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METRICAL DISTORTIONS AND GEOMETRIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Metrical distortions are introduced as generalizations of diffeomorphisms as point set transformations within Riemannian geometry. Especially, synthetical coordinates as an explicit Euclidean coordinate system, are derived from the natural affine space over the manifold. As an immediate consequence, Riemann-Lebesgue integration is generalized and a natural integration theory for a special type of partial differential equations is established. As a substantial result, the naming "rubber sheet geometry" for topology is justified.

I. Introduction

This paper intends to state a detailed analysis of metrical distortions and synthetical coordinates and derives a natural integration theory for geometric PDEs

$$Df(x) = \varkappa(f(x))$$

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as immediate consequence (Theorem 14). Furthermore, the Riemann-Lebesgue integration theory, in the sense of a generalization of the transformation theorem is extended (Theorem 9). The integration of differential forms becomes, thereby, a separate meaning. By metrical distortions, the differential topology of a manifold can be understood intuitively as the geometry of invariants which are stable which respect to "rubber sheet transformations" (Remark 10(III)).

Parametric manifolds, introduced by Riemann, were unified under one definition, especially, by the work of Whitney [4], who showed that the type of submanifolds of the Euclidean space and the type of abstract manifolds, without any exterior space, built the same class in a differential topological sense, i.e., without respecting a Riemannian metric. The geodesic system of the Riemannian manifold provides a natural analysis of the shortest connection by ODE methods.

The affine space over the manifold *M*, isomorph to the double tangential space TT_pM, with an arbitrary point $p \in M$, provides a natural Euclidean coordinate system in a curved Riemannian manifold. The only thing that has to be considered for a rigorous treatment is that the space exists consistently, and that points in the Euclidean space and in the manifold can be identified one to one by the geodesic system. The thereby obtained mapping is, as a homeomorphism part of the pseudo group structure of differentiable chart transformations and defines a metrical distortion, a construction which seems to be the most natural thing in the world, since, physically one can metrically distort a rubber sheet and can take stock of the associated point set transformation.

Then this surprisingly simple construction leads to many deep consequences. Let alone that the integration of geometric PDEs is possible, the natural and powerful applications of metrical distortions are their application to Itô diffusions [7, 8]

$$Y_t^i = Y_0^i + \int_0^t b^i(s, Y_s)ds + \int_0^t \varkappa^i_j \circ O^j_k(s, Y_s)dX_s^k, \quad i = 1, ..., n,$$

where the $\varkappa$ is defined in Proposition 6(I) and O is a Euclidean orthonormal mapping such that

$$(\varkappa \circ O) \circ (\varkappa \circ O)^T = \varkappa \circ \varkappa^T$$

is the reciprocal of the metrical tensor. The intrinsic spin torsion degree of freedom, established by O, has a natural geometric orthonormal principal bundle structure and allows a natural spin orbit coupling theory [8].

II. Metrical Distortions

For simplicity, let

$$(\mathbb{R}^n, (g_{ij})^{ij}) = (\mathbb{R}^n, \langle ., . \rangle^g) \tag{1}$$

denote a geodesically complete n-dimensional Riemannian manifold, homeomorphic to $\mathbb{R}^n$ (see [1]). Every point p in the Riemannian manifold $(\mathbb{R}^n, (g_{ij})^{ij})$, gives rise to the *affine Riemannian normal coordinate system* by the choice of a basis $(b_1, ..., b_n) \in T_p\mathbb{R}^n$, which can be assumed to be positively oriented without restriction, i.e.,

$$\exp_p(v^i b_i) : T_p\mathbb{R}^n \to \mathbb{R}^n, \tag{2}$$

= endpoint $c(1)$ of the (unique) arc length proportional geodesic c, from p with tangential vector $v^i b_i$ in p (see [1]).

$\exp_p^{-1}$ is the transformation in the normal coordinate system with origin p, which is given abstract by $T_p\mathbb{R}^n$ and concretely, i.e., in a chart, by the choice of the basis via

$$v^i b_i \to (v^1, ..., v^n)^T \in \mathbb{R}^n. \tag{3}$$

If the basis $b_1, ..., b_n = n_1, ..., n_n$ is orthonormal, then the coordinate system is called *Riemannian orthonormal coordinate system* or *orthonormal* *coordinate system*. Affine normal coordinates and orthonormal normal coordinates are connected by a global linear mapping.

The solution of the problem in (2) is obtained by a treatment of the *geodesic ODE*

$$\ddot{c}^k(t) + \Gamma_{ij}^k(c(t))\dot{c}^j(t)\dot{c}^i(t) = 0, \quad k = 1, \ldots, n, \tag{4}$$

with the geodesic $c : [0, \infty] \to \mathbb{R}^n$ and the Christoffel symbols Γ and $\dot{c} := \frac{d}{dt}c$.

Proposition 1. *In affine normal coordinates, the tangential space* $T_p\mathbb{R}^n$ *and the manifold as coordinate system are naturally identified by the choice of the basis in* (3). *Moreover, the differential*

$$D_0 \exp_p : T_0T_p\mathbb{R}^n = T_pT_p\mathbb{R}^n \to T_p\mathbb{R}^n \tag{5}$$

is a natural identification. We can choose $(D_0 \exp_p)^{-1}b_1, \ldots, (D_0 \exp_p)^{-1}b_n$ *as a basis of* $T_pT_p\mathbb{R}^n$.

$\exp_p^{-1}$, *as a coordinate change, induces the trivial metric in* $T_0\mathbb{R}^n$, *where* $\mathbb{R}^n$ *is the coordinate space via* (3), *where an orthonormal basis was chosen*.

The differential

$$(D_0 \exp_p)^{-1} : (T_p\mathbb{R}^n, \langle.,.\rangle_{T_p\mathbb{R}^n}^{eucl}) \to (T_pT_p\mathbb{R}^n, \langle.,.\rangle_{T_pT_p\mathbb{R}^n}^{eucl})$$

is a natural identification.

Proof. Follows by the triviality of the metric in normal coordinates [1].

Let $v \in T_p\mathbb{R}^n$. If

$$[t \to tv] \in T_0T_p\mathbb{R}^n$$

denotes the equivalence class, then

$$D_0 \exp_p([t \to tv]) = [t \to \exp_p(tv)] = v \in T_p\mathbb{R}^n$$

by definition of the differential such that $D_0 \exp_p$ is the canonical identification. $\square$

With the choice of an origin $p \in \mathbb{R}^n$, the double tangential space $TT_p\mathbb{R}^n$ gives rise to an affine space

$$(M, (\Pi, +), \circ), \tag{6}$$

with the set of points $M \overset{\text{homeomorphic}}{\simeq} \mathbb{R}^n$, the vector space of translations $(\Pi, +)$ and the sharply transitive group operation

$$(\vec{v} + \vec{w}) \circ p = \vec{v} \circ (\vec{w} \circ p) \tag{7}$$

of the translations $\vec{v}, \vec{w} \in \Pi$ on the points of the space p. The difference $\overrightarrow{pq} \in \Pi$ between the two points p and q is the unique translation such that $\overrightarrow{pq} \circ p = q$. The parallel translation of $\vec{v} \in \Pi$ along $\overrightarrow{pq}$ is $\vec{v} \circ (\overrightarrow{pq} \circ p)$ as point.

$TT_p\mathbb{R}^n$ is an affine space simply because it is a vector space.

After the choice of an origin in an affine space, the space restricts to a vector space. The choice of an origin q restricts $TT_p\mathbb{R}^n$ to the tangential space $T_qT_p\mathbb{R}^n$.

Definition 2. The orthonormal *origin* $(p, n_1, ..., n_n)$, with $n_i \in T_pT_p\mathbb{R}^n$ by Proposition 1, gives rise to a frame $(q, n_1, ..., n_n)$, $n_i \in T_qT_p\mathbb{R}^n$, $\forall q$ by Euclidean parallel translation in (6). This frame is called *synthetical frame* induced by the origin $(p, n_1, ..., n_n)$.

The synthetical frame can be assumed to be positively oriented without restriction. □

Lemma 3. (I) *Let* $\exp_p(w) = q,\ \ w \in T_p\mathbb{R}^n$. *If it is not singular, then differential of the mapping*

$$\Theta_q^p : v \to \exp_p(v + w), \quad v \in T_p\mathbb{R}^n \tag{8}$$

is a canonical linear affine transformation from the affine space $TT_p\mathbb{R}^n$, *with the choice of the origin q, in the tangential space* $T_q\mathbb{R}^n$, *i.e.,*

$$D_w\Theta_q^p = D_w \exp_p \circ\, id_{pq}(v) : T_pT_p\mathbb{R}^n \to T_qT_p\mathbb{R}^n \to T_q\mathbb{R}^n, \tag{9}$$

where $id_{pq} : T_0T_p\mathbb{R}^n \to T_qT_p\mathbb{R}^n$ *denotes the Euclidean parallel translation in* $TT_p\mathbb{R}^n$.

(II) *The concept of the synthetical frame is locally consistent. The synthetical frame does not depend on the choice of the origin. If we choose* $(p, (n_i)_{i=1,\dots,n})$ *and* $(q, (n_i')_{i=1,\dots,n})$ *as origins for the construction of a synthetical frame for q in the vicinity of p, then the associated affine spaces* $TT_p\mathbb{R}^n$ *and* $TT_q\mathbb{R}^n$ *are connected by a global affine mapping, i.e., the Euclidean parallel translation and the affine linear mapping in* (9) *inclusively the identification in Proposition* 1.

(III) *Let an orthonormal synthetical frame, induced by the origin* $(p, (n_i)_{i=1,\dots,n})$, *be given at the outset. In the vicinity of p, the linear mapping*

$$D_w \exp_p : T_qT_p\mathbb{R}^n \to T_q\mathbb{R}^n \tag{10}$$

is a linear coordinate change $(T_qT_p\mathbb{R}^n, \langle .,.\rangle^{eucl}_{T_qT_p\mathbb{R}^n}) \simeq (T_q\mathbb{R}^n, \langle .,.\rangle^{g}_{T_q\mathbb{R}^n})$.

The synthetical frame $n_1, \ldots, n_n \in T_q T_p \mathbb{R}^n$, *as orthonormal coordinate basis of the space which is obtained by the choice of the origin q in the affine space* $TT_p \mathbb{R}^n$, *induces orthonormal coordinates in the manifold* $\forall q$, *locally*.

Proof. (I) Is clear, id_{pq} appears by the chain rule.

(II) Follows by (I) and the global character of the Euclidean parallel translation, respectively, Proposition 1, i.e.,

$$(D_0 \exp_q)^{-1} \circ D_w \exp_p \circ id_{pq}(v):$$

$$T_p T_p \mathbb{R}^n \to T_q T_p \mathbb{R}^n \to T_q \mathbb{R}^n \to T_q T_q \mathbb{R}^n.$$

Because of Proposition 1, $\exp_p$ is always a local diffeomorphism.

(III) The differential $D_w \exp_p : T_q T_p \mathbb{R}^n \to T_q \mathbb{R}^n$ of the exponential mapping is a linear transformation from $T_q T_p \mathbb{R}^n$ to $T_q \mathbb{R}^n$. We can choose $D_w \exp_p n_i$, $i = 1, \ldots, n$ as a basis of $T_q \mathbb{R}^n$. This basis gives rise to an affine normal coordinate system in q with the differential $D_w \exp_p : T_q T_p \mathbb{R}^n \to T_q \mathbb{R}^n$ of the exponential mapping as linear coordinate change from $T_q T_p \mathbb{R}^n$ to $T_q \mathbb{R}^n$.

It is easy to conclude that the coordinate change in the manifold from the orthonormal coordinate space $T_p \mathbb{R}^n$ to the orthonormal coordinate space $T_q \mathbb{R}^n$ via (8) has an orthonormal differential in coordinates with respect to the Riemannian metric such that

$$(D_0 \exp_q)^{-1} \circ D_w \exp_p \circ id_{pq}(v)$$

is, as Jacobian, a Euclidean orthonormal mapping if we choose orthonormal coordinates via (3) and use the canonical identity in Proposition 1.

Because of the consistency (II), the, by coordinate change induced basis in $T_q\mathbb{R}^n$ is independent from the choice of p in the affine space $TT_p\mathbb{R}^n$. If we choose $TT_q\mathbb{R}^n$, then the assertion that the induced basis is orthonormal is obvious by Proposition 1, i.e., it follows that the basis $D_w \exp_p n_i$, $i = 1, \ldots, n$ is orthonormal in the Riemannian structure of $(T_q\mathbb{R}^n, (g_{ij})^{ij})$.

Because of the choice of orthonormal coordinates in p, we have

$$\delta_{ij} = \langle n_i, n_j \rangle^{eucl}_{T_pT_p\mathbb{R}^n} = \langle id_{pq} n_i, id_{pq} n_j \rangle^{eucl}_{T_qT_p\mathbb{R}^n}$$

$$= \langle D_w \exp_p n_i, D_w \exp_p n_j \rangle^{g}_{T_q\mathbb{R}^n},$$

since the by $D_w \exp_p \circ id_{pq}$ induced metric from $(T_pT_p\mathbb{R}^n, (\delta_{kl})^{kl})$ to $(T_q\mathbb{R}^n, (g_{ij})^{ij})$ is

$$((D_w \exp_p \circ id_{pq})^{-1})^k_i \delta_{kl} ((D_w \exp_p \circ id_{pq})^{-1})^l_j$$

$$= ((D_w \exp_p)^{T,-1} (D_w \exp_p)^{-1})_{ij} = g_{ij},$$

where, e.g., $((D_w \exp_p \circ id_{pq})^{-1})^l_j$ denotes the Jacobian of the mapping

$$v \to \exp_p(v + w), \quad v \in T_p\mathbb{R}^n$$

referring $D_w \exp_p n_1, \ldots, D_w \exp_p n_n$ and an arbitrary coordinate frame $\partial_1, \ldots, \partial_n$. id_{pq} is, as before, the Euclidean parallel translation.

The synthetical frame, induced by the origin $(p, (n_i)_{i=1, \ldots, n})$, induces orthonormal coordinates in every point q. $\square$

One has to ensure that the differential in (9) is not singular what is possible in conjugated points respectively in the cut locus [1]. However, because of Proposition 1, $\exp_p$ is always locally not singular such that Lemma 3 provides local coordinates.

Corollary 4. (I) (*Affine space over the manifold*). *Let* $(\mathbb{R}^n, (g_{ij})^{ij})$ *be a geodesically complete manifold. There is a unique affine space*

$$(\mathbb{R}^n, (\Pi, +), \circ) \simeq TT_p\mathbb{R}^n \tag{11}$$

called the affine space over the manifold.

With the choice of an orthonormal origin $(p, n_1, ..., n_n)$ *in* (11), *the resulting space is called synthetical Euclidean space* $\mathbb{R}^n_{synth}$, *derived from the Riemannian manifold* $(\mathbb{R}^n, (g_{ij})^{ij})$.

Locally for q in the vicinity of p, the synthetical frame $n_1, ..., n_n \in T_qT_p\mathbb{R}^n$, *as coordinate basis of the space which is obtained by the choice of the origin q in the affine space* $TT_p\mathbb{R}^n$, *induces orthonormal coordinates in the manifold, where* $T_q\mathbb{R}^n$ *and the manifold are identified via normal coordinates. The freedom of the choice of the origin* $(p, (n_i)_{i=1, ..., n})$ *expresses a global affine symmetry of* $(\mathbb{R}^n, (\Pi, +), \circ)$.

(II) (*Explicit point set association*). *The local explicit point set association with origin p in* (11)

$$\Phi_g : (\mathbb{R}^n, (\Pi, +), \circ) \to (\mathbb{R}^n, (g_{ij})^{ij})$$

is differentiable because it is linearizable by Proposition 1. *If normal coordinates are chosen along any straight line, then the point set mapping is id along the line. The point set association is computed by the lift in synthetical coordinates of the Euler scheme associated to a geodesic, respectively, its inverse operation.* Φ_g *is called a metrical distortion in* *coordinates*. *The exterior differential is obtained by curve transportation and obeys* $D\Phi_g n_i = (D\exp_p)^i_j \partial_i$, *with a coordinate frame* ∂_i. □

We could have derived a complete Euclidean space from the Riemannian manifold $(\mathbb{R}^n, (g_{ij})^{ij})$ by showing Hilbert's foundation of Euclidean geometry axiomatic system [2]. However, the affine space property is indeed the essence of the idea of synthetical coordinates. It is all one needs to work out the full program.

Every coordinate chart in the manifold gives rise to a coordinate frame $\partial_1, ..., \partial_n$. The normal coordinate frame $\partial_1, ..., \partial_n \in T(T_p\mathbb{R}^n)$ may not be confused with the synthetical coordinate frame $n_1, ..., n_n \in TT_p\mathbb{R}^n$. Both frames are connected by the endomorphism $D_w \exp_p$, which gives rise to a Jacobian matrix $(D_w \exp_p)^i_j$ after the choice of bases. The parentheses in $T(T_p\mathbb{R}^n)$ emphasize that the coordinate system $T_p\mathbb{R}^n \simeq \mathbb{R}^n$ has to be considered. A differentiation of a smooth curve in a coordinate system always leads in the tangential space. The metric is defined on $T\mathbb{R}^n$. The double tangential space $TT_p\mathbb{R}^n$ is free to choose coordinates on the manifold via the exponential mapping such that the isometry in Lemma 3(III) is nothing but a coordinate change isometry.

It is crucial for the understanding that $\mathbb{R}^n_{synth} \simeq T_0T_p\mathbb{R}^n$ has to be separated in Cartesian coordinates from the $\mathbb{R}^n \simeq T_p\mathbb{R}^n$ that is described by the normal coordinate system. The differential topological structure is different by construction. It differs by a metrical distortion. If one travels along the grid in $\mathbb{R}^n_{synth}$ one travels along straight lines, whereas the grid in Riemannian normal coordinates is, in general, not composed by geodesics. All one has to know for a rigorous treatment is, that $\mathbb{R}^n_{synth}$ exists explicitly and fulfills any demands of a Euclidean space. Moreover, one can identify points in the synthetical and in the curved coordinate system.

It is not surprising that synthetical coordinates of a manifold $(\mathbb{R}^n, (g_{ij})^{ij})$ exist, because it is very clear that one can take stock of the coordinate change if one introduces a metrical distortion in a Euclidean space. These kind of metrical distortions naturally operate on manifolds.

Definition 5. Let $(M^{1/2}, \langle.,.\rangle^{1/2})$ denote geodesically complete n-dimensional Riemannian manifolds, homeomorphic to $\mathbb{R}^n$ (for simplicity). Let Φ_{g_1/g_2} denote the (local) point set association $\mathbb{R}^n_{synth} \to M^{1/2}$. Then $\Phi_{g_2} \circ \Phi^{-1}_{g_1}$ is called *metrical distortion*. □

A metrical distortion of a Euclidean space $\mathbb{R}^n_{synth}$ therefore induces a metrical tensor $(g_{ij})^{ij}$ in the tangential space of the image space.

Proposition 6. (I) *Let* $n_1, ..., n_n \in TT_p\mathbb{R}^n$ *denote the synthetical coordinate frame in* $\mathbb{R}^n_{synth}$ *and* $\partial_1, ..., \partial_n$ *denote any coordinate frame. Let, furthermore,* $\varkappa$ *denote the Jacobian of the exponential mapping referring to the frames such that*

$$D_w \exp_p(n_i) = \varkappa_i^j \partial_j$$

with the sum convention. Then

$$\sum_k \varkappa^i_k \varkappa^j_k = g^{ij} \ \ or \ \ (\varkappa\varkappa^T)^{ij} = g^{ij} \qquad (12)$$

in coordinates, with the reciprocal of the metrical tensor, i.e.,

$$g^{ik} g_{kj} = \delta^i_j.$$

(II) *The Levy-Civita connection* [1]

$$(\nabla^g_{\partial_i} \xi)^j = \partial_i \xi^j + \Gamma^j_{ik} \xi^k \qquad (13)$$

with an arbitrary vector field ξ *is invariant with respect to the metrical distortions*.

(III) *In normal coordinates with origin p* (*and then in every coordinate system*), $\varkappa_i^j \partial_j$ *in any point q*, *is equal to the Riemannian parallel translation of* $\partial_j \in T_p\mathbb{R}^n$, *from p to q along the unique arc length geodesic connecting p and q*.

The Euclidean parallel translation in $\mathbb{R}^n_{synth}$ *commutes with the Riemannian parallel translation along geodesics in the manifold*.

Proof. (I) Is clear with Lemma 3(III), respectively, observation of the induced metrical tensor as in Lemma 3(III).

(II) Take an arbitrary point p. If we choose an arbitrary coordinate system, then we have to show that

$$(D_w \exp_p(\partial_{n_i}\xi^{n,k}n_k))^j = \partial_i \xi^j + \Gamma^j_{ik}\xi^k$$

in p, where $\xi = D_w \exp_p(\xi^{n,k}n_k)$. However, this is obvious, if we choose normal coordinates in p, since the Γ-symbols vanish and $D_w \exp_p$ is trivial.

(III) Is obvious with (II) and Lemma 3(III) by the equation of the parallel translation (linear ODE)

$$(\nabla^g_{\dot{c}}\xi)^j = \left(\frac{\nabla^g}{\dot{c}}\xi(c(t))\right)^j = \dot{\xi}^j(c(t)) + \Gamma^j_{ik}(c(t))\xi^k(c(t))(\dot{c}(t))^i = 0 \qquad (14)$$

of the vector field ξ along the predetermined geodesic c. □

Remark 7 (Computational differential geometry)**.** With Proposition 6(III), the most easy way to compute $\varkappa$ is to state the solution of the parallel translation problem in Proposition 6(III) (linear ODE, geodesic ODE).

If a coordinate system $(\mathbb{R}^n, (g_{ij})^{ij})$ with the metrical tensor is given at the outset, then it can be assumed without restriction by the orthogonalization scheme of linear algebra that $(g_{ij})^{ij}$ is trivial in the origin 0, i.e., $\partial_1, ..., \partial_n$ is orthonormal in 0. Then the unit speed geodesics have to be determined by solving equation (4), which is a second order ODE boundary problem. Finally, $\varkappa$ is obtained in any point q by solving the linear parallel translation ODE (14) of the basis $\partial_1, ..., \partial_n(0) \in T_0\mathbb{R}^n$ along the geodesic connecting 0 and q. $\square$

The coordinate change in Lemma 3(III) is the radial isometry in normal coordinates in the Gauß lemma [1] from differential geometry. A contravariant coordinate change is trivially an isometry, since it is *id* on the manifold. One may not confuse $T_qT_p\mathbb{R}^n$ with $T_q\mathbb{R}^n$.

Corollary 8 (Generalized Gauß lemma)**.** *If* $(\mathbb{R}^n, (g_{ij})^{ij})$ *denotes a Riemannian manifold, then locally*

$$D_{tw}\exp_p : (T_{q(t)}T_p\mathbb{R}^n, \delta_{ij}) \to (T_{q(t)}\mathbb{R}^n, g_{ij}^{q(t)})$$

is an isometry between the synthetical space $(T_qT_p\mathbb{R}^n, \delta_{ij})$ *and the tangential space* $T_q\mathbb{R}^n$ *equipped with the metric* g_{ij}^q, *where* $\exp_p(tw) = q(t)$ *is the arc length proportional geodesic with tangential vector w in p.*

Especially,

$$D_{tw}\exp_p \circ id_{pq(t)} \circ (D_0\exp_p(w))^{-1}(w) = \frac{d}{dt}q(t). \tag{15}$$

Proof. Lemma 3(III). Follows by the orthonormality of the differential of a mapping from an orthonormal coordinate tangential space in an orthonormal coordinate tangential space as in (9) with the canonical identification

$$(D_0\exp_p(w))^{-1} : T_p\mathbb{R}^n \simeq T_pT_p\mathbb{R}^n$$

from Proposition 1.

(15) follows because the metrical distortion is a differentiable mapping and geodesics are transferred to geodesics. □

Theorem 9 (Transformation formula for metrical distortions)**.** *Metrical distortions, as point set transformations, are contravariant coordinate change isometries. For any open Euclidean ball* $O \subset \mathbb{R}^n_{synth}$, *the transformation formula*

$$\int_O dvol^{eucl} = \int_{\Phi_g(O)} dvol^g \tag{16}$$

holds with the metrical distortion

$$\Phi_g : (\mathbb{R}^n_{synth}, (\delta_{ij})^{ij}) \to (\mathbb{R}^n, (g_{ij})^{ij}).$$

Proof. The homeomorphic point set transformation Φ_g^{-1} transforms every Lebesgue zero set to a zero set. The image measure

$$(\Phi_g)_{\#}^{-1}(\sqrt{\det(g_{ij})}\,\lambda)$$

with the Lebesgue measure λ, which obeys (16), is thus, absolutely continuous and has thereby a Lebesgue density $f\lambda$ in $\mathbb{R}^n_{synth}$ (see [3]). We have to show that this density is constant 1.

It suffices to prove that this is true in any point q. By the ordinary transformation formula, the image measure is independent from the choice of coordinates. Choose normal coordinates with origin $q = 0 \in \mathbb{R}^n_{synth}$ (without restriction for the simplicity of the construction of φ). Let $\varphi(x) \in C_0^0(\mathbb{R}^n_{synth})$ (continuous compact support), $\int \varphi(x)dx = 1$ and $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$ such that $\int \varphi_\varepsilon(x)dx = 1$ and $\varphi_\varepsilon \underset{\varepsilon\to 0}{\to} \delta_0$, with the delta distribution in q. With the image measure theorem, it follows that necessarily $f(q) = 1$, because

$$f(q) = \lim_{\varepsilon\to 0}\int \varphi_\varepsilon(x)\,f(x)dvol^{eucl}(x) = \lim_{\varepsilon\to 0}\int \varphi_\varepsilon(\Phi_g^{-1}(y))dvol^g(y) = 1,$$

where Proposition 1 and that the volume form answers with +1 on any positively oriented orthonormal basis was used. □

Remark 10. (I) A metrical distortion Φ_g associated to $\varkappa$ is said to be *induced* by a *coordinate change*, i.e., by a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ if

$$D\Phi D\Phi^T = \varkappa\varkappa^T, \tag{17}$$

i.e., if the metric is induced by a diffeomorphism.

The comparison (17) of $D\Phi$ and $\varkappa$ has to be treated with care, since

$$D\Phi(x) : T_x\mathbb{R}^n \to T_{\Phi(x)}\mathbb{R}^n$$

and $\varkappa$ is an endomorphism of the tangential space $T_x\mathbb{R}^n$ of any point $x \in \mathbb{R}^n$.

(II) (Affine space over compact manifolds). Let

$$(M^n, (g_{ij})^{ij}) = (M^n, \langle ., .\rangle^g) \tag{18}$$

denote a compact n-dimensional Riemannian manifold (see [1]) or more generally any Riemannian manifold.

The *cut locus* denoted as $cutlocus(p)$ referring to any point p is the set of all points where geodesics with start in p are not unique connections. $cutlocus(p)$ is a Lebesgue zero set (see [1]).

The affine space over the manifold (6) is the group operation of the translations on the compact set of points M^n that is only locally sharply transitive if M^n is compact.

The differential of $\exp_p$ is eventually singular in the cut locus, which is indicated by the interplay of the Jacobi-field formalism and the possibility of closed geodesics [1]. However, the complete construction is locally consistent and provides Euclidean synthetical coordinates locally.

The affine space over the manifold is an *affine covering*

$$\pi_p : TT_pM^n \to (M^n, (\Pi, +), \cdot). \tag{19}$$

For any point $q \in M^n$, we can choose an origin p in (6) such that in an environment of q the covering conditions (see [6]) are fulfilled.

The synthetical frame $n_1, ..., n_n \in TT_pM^n$ according to Definition 2 exists globally. But, because of the eventual singularity of the exponential function in $cutlocus(p)$, Lemma 3 is only locally rigorous. If Proposition 6(III) is considered, then holonomy effects take place (closed geodesics!).

(III) (Rubber sheet geometry). The set of metrics $\langle ., .\rangle^g$ on the tangential space TM^n of a fixed manifold M^n is convex. For any two metrics $\langle ., .\rangle^{g_0}$ and $\langle ., .\rangle^{g_1}$, there is the convex combination $\langle ., .\rangle^{g_t}$, $t \in [0, 1]$ connecting them in every tangential space T_pM^n with $p \in M^n$.

By metrical distortion, one can obtain a *metrical homotopy*

$$\Phi_t : (M^n, \langle ., .\rangle^{g_0}) \hookrightarrow (M^n, \langle ., .\rangle^{g_t}).$$

The point set mapping is called *rubber sheet transformation* alternatively to metrical distortion.

All the properties and invariants which are stable with respect to metrical homotopies (and their continuous closure) are called the *differential topology* of the manifold. These invariant properties constitute the *exterior geometry* of the manifold. For instance, the possibility to reach through a torus (donut) is such an invariant property. Invariants of (contravariant) coordinate changes, like, e.g., the Riemannian curvature tensor or the Lie-derivative built the *inner geometry*.

A rubber sheet transformation which is induced by a diffeomorphism (see (I)) is called *bending without stretching*. $\square$

The construction of metrical homotopies justifies the naming “rubber sheet geometry” for topology with the exterior geometry as invariants. The complete formalism is independent from any embedding in an exterior space. An embedding fulfills a special PDE such that a graph mapping projection is possible.

III. Integration of Geometric PDEs

As in Definition 5, let $(M^n, \langle ., .\rangle^g)$ denote a geodesically complete n-dimensional Riemannian manifold, homeomorphic to $\mathbb{R}^n$ (for simplicity). Let Φ_g denote the point set association $\mathbb{R}^n_{synth} \to M^n$.

Theorem 11 (Differential geometry, see [1])**.** *The Levy-Civita connection is the unique metrical connection* ∇^g [1]

$$\partial_v \langle w, u\rangle = \langle \nabla^g_v w, u\rangle + \langle w, \nabla^g_v u\rangle \tag{20}$$

which fulfills

$$\nabla^g_v w - \nabla^g_w v - [v, w]^g = 0, \tag{21}$$

where $[v, w]^g$ *is the Lie-derivative* [1] *in curved coordinates*. *The equation of the geodesic* (4) *is*

$$\nabla^g_{\dot{c}} \dot{c} = 0. \tag{22}$$

$\square$

If $\varkappa$ is torsion free, i.e., if it is the $\varkappa$ of the geodesic system associated with $\varkappa\varkappa^T$ $(\varkappa \circ O$ with O orthonormal leads to the same metrical tensor according to Proposition 6(III)), then the metrical distortion $\Phi_g = f$, as point set mapping, is the solution of the first-order PDE

$$Df(x) = \varkappa(f(x)) \tag{23}$$

with $x \in \mathbb{R}^n_{synth}$.

It is an easy exercise to realize that the complete construction of metrical distortions also works with metrical affine connections ∇ [1, 5], what leads to metrical ∇-distortions.

The ∇-parallel translation ODE (14) with metrical ∇ fulfills the isometry property. Because of (20) $\left(\text{with } \partial_k g_{ij} = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{li}\right)$

$$\frac{d}{dt}\langle v(c(t)),\, w(c(t))\rangle \overset{(20)}{=} \langle \nabla_{\dot{c}} v(c(t)),\, w(c(t))\rangle + \langle v(c(t)),\, \nabla_{\dot{c}} w(c(t))\rangle = 0, \tag{24}$$

if v and w are parallel along c.

∇-geodesics according to the ODE (22) are automatically arc length proportional because

$$\frac{d}{dt}\langle \dot{c}(t),\, \dot{c}(t)\rangle = \langle \nabla_{\dot{c}} \dot{c}(t),\, \dot{c}(t)\rangle + \langle \dot{c}(t),\, \nabla_{\dot{c}} \dot{c}(t)\rangle = 0. \tag{25}$$

It is an immediate consequence of (25), that ∇-geodesics are autoparallel. The parallel translation along geodesics of the tangential vector is the tangential vector.

Lemma 12. (I) *Let* ∇^1 *and* ∇^2 *denote affine connections. There is a unique tensor*

$$(\nabla^1_{\partial_i} \partial_j - \nabla^2_{\partial_i} \partial_j)^k = T^k_{ij}, \tag{26}$$

the relative torsion tensor (*also known as contorsion in the literature*).

(II) *Let* ∇ *be a metrical connection. The* ∇ *-geodesic problem associated to* ∇ *via the ODE* (22) *is locally uniquely solvable. For any point p, there is* *an open ball* $p \in B_p$ *such that for any* $q \in B_p$ *there is a unique arc length parametrized* ∇*-geodesic connecting them.*

Proof. (I) Γ-symbols of an affine connection are defined by

$$(\nabla_{\partial_i}\partial_j)^k = \Gamma_{ij}^k.$$

The transformation formula for Γ-symbols can be computed explicitly if we consider that the symbols as functions lead to a vectorial transformation behavior of the result of an affine connection, i.e., if

$$\Phi : q^i \to q^{i'}$$

is a coordinate change

$$\frac{\partial q^i}{\partial q^{i'}}\frac{\partial q^{j'}}{\partial q^j}(\nabla_{\partial_i}\xi)^j = \frac{\partial q^i}{\partial q^{i'}}\frac{\partial q^{j'}}{\partial q^j}(\partial_i\xi^j + \Gamma_{ik}^j\xi^k) \overset{!}{=} \partial_{i'}\left(\frac{\partial q^{j'}}{\partial q^j}\xi^j\right) + \Gamma_{i'k'}^{j'}\xi^{k'}$$

such that

$$\frac{\partial q^i}{\partial q^{i'}}\frac{\partial q^{j'}}{\partial q^j}\partial_i\xi^j + \frac{\partial q^i}{\partial q^{i'}}\frac{\partial q^{j'}}{\partial q^j}\Gamma_{ik}^j\xi^k = \partial_{i'}\left(\frac{\partial q^{j'}}{\partial q^j}\right)\xi^j + \frac{\partial q^{j'}}{\partial q^j}\partial_{i'}\xi^j + \Gamma_{i'k'}^{j'}\xi^{k'},$$

i.e.,

$$\frac{\partial q^i}{\partial q^{i'}}\frac{\partial q^{j'}}{\partial q^j}\Gamma_{ik}^j\xi^k = \frac{\partial^2 q^{j'}}{\partial q^i\partial q^j}\frac{\partial q^i}{\partial q^{i'}}\xi^j + \Gamma_{i'k'}^{j'}\xi^{k'}, \quad \forall\xi,$$

or obviously,

$$\frac{\partial q^i}{\partial q^{i'}}\frac{\partial q^{j'}}{\partial q^j}\Gamma_{ik}^j\frac{\partial q^k}{\partial q^{k'}}\xi^{k'} = \frac{\partial^2 q^{j'}}{\partial q^i\partial q^j}\frac{\partial q^i}{\partial q^{i'}}\frac{\partial q^j}{\partial q^{k'}}\xi^{k'} + \Gamma_{i'k'}^{j'}\xi^{k'}, \quad \forall\xi$$

such that Γ-symbols transform as follows:

$$\frac{\partial q^i}{\partial q^{i'}}\frac{\partial q^{j'}}{\partial q^j}\Gamma_{ik}^j\frac{\partial q^k}{\partial q^{k'}} - \frac{\partial^2 q^{j'}}{\partial q^i\partial q^j}\frac{\partial q^i}{\partial q^{i'}}\frac{\partial q^j}{\partial q^{k'}} = \Gamma_{i'k'}^{j'}.$$

Any difference of two affine connections ∇^1 and ∇^2

$$(\nabla^1_{\partial_i}\partial_j - \nabla^2_{\partial_i}\partial_j)^k = T^k_{ij}$$

is a unique tensor $T = T^k_{ij}\partial_k d^i d^j$, the relative torsion tensor because the second derivative terms in the transformation behavior are compensated in the difference and the rest has vectorial transformation behavior. This justifies the naming affine connections.

(II) The initial value problem of the geodesic ODE is uniquely solvable. The normal coordinate mapping for the ∇-geodesic system exists for the direction $T_p\mathbb{R}^n \to \mathbb{R}^n$ and is differentiable. The differential is *id* in *p* by construction. Therefore, the mapping is locally invertible what includes the assertion. □

The geodesic system (or also geodesic spray in the literature) associated to ∇ gives rise to local ∇-normal coordinates via the ODE (22) and Lemma 12(II). By the special form of the ODE, geodesics are straight lines in normal coordinates. The Γ-symbols associated to ∇ vanish in the origin, which is an easy consequence of the straight lines with origin 0 as geodesics in normal coordinates. Moreover, by Taylor-series

$$g_{ij}(x) = \delta_{ij} + O(x^2) \tag{27}$$

by the metrical property (20), since

$$\partial_k g_{ij}(0) \overset{(20)}{=} \Gamma^l_{ki}(0)g_{lj}(0) + \Gamma^l_{kj}(0)g_{li}(0) \overset{\Gamma=0}{=} 0.$$

Especially, $\mathbb{R}^n_{\varkappa-synth}$ is established completely analogous to the Levy-Civita case and Proposition 6 holds in full generality with additional relative torsion.

The following lemma establishes the correspondence between an arbitrary $\varkappa$ and a unique associated metrical connection ∇.

Lemma 13. *Let* $\varkappa : \mathbb{R}^n \to \mathbb{R}^{n^2}$ *be nondegenerate. The Levy-Civita geodesic system associated to* $\varkappa\varkappa^T$ (*see Proposition* 6(I)) *establishes a unique* $\varkappa^g$ *as the differential of the exponential mapping.*

Then, with respect to the metric $\langle ., . \rangle^g$ *-orthonormal automorphism field*

$$O^{\varkappa}(x) := (\varkappa^g)^{-1} \circ \varkappa(x) \in \mathit{Automorphisms}\ (T_x \mathbb{R}^n) \tag{28}$$

determines a unique metrical connection $\nabla^{\varkappa}$, *which is determined by its* $\nabla^{\varkappa}$*-parallel translation system along* $\nabla^{\varkappa}$*-geodesics, whereby* (28) *is recovered relative to the Levy-Civita system.*

Proof. We have to show two directions of unique correspondence:

"$\Leftarrow$" If a metrical connection is given at the outset, (28) respectively $\varkappa$ in any point x is obtained uniquely by its parallel translation system along geodesics relative to the Levy-Civita connection, because it is orthonormal by (24) and Proposition 6(III) also holds for general metrical connections.

"$\Rightarrow$" (a) By Lemma 12(I), any difference of two affine connections ∇^1 and ∇^2

$$(\nabla^1_{\partial_i} \partial_j - \nabla^2_{\partial_i} \partial_j)^k = T^k_{ij}$$

is a unique tensor $T = T^k_{ij} \partial_k d^i d^j$, the relative torsion tensor.

(b) The connection form ϑ^s of an affine connection ∇ referring to the frame $s_1, ..., s_n$ is defined by

$$\nabla_{\partial_i} s_j =: (\vartheta^s)^k_j (\partial_i) s_k.$$

The connection form is a matrix valued 1-form with affine transformation behavior and determines the connection uniquely.

Let $e_1, ..., e_n$ denote an arbitrary orthonormal frame. The connection form referring to $e_1, ..., e_n$ of a metrical connection is skew symmetric

$$\begin{aligned} 0 = \partial_i \delta_{kj} &= \partial_i \langle e_k, e_j \rangle \\ &= \langle \nabla_{\partial_i} e_k, e_j \rangle + \langle e_k, \nabla_{\partial_i} e_j \rangle \\ &= (\vartheta^e)^j_k(\partial_i) + (\vartheta^e)^k_j(\partial_i). \end{aligned}$$

On the other hand, any with respect to an orthonormal frame skew symmetric connection form is associated to a metrical connection

$$\begin{aligned} &\langle \nabla_{\partial_i} v, w \rangle + \langle v, \nabla_{\partial_i} w \rangle \\ &= \langle \partial_i v^k e_k + v^k \nabla_{\partial_i} e_k, w^j e_j \rangle + \langle v^k e_k, \partial_i w^j e_j + w^j \nabla_{\partial_i} e_j \rangle \\ &= \delta_{jk} w^j \partial_i v^k + \delta_{jk} v^k \partial_i w^j + \langle v^k (\vartheta^e)^l_k(\partial_i) e_l, w^j e_j \rangle \\ &\quad + \langle v^k e_k, w^j (\vartheta^e)^l_j(\partial_i) e_l \rangle \\ &= \partial_i \langle v^k e_k, w^j e_j \rangle + v^k w^j \delta_{jl} (\vartheta^e)^l_k(\partial_i) \\ &\quad + v^k w^j \delta_{kl} (\vartheta^e)^l_j(\partial_i) = \partial_i \langle v, w \rangle. \end{aligned}$$

(c) The unique *torsion tensor* associated to $O^{\varkappa}$

$$T^{\varkappa} = (T^{\varkappa})^j_{ik} \partial_j d^i d^k$$

is defined by the $\langle .,. \rangle^g$-orthonormal automorphism field (28)

$$(T^{\varkappa})_k := \frac{d}{dt}\Big|_{t=0} (O^{\varkappa})^{-1}(c_k(0)) O^{\varkappa}(c_k(t)) \in \mathfrak{A}(O(n), \langle .,. \rangle^g),$$

where

$$c_k(t) : [0, T] \to \mathbb{R}^n$$

is a differentiable curve with

$$\frac{d}{dt}\Big|_{t=0} c_k(t) = \partial_k, \ \mathfrak{A}(O(n), \langle ., . \rangle^g)$$

denotes the Lie-algebra of skew symmetric matrices and $\partial_1, ..., \partial_n$ expresses the coordinate frame.

$\mathfrak{A}(O(n), \langle ., . \rangle^g)$-skew symmetry means precisely skew symmetry with respect to an orthonormal frame.

(d) If we define

$$(\vartheta^{\varkappa})^j_k(\partial_i) := (\vartheta^g)^j_k(\partial_i) + (T^{\varkappa})^j_{ik},$$

where $(\vartheta^g)^j_k(\partial_i)$ is the connection form of the Levy-Civita connection referring to $\partial_1, ..., \partial_n$, i.e., the Γ-symbols, $\nabla^{\varkappa}$ is determined uniquely by $O^{\varkappa}$ because $O^{\varkappa}$ is uniquely determined by its linearization $T^{\varkappa}$.

With Proposition 6, a metrical connection is determined by its parallel translation system (what is a well-known result in differential geometry [1], without metrical distortions). And

$$(\nabla^{\varkappa}_{\dot{c}}\xi)^j = \left(\frac{\nabla^{\varkappa}}{\dot{c}}\xi(c(t))\right)^j$$

$$= \dot{\xi}^j(c(t)) + (\Gamma^j_{ik} + (T^{\varkappa})^j_{ik})(c(t))\xi^k(c(t))(\dot{c}(t))^i = 0$$

is the $\nabla^{\varkappa}$-parallel translation ODE of ξ along an arbitrary curve c such that $O^{\varkappa}$ is recovered relative to the Levy-Civita system along Levy-Civita geodesics and the proof is finished. □

The tensor T_{ij}^{k} in Lemma 12(I) is an endomorphism valued one form in the index i. The endomorphism can be seen as element of the Lie-algebra of endomorphisms $(T_{ij}^{k})_{k}^{j} \in \mathfrak{A}$ (tangential space in id), $\forall i$. For any differentiable curve

$$c(t) \in C^{1}([0, T] \to \mathbb{R}^{n}),$$

there is a unique endomorphism field $O_c(c(t))$ along c such that

$$\left(\frac{d}{d\varepsilon}|_{\varepsilon=0}\, O_c^{-1}(c(t))O_c(c(t+\varepsilon))\right)_{k}^{j} = (T_{ij}^{k}(c(t)))_{k}^{j} \dot{c}^{i}(t),$$

$$O_c(c(0)) = id,$$

what justifies the naming relative torsion tensor. It describes how the space is twisted by a metrical ∇-distortion relative to a Levy-Civita distortion.

The usual variational functional

$$L(T, c, \dot{c}) := \frac{1}{2}\int_{0}^{T} \langle \dot{c}, \dot{c}\rangle^{g}(t)\,dt \tag{29}$$

immediately leads to the equation of the ∇-geodesic

$$\ddot{c}^{l} + T_{sj}^{l}\dot{c}^{s}\dot{c}^{j} + (\Gamma^{g})_{sj}^{l}\dot{c}^{s}\dot{c}^{j} = 0, \tag{30}$$

if the variation is performed in $R^{n}_{\varkappa-synth}$, since straight lines are solutions there.

See also [9-11] for attempts to establish variational formalisms for ∇-geodesics without explicit synthetical coordinates.

With the preliminaries above, it is obvious to state the following integration theory of the geometric PDE.

Theorem 14 (Integration of geometric PDEs)**.** *Let* $\varkappa : \mathbb{R}^n \to \mathbb{R}^{n^2}$ *be nondegenerate.*

The geometric PDE

$$Df(x) = \varkappa(f(x)) \tag{31}$$

is locally integrated by the metrical $\nabla^{\varkappa}$ *-distortion*

$$f|_B = \Phi_{\varkappa} : \mathbb{R}^n_{\varkappa-synth}|_B \to (\mathbb{R}^n, \langle ., .\rangle^g)$$

associated to $\varkappa$*, where* $R^n_{\varkappa-synth}|_B$ *denotes the restriction to a ball B.* □

It is crucial that the parameter x in (31) is synthetical and is not induced as in Remark 10(I).

Remark 15. If the geometric PDE (31) is fulfilled, then the reciprocal PDE is

$$Dh(y) = \varkappa^{-1}(y) \tag{32}$$

if $f(x) = y$.

The reciprocal PDE implicitly demands that h is a diffeomorphism. It is only integrable if $\varkappa^{-1}$ is induced. □

The reciprocal PDE describes a lift operation of tangential structures and not an explicit mapping.

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