Far East Journal of Mathematical Sciences (FJMS)

Published Online: December 2013
Available online at http://pphmj.com/journals/fjms.htm
Volume 81, Number 2, 2013, Pages 245-252

ON MAXIMAL α-IDEALS IN ORDERED SEMIGROUPS

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Abstract

It is known that every maximal ideal in a commutative ring with identity is prime; this result is also valid for any commutative ordered semigroup with identity and for any commutative semigroup with identity as well. In this paper, we show that every maximal α-ideal in a commutative, positive ordered semigroup is prime.

1. Preliminaries

Algebraic properties of ordered semigroups have been studied, for an example, the homomorphism theorems for ordered groupoids and ordered semigroups have been considered by Kehayopulu et al. [7] after considering the ideals in ordered groupoids. Relations between maximal ideals and prime ideals in any ring have been widely studied by many authors; one of the important results is that every maximal ideal in any commutative ring with identity is prime [3, p. 128]. Schwarz [10] showed that this result is also valid

Received: September 25, 2013; Revised: October 7, 2013; Accepted: October 16, 2013
2010 Mathematics Subject Classification: 06F05.
Keywords and phrases: commutative ordered semigroup, positive ordered semigroup, maximal bi-ideal, maximal α-ideal, prime.
Communicated by K. P. Shum

for any commutative semigroup with identity. Recently, Kehayopulu et al. [6] proved that the result holds for any commutative ordered semigroup with identity as well.

In [9], the authors introduced and studied α-ideals and generalized α-ideals in semigroups; the results obtained generalized the results on (m, n)-ideals introduced and described by Lajos [8] and on bi-ideals.

In this paper, we consider α-ideals in any ordered semigroup. We prove that, in a commutative positive ordered semigroup with identity, every maximal α-ideal is prime.

If X is any nonempty set, then the *free monoid* over an alphabet X will be denoted by X^*.

Let $(S, \cdot)$ be any semigroup. If A and B are nonempty subsets of S, then we write AB for the set of all elements ab of S with $a \in A$ and $b \in B$. The set of all subsets of S will be denoted by $P(S)$. For any $\alpha \in \{0, 1\}^*$, a mapping $f_\alpha^S : P(S) \to P(S)$ is defined by:

(i) $f_\alpha^S(A) = \varnothing$ if α is the empty word;

(ii) $f_\alpha^S(A) = A_1A_2 \cdots A_n$ if $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$, $\alpha_i \in \{0, 1\}$, where

$$A_i = \begin{cases} A, & \text{if } \alpha_i = 1; \\ S, & \text{if } \alpha_i = 0. \end{cases}$$

Hereafter, we let

$$\Lambda = \{0, 1\}^* \backslash \{1\}^*.$$

If $\alpha \in \Lambda$ (hence α is not the empty word) such that $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$, where $\alpha_i \in \{0, 1\}$, then the *length* of α, denoted by $l(\alpha)$, is defined to be n.

Lemma 1.1 [2, 9]**.** *Let* $(S, \cdot)$ *be any semigroup. If A and B are nonempty subsets of S and* $\alpha, \beta \in \Lambda$, *then the following conditions hold*:

(1) $A \subseteq B \Rightarrow f_\alpha^S(A) \subseteq f_\alpha^S(B)$;

(2) $f_{\alpha\beta}^S(A) = f_\alpha^S(A) f_\beta^S(A)$;

(3) $Af_\alpha^S(A) \subseteq f_\alpha^S(A),\ f_\alpha^S(A)A \subseteq f_\alpha^S(A)$;

(4) $f_\alpha^S(A) f_\alpha^S(A) \subseteq f_\alpha^S(A)$;

(5) $f_\alpha^S(A \cup f_\alpha^S(A)) \subseteq f_\alpha^S(A)$;

(6) $f_\alpha^S(A \cup A^2 \cup \ldots \cup A^k \cup f_\alpha^S(A)) \subseteq f_\alpha^S(A)$ *for any positive integer* k.

A semigroup $(S, \cdot)$ together with a partial order $\leq$ that is *compatible* with the semigroup operation, meaning that for $x, y, z \in S$,

$$x \leq y \Rightarrow zx \leq zy,\ xz \leq yz,$$

is called an *ordered semigroup* [1].

By a *subsemigroup* of an ordered semigroup $(S, \cdot, \leq)$, we mean a nonempty subset A of S such that $AA \subseteq A$, i.e., that $xy \in A$ for all $x, y \in A$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. If A is a nonempty subset of S, then we write

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

For any nonempty subsets A and B of S, we have (1) $A \subseteq B$ implies $(A] \subseteq (B]$, (2) $(A](B] \subseteq (AB]$ and (3) $((A]] = (A]$.

Lemma 1.2 [13]**.** *Let* $(S, \cdot, \leq)$ *be an ordered semigroup*. *If* A *is a nonempty subset of* S *and* $\alpha \in \Lambda$, *then*

$$f_\alpha^S((A]) \subseteq (f_\alpha^S(A)].$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. A subsemigroup A of S is called an α-*ideal* of S if the following conditions hold:

(i) $f_\alpha^S(A) \subseteq A$;

(ii) $(A] = A$, i.e., if $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

This concept was defined in [13] to be a generalization of bi-ideals in ordered semigroups. This is because if $\alpha = 101$, then *A* is a *bi-ideal* [5] of *S*.

Let $(S, \cdot, \leq)$ be an ordered semigroup. It was proved in [13] that the principal α-ideal generated by a nonempty subset *A* of *S* is of the form

$$(A)_\alpha = (A \cup A^2 \cup \cdots \cup A^{l(\alpha)-1} \cup f_\alpha^S(A)]. \tag{1}$$

Theorem 1.3. *Let* $(S, \cdot, \leq)$ *be any ordered semigroup. If A is a nonempty subset of S and* $\alpha = 1\beta$ *for some* $\beta \in \Lambda$, *then* $(f_\alpha^S(A)]$ *is an* α*-ideal of S.*

Proof. Let *A* be a nonempty subset of *S*. By Lemma 1.1, we have

$$(f_\alpha^S(A)](f_\alpha^S(A)] \subseteq (f_\alpha^S(A)\, f_\alpha^S(A)] \subseteq (f_\alpha^S(A)].$$

Thus, $(f_\alpha^S(A)]$ is a subsemigroup of *S*. We will show that

$$f_\alpha^S((f_\alpha^S(A)]) \subseteq (f_\alpha^S(A)].$$

By Lemma 1.2, it suffices to show that

$$f_\alpha^S(f_\alpha^S(A)) \subseteq f_\alpha^S(A). \tag{2}$$

We assert this condition by induction on the length of α. Clearly, if $l(\alpha) = 1$, then (2) holds. Assume that $l(\alpha) > 1$ and (2) holds for any γ with $l(\gamma) = l(\alpha) - 1$. Since $\alpha = 1\beta$ with $\beta \in \Lambda$ and Lemma 1.1, we have

$$\begin{aligned} f_\alpha^S(f_\alpha^S(A)) &= f_1^S(f_\alpha^S(A))\, f_\beta^S(f_\alpha^S(A)) \\ &= f_\alpha^S(A)\, f_\beta^S(Af_\beta^S(A)) \\ &\subseteq f_\alpha^S(A)\, f_\beta^S(A) \end{aligned}$$

$$= f_1^S(A)\, f_\beta^S(A)\, f_\beta^S(A)$$

$$= f_1^S(A)\, f_\beta^S(A)$$

$$= f_\alpha^S(A).$$

Finally, since $((f_\alpha^S(A)]] = (f_\alpha^S(A)],$ so $(f_\alpha^S(A)]$ is an α-ideal of *S*. This completes the proof.

2. Main Results

Let $(S, \cdot, \leq)$ be any ordered semigroup, and let $\alpha \in \Lambda.$ An α-ideal *A* of *S* is said to be *prime* if for all $x, y \in S,$ $xy \in A$ implies $x \in A$ or $y \in A.$ An α-ideal *A* of *S* is said to be *maximal* if for any α-ideal *B* of *S*, $A \subseteq B \subseteq S$ implies $A = B$ or $B = S.$

An ordered semigroup $(S, \cdot, \leq)$ is said to be *positive* [11] if for all $x \in S,$ $x < x^2.$ The notation $x < x^2$ stands for $x \leq x^2,$ but $x \neq x^2.$

We now prove the main result:

Theorem 2.1. *Let* $(S, \cdot, \leq)$ *be any commutative*, *positive ordered semigroup with identity*. *Let* $\alpha \in \Lambda.$ *If M is a maximal* α*-ideal of S*, *then M is prime*.

Proof. Let *e* be an identity element of *S*. Let *r* be the number of occurrences of 1 in α. Assume that *M* is a maximal α-ideal of *S*. Since *S* is commutative, it follows that

$$S^{l(\alpha)-r}M^r = f_\alpha^S(M). \tag{3}$$

To show that *M* is prime, we let $x, y \in M$ be such that $xy \in M$ and $x \notin M.$ Then

$$M \subset M \cup \{x\} \subseteq (M \cup \{x\})_\alpha \subseteq S.$$

By the maximality of M, we have $S = (M \cup \{x\})_{\alpha}$. Since S is commutative, we have

$$S^{l(\alpha)-r}(M \cup \{x\})^{r} = f_{\alpha}^{S}(M \cup \{x\}). \tag{4}$$

Using equations (1) and (4),

$$S = ((M \cup \{x\}) \cup (M \cup \{x\})^{2} \cup \cdots \cup (M \cup \{x\})^{l(\alpha)-1}$$

$$\cup S^{l(\alpha)-r}(M \cup \{x\})^{r}].$$

We consider the following three cases:

Case 1. $e \leq u$ for some $u \in M \cup \{x\}$. Then $y \leq uy$. If $u = x$, then $y \leq xy \in M$, and hence $y \in M$. Suppose that $u \in M$. By (3), we get

$$y \leq uy \leq u^{2}y \leq \cdots \leq yu^{r} \in S^{l(\alpha)-r}M^{r} \subseteq M.$$

Thus, $y \in M$.

Case 2. $e \leq u \in (M \cup \{x\})^{k}$ for some positive integer $1 < k \leq l(\alpha) - 1$. Then $e \leq u_1 u_2 \cdots u_k$ for some $u_1, u_2, ..., u_k \in M \cup \{x\}$, and hence

$$y \leq u_1 u_2 \cdots u_k y.$$

If $u_i \in M$ for some $1 \leq i \leq k$, then we proceed in the same manner as Case 1. Suppose that $u_1 = u_2 = \cdots = u_k = x$. Since S is positive, we have

$$y \leq y^{k} \leq (xy)^{k} \in M.$$

Then $y \in M$.

Case 3. $e \leq u \in f_{\alpha}^{S}(M \cup \{x\})$. By (4), there exist $a_1, a_2, ..., a_{l(\alpha)-r} \in S$ and $z_1, z_2, ..., z_r \in M \cup \{x\}$ such that

$$y \leq a_1 a_2 \cdots a_{l(\alpha)-1} z_1 z_2 \cdots z_r y.$$

If $z_i \in M$ for some $1 \le i \le r$, then we proceed in the same manner as Case 1. Assume that $z_1 = z_2 = \cdots = z_r = x$. Since S is positive,

$$y \le y^r \le a_1 a_2 \cdots a_{l(\alpha)-1}(xy)^r \in M.$$

Then $y \in M$. This completes the proof. $\square$

The following generalize the concept of prime bi-ideal in any semigroup introduced and described by Shabir and Kanwal [12]. A bi-ideal A of an ordered semigroup $(S, \cdot, \le)$ is said to be *prime* if for all $x, y \in A$, $xy \in A$ implies $x \in A$ or $y \in A$.

By Theorem 2.1, if we take α to be 101, then we have the following corollary:

Corollary 2.2. *Let* $(S, \cdot, \le)$ *be a commutative, positive ordered semigroup with identity. If M is a maximal bi-ideal of S, then M is prime.*

References

[1] G. Birkhoff, Lattice Theory, 25, Rhode Island, American Mathematical Society Colloquium Publications, Am. Math. Soc., Providence, 1984.

[2] F. Catino, On complete α-ideals in semigroups, Czechoslovak Math. J. 40 (1990), 155-158.

[3] I. N. Herstein, Topics in Algebra, Blaisdell Publishing Company, New York, 1964.

[4] N. Kehayopulu, On weakly prime ideals of ordered semigroups, Math. Japonica 35(6) (1990), 1051-1056.

[5] N. Kehayopulu, On completely regular poe-semigroups, Math. Japonica 37(1) (1992), 123-130.

[6] N. Kehayopulu, J. Ponizovskii and M. Tsingelis, A note on maximal ideals in ordered semigroups, Algebra Discrete Math. 2003, no. 1, 32-35.

[7] N. Kehayopulu, M. Tsingelis and K. P. Shum, Homomorphism theorems for ordered groupoids, Algebras Groups Geom. 22 (2005), 25-36.

[8] S. Lajos, Generalized ideals in semigroups, Acta. Sci. Math. 22 (1961), 217-222.

[9] M. M. Miccoli and B. Pondelicek, On α-ideals and generalized α-ideals in semigroups, Czechoslovak Math. J. 39 (1989), 522-527.

[10] Št. Schwarz, Prime ideals and maximal ideals in semigroups, Czechoslovak Math. J. 19 (1969), 72-79.

[11] T. Saitô, Ordered inverse semigroups, Trans. Amer. Math. Soc. 153 (1971), 99-138.

[12] M. Shabir and N. Kanwal, Prime bi-ideals of semigroups, Southeast Asian Bull. Math. 31 (2007), 757-764.

[13] P. Towwan and T. Changphas, On complete α-ideals in ordered semigroups, Int. J. Pure Appl. Math. 86 (2013), 465-470.