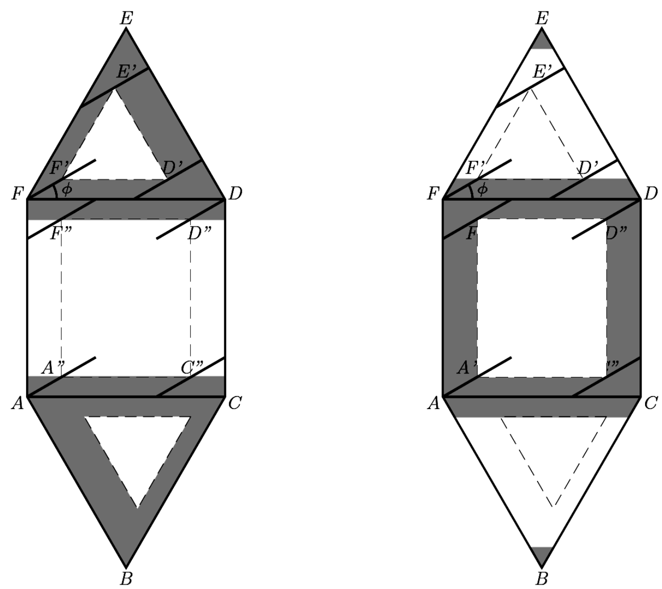
Hence we have if $a \leq l < a\sqrt{2}$

$$p_{K,\mathcal{R}} = \frac{(4a^2 - 8a\sqrt{l^2 - a^2} + 2l^2 + \sqrt{3}\pi a^2 + 8a^2\arccos(a/l))}{(2+\sqrt{3})\pi a^2}. \qquad (9)$$

Let us consider now the problem of the independence of the two events I_1 = "The body *K* meets some of the boundary points of the triangles of the tile $\mathcal{R}$" and I_2 = "The body *K* meets some of the boundary points of the squares of the tile $\mathcal{R}$".

In the same way as above we compute the measure of the line segments K whose center is in T_0 and intersect or a triangle or a square of $\mathcal{R}$ (or both).

Proposition 4. *The events* I_1 *and* I_2 *are independent only for* $l \neq \bar{t}a$, *where* $\bar{t} \approx 0.958473$ *is the unique solution of equation* (12) *in the interval* $a\frac{\sqrt{3}}{2} \leq l < a$.



(a) (b)

Figure 5

Proof. Using the same notation of Proposition 3, let us note that if the centroid of the line segment is in a triangle, then the segment intersects the triangle if the centroid is not in the region $\mathcal{N}_1$ and, analogously, it intersects a square if the centroid is in the square and not in the region $\mathcal{N}_2$. Moreover it intersects a square if the centroid is in a triangle at a distance less than $\frac{l}{2}\sin\phi$ from the side of a square and it intersects a triangle if its centroid is in a square at a distance less than $\frac{l}{2}\sin\phi$ from the upper or the lower side of a square.

Finally we remember that $p(I_1 \cap I_2) = p(I_1) + p(I_2) - p(I_1 \cup I_2)$ and $p(I_1 \cup I_2)$ is the probability we found in Proposition 3.

Then we have to consider four cases: $0 \leq l < a\sqrt{3}/2$; $a\sqrt{3}/2 \leq l < a$; $a \leq l < a\sqrt{2}$; $l \geq a\sqrt{2}$

(i) Let $0 \leq l < a\sqrt{3}/2.$ In this case the measure of the line segments with centroid in a square that intersect a triangle is $2\int_0^\pi \frac{al}{2}\sin\phi d\phi = 2al$ and this is also the measure of the line segments with centroid in a triangle that intersects a square.

So we obtain:

$$p(I_1) = \frac{l(48a - l(9 + 2\sqrt{3}\pi))}{3(2 + \sqrt{3})a^2\pi},$$

$$p(I_2) = \frac{2(6a - l)l}{(2 + \sqrt{3})a^2\pi}$$

and

$$p(I_1 \cap I_2) = \frac{8l}{(2 + \sqrt{3})a\pi}.$$

In order to evaluate the independence of the events I_1 and I_2 we have to study the equation:

$$p(I_1)\,p(I_2) = p(I_1 \cap I_2),$$

i.e.,

$$l(6a - l)(48a - (9 + 2\sqrt{3}\pi)l) = 12\pi a^3(2 + \sqrt{3}). \tag{10}$$

Let $D = a \cdot t$ be, with $0 < t < 1/\sqrt{3}$: the equation (10) is satisfied if in such interval there exists a zero of the function

$$f(t) = t(6-t)(48 - (9 + 2\sqrt{3}\pi)t) - 12\pi(2+\sqrt{3}).$$

Since in the interval $[0, \sqrt{3}/2]$ the function $f(t)$ is increasing, we obtain $f(t) \leq f(\sqrt{3}/2) < 0$ and so the events I_1 and I_2 are dependent.

(ii) Let now $a\sqrt{3}/2 \leq l < a$. In this case the only difference with the previous one is that now a line segment with center in a triangle intersect always a square if $\sin\phi \geq \dfrac{a\sqrt{3}}{2l}$ and, if $\sin\phi < \dfrac{a\sqrt{3}}{2l}$, then the area of the region in which the center have to be for the line segment intersects a square is, as before, $al\sin\phi$ so:

$$p(I_1) = \frac{6\sqrt{3}(3a^2 + 2l^2)\arccos\dfrac{a\sqrt{3}}{2l} - 3a(9\sqrt{4l^2 - 3a^2} - 16l) - (9 + 2\sqrt{3}\pi)l^2}{3a^2\pi(2+\sqrt{3})},$$

$$p(I_2) = \frac{12al - 2a\sqrt{4l^2 - 3a^2} - 2l^2 + 2\sqrt{3}a^2\arccos\dfrac{a\sqrt{3}}{2l}}{a^2\pi(2+\sqrt{3})}$$

and

$$p(I_1 \cap I_2) = \frac{8l - 2\sqrt{4l^2 - 3a^2} + 2\sqrt{3a}\arccos\dfrac{\sqrt{3a}}{2l}}{\pi a(\sqrt{3}+2)}.$$

In order to evaluate the independence of the events I_1 and I_2 we have to solve the equation

$$p(I_1)\,p(I_2) = p(I_1 \cap I_2),$$

i.e.,

$$\left(-3a(-16l+9\sqrt{-3a^2+4l^2})\right.$$

$$\left.-l^2(9+2\sqrt{3}\pi)+6\sqrt{3}(3a^2+2l^2)\arccos\left(\frac{\sqrt{3}a}{2l}\right)\right)$$

$$\times\left(6al-l^2-a\sqrt{-3a^2+4l^2}+\sqrt{3}a^2\arccos\left(\frac{\sqrt{3}a}{2l}\right)\right)$$

$$=3(2+\sqrt{3})a^3\pi\left(4l-\sqrt{-3a^2+4l^2}+\sqrt{3}a\arccos\left(\frac{\sqrt{3}a}{2l}\right)\right). \qquad (11)$$

By setting as above $l = t \cdot a$, we obtain

$$\left(-48+(9+2\sqrt{3}\pi)t^2+27\sqrt{-3+4t^2}-6\sqrt{3}(3+2t^2)\arccos\frac{\sqrt{3}}{2t}\right)$$

$$\times\left(-6t+t^2+\sqrt{-3+4t^2}-\sqrt{3}\arccos\frac{\sqrt{3}}{2t}\right)$$

$$=3(2+\sqrt{3})\pi\left(4t-\sqrt{-3+4t^2}+\sqrt{3}\arccos\frac{\sqrt{3}}{2t}\right). \qquad (12)$$

With a computer simulation (for example using a mathematical software like Maxima or Mathematica) it is possible to see that the above equation has a unique solution in the interval $a\frac{\sqrt{3}}{2} \leq l < a$ given by $l \approx 0.958473a$.

(iii) Let now $a \leq l < a\sqrt{2}$. In this case a line segment with the centroid in a square intersects always a triangle if $\sin\phi \geq \frac{a}{l}$ and, if $\sin\phi < \frac{a}{l}$, then the area of the region in which the center has to be for the line segment to intersect a square is, as before, $al\sin\phi$ so:

$$p(I_1) = \frac{4l - 4\sqrt{l^2-a^2} + \sqrt{3}a\pi + 4a\arccos\frac{a}{l}}{2a\pi + \sqrt{3}a\pi},$$

$$p(I_2) = \frac{2\left(2a^2 + 2al + l^2 - 4a\sqrt{l^2 - a^2} - a\sqrt{4l^2 - 3a^2} + 4a^2 \arccos\frac{a}{l} + \sqrt{3}a^2 \arccos\frac{\sqrt{3}a}{2l}\right)}{(2+\sqrt{3})a^2\pi}$$

and

$$p(I_1 \cap I_2)$$

$$= \frac{8l - 4\sqrt{l^2 - a^2} - 2\sqrt{4l^2 - 3a^2} + 4a\arccos\frac{a}{l} + 2\sqrt{3}a\arccos\frac{\sqrt{3}a}{2l}}{2a\pi + \sqrt{3}a\pi}.$$

Also in this case in order to solve the equation $p(I_1)\,p(I_2) - p(I_1 \cap I_2) = 0$ we use a computer simulation and we find that in the interval under consideration we have $p(I_1)\,p(I_2) > p(I_1 \cap I_2)$ and so the events I_1 and I_2 are dependent in this case.

(iv) Finally let us consider $l \geq a\sqrt{2}$. As above we find:

$$p(I_1) = \frac{4l - 4\sqrt{l^2 - a^2} + \sqrt{3}a\pi + 4a\arccos\frac{a}{l}}{2a\pi + \sqrt{3}a\pi}$$

$$= 1 - \frac{2\left(2\sqrt{l^2 - a^2} - 2l + a\pi - 2a\arccos\frac{a}{l}\right)}{(2+\sqrt{3})a\pi},$$

$$p(I_2) = \frac{4l - 2\sqrt{4l^2 - 3a^2} + 2a\pi + 2\sqrt{3}a\arccos\frac{\sqrt{3}a}{2l}}{2a\pi + \sqrt{3}a\pi}$$

$$= 1 - \frac{2\left(2\sqrt{l^2 - a^2} - 2l + a\pi - 2a\arccos\frac{a}{l}\right)}{(2+\sqrt{3})a\pi}$$

and

$$p(I_1 \cap I_2)$$

$$= \frac{2\left(4l - 2\sqrt{l^2 - a^2} - \sqrt{4l^2 - 3a^2} + 2a \arccos \frac{a}{l} + \sqrt{3}a \arccos \frac{\sqrt{3}a}{2l}\right)}{(2 + \sqrt{3})a\pi}$$

$$= 1 - 2 \frac{2\left(2\sqrt{l^2 - a^2} - 2l + a\pi - 2a \arccos \frac{a}{l}\right)}{(2 + \sqrt{3})a\pi}.$$

Since $(1 - \alpha)^2 - (1 - 2\alpha) > 0,$ we obtain also in this case $p(I_1)\, p(I_2) > p(I_1 \cap I_2)$ and the events I_1 and I_2 are dependent.

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