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COMPACTNESS RESULTS FOR TRANSPORT OPERATOR WITH BOUNDARY CONDITIONS

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Abstract

The purpose of the present work is to provide a systematic analysis of some compact results dealing with the transport equation where the incoming flow is related to the outgoing one by a bounded operator for a large class of measures with support of the velocities space and under some conditions on the collision operator. We present a new method shorter than that established in (cf. [5]) and we emphasize, also, the dominant role played by the neutronics theory in our proof techniques. Finally, our results are independent from the boundary operator and permit us to understand the time asymptotic behavior of some solutions to Cauchy problems.

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1. Introduction

Let H be a linear bounded operator on some chosen trace spaces covering, in particular, all well known physical models: periodicals, reflexives or mixed.

We study the compactness of the transport operator for bounded geometry, in mono and multidimensional cases, which can be modeled by the integro-differential operator A_H stated as:

$$\begin{cases} A_H\varphi(x, v) := \underbrace{-v \cdot \nabla_x - \sigma(v)}_{T_H\varphi(x,v)}\varphi(x, v) \\ \qquad\qquad\qquad + \underbrace{\int_V k(x, v, v')\varphi(x, v')d\mu(v')}_{K\varphi(x,v)} \\ \psi_- := H(\psi_+) \qquad\qquad \text{(boundary conditions)}, \end{cases}$$

where $(x, v) \in \Omega \times V$:

- Ω is a domain set of $\mathbb{R}^n$, $n \geq 1$.
- $d\mu(\cdot)$ is a positive measure on $\mathbb{R}^n$ with support of the velocities space V such that $d\mu(0) = 0$.
- The function $\varphi(x, v)$ is the density of the probability of the gas particles in the position x and velocity v.
- φ_- (resp. φ_+) is the restriction of φ to Γ_- (resp. Γ_+), where Γ_- (resp. Γ_+) is the incoming part (resp. outgoing) of the phase boundary space $\Omega \times V$ defined as:

  $$\Gamma_\pm = \{(x, v) \in \partial\Omega \times V, \pm v.n_x \geq 0\},$$

  where n_x is the outward normal at $x \in \partial\Omega$.
- The functions $\sigma(v)$ is the frequency collision with velocity v.
- The collision linear operator K, with nucleus the function k, reports

the reflection (resp. scattering) and the production (resp. in the presence of fissile materials) of neutrons by fission. It has the peculiarity of being local with respect to the space variable $x \in \Omega$.

- The operator T_H is intended to represent the streaming operator while the operator $A_H = T_H + K$ describes the transport of particles (neutrons, photons, gas molecular, ...) in Ω perturbed by K.

The evolution of the transport theory is related to the development of the nuclear industry since the second world war and its origin became from the radiative transfer (cf. [4]). The neutronics theory has benefited from the major interest of pioneer works in physics and mathematics, especially, Jörgens et al. (cf. [2, 3, 7, 8, 13-16]). The transport theory began to have connections with semigroups, the spectral theory of self-adjoint operators, the positivity and, in general, with functional analysis.

Since the transport equations are linear in nature and the proportion of neutrons is infinitesimal than the proportion of atoms in the environment propagation (in order of 10^{-11}) so the interaction neutron-neutron are negligible by comparison to the interaction neutron-environment where the properties are independent with the neutronic populations.

For general boundary conditions $(H \neq 0)$, the interactions particles-environment are complex in nature and they are established by many concurrent factors so the precise mathematics formulations present some controversies. Nevertheless, the model frequently used is to suppose that the part of outgoing flow is re-emitted in a deterministic direction (*specular reflection*) whereas the part of the incoming flow is re-emitted in randomly directions (*diffusive reflection*).

The results of (cf. [9]) in the absorbent case $(H = 0)$ will be extended, in this work, to the general one via a new method easier than that proposed in (cf. [5]) where the transport operator with the boundary absorbent conditions has more importance than that obtained in bounded geometry, notably in greater dimension, $n > 1$.

Recall, first, that the resolvent set of the bounded operator $T : D(T) \subset X \to X$ with range dense in X, is defined as follow:

$$\rho(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda.Id_X - T : D(T) \to X \text{ is a bijection}\}.$$

Its complement $\sigma(T)$ in $\mathbb{C}$, known as the spectrum of T, is compact. Hence, the set $\rho(T)$ is an open set and the application

$$\lambda \in \rho(T) \mapsto R(\lambda, T) := [\lambda.Id_X - N(\lambda)]^{-1}$$

is analytic on every connected component of $\rho(T)$. Finally, we define an important subset of $\sigma(T)$ known as punctual spectrum:

$$\sigma_p(T) := \{\lambda;\ \lambda.Id_X - T : D(T) \to X \text{ is an injection}\}.$$

A complex number $\lambda_0 \in \sigma_p(T)$ is said to be *eigenvalue* and each $x_0 \in D(T)\backslash\{0\}$ such that

$$(\lambda.Id_X - T)x_0 = 0$$

is said to be *eigenvector* corresponding to λ_0.

Now, we enounce the following result, (cf. [11, 14]), usually known as the:

Theorem 1.1 (Gohberg-Shmulyan alternative)**.** *Let X be a complex Banach space and a family* $\lambda \mapsto T(\lambda)$ *of compact linear holomorphic operators in X*, *defined on a connected part of* $\mathbb{C}$. *We have the following alternative*:

1. *The number* 1 *is an eigenvalue of* $T(\lambda)$ *for all* $\lambda \in \mathbb{C}$.

2. *The set* $R(1, T(\lambda))$ *exists unless on a discrete set of* λ *-pole of* $T(\cdot)$ *for degenerate principal parts* (*i.e.*, *the associated coefficients have finite rank*).

Corollary 1.2. *If the family* $\lambda \mapsto T(\lambda)$ *has only compact power of order m*, *then the alternative still remains true*.

Remark 1.3. Let C be an open connected subset of $\rho(T)$. If the power of $KR(\lambda.Id, T)$ is compact for all $\lambda \in \rho(T)$, then we prove that

- either $C \bigcap \sigma(T + K) \subset \sigma_p(T + K)$
- or $C \bigcap \sigma(T + K)$ is reduced to the isolated eigenvalues with finite algebraic multiplicities.

For this end, we apply Corollary 1.2 to $N(\lambda) = KR(\lambda, T)$ and the fact that 1 is an eigenvalue of $KR(\lambda, T)$ if and only if λ is an eigenvalue of $T + K$ (cf. [6, Theorem 1]). We state that the second situation is produced for the set

$$\sigma(T + K) \bigcap \{\lambda : \operatorname{Re} \lambda > \omega_0(T)\}.$$

This means that it contains, at least, the eigenvalues of finite algebraic multiplicities. This enables us, with the help of the Dunford formula and some supplementary hypothesis, to have an asymptotical description of $V(t)\varphi_0$ (semigroup perturbed at initial data) when $\varphi_0 \in D[(T + K)^2]$ (cf. [6]). For this reason, we study the compactness power of $KR(\lambda, T)$. □

2. Notation

Fix $(x, v) \in \overline{\Omega} \times \overline{V}$. Let

$$t^{\pm}(x, v) = \sup\{t > 0,\ x \pm sv \in \Omega,\ 0 < s < t\}.$$

Hence, for $(x, v) \in \Gamma_{\pm}$, we have $t^{\pm}(x, v) = 0$ and $t^{\pm} > 0$ otherwise. In fact, in all cases, we have

$$(x \pm t^{\pm}(x, v), v) \in \Gamma_{\pm}.$$

Letting $1 < p < +\infty$, we define the following functional spaces:

$$W_p = \{\varphi \in X_p \text{ such that } v.\nabla_x \varphi \in X_p\},$$

where

$$X_p = L_p(\Omega \times V,\ dxd\mu(v)).$$

A suitable traces space is defined by:

$$L^{p,\pm} := L^p(\Gamma_\pm;\ |v.n_X|\, d\gamma_x d\mu(v)),$$

where $d\gamma_x$ is the Lebesgue measure on $\partial\Omega$. The traces $\varphi_\pm$ of $\varphi \in W_p$ on $\Gamma_\pm$ can be defined. We noted, in general, that these traces are not in $L^{p,\pm}$ but in $L^{p,\pm}_{loc}$. Precisely, they are in the so called space L^p *with weight* (c.f. [1]). We define a new subset of W_p as:

$$\hat{W}_p = \{\varphi \in W_p,\ \varphi_- \in L^{p,-}\}.$$

Remark 2.1. This subset can be written as:

$$\hat{W}_p = \{\varphi \in W_p,\ \varphi_- \in L^{p,-}\} = \{\varphi \in W_p,\ \varphi_+ \in L^{p,+}\} \subseteq W_p$$

because for $\varphi \in W_p$, we have $\varphi_+ \in L^{p,+}$ if and only if $\varphi_- \in L^{p,-}$. □

Let $H : L^{p,+} \mapsto L^{p,-}$ be a bounded operator. The advection operator T_H and its domain are defined as:

$$\begin{cases} T_H\varphi(x, v) = -v.\nabla_x\varphi(x, v) - \sigma(v)\varphi(x, v) \\ D(T_H) = \{\varphi \in \hat{W}_p : \varphi_- = H(\varphi_+)\}, \end{cases}$$

where the collision frequency $\sigma(\cdot)$ must be in $L^\infty(V)$. Let $\lambda \in \mathbb{C}$ and consider the following limits problem:

$$\begin{cases} \lambda\varphi(x, v) + v.\nabla_x\varphi(x, v) + \sigma(v)\varphi(x, v) = \psi(x, v) \\ \varphi_- = H(\varphi_+), \end{cases} \tag{2.1}$$

where ψ is a given function in X_p and the unknown function φ must be in $D(T_H)$.

Let

$$\lambda^* = \mu - \operatorname*{ess\,inf}_{v \in V} \sigma(v).$$

For $\lambda^* + \operatorname{Re}\lambda > 0,$ equation (2.1) must be resolved formally as:

$$\varphi(x, v) = \varphi(x - t^-(x, v)v, v)e^{-(\lambda+\sigma(v))t^-(x, v)}$$

$$+ \int_0^{t^-(x, v)} e^{-(\lambda+\sigma(v))s}\psi(x - sv, v)ds. \tag{2.2}$$

For $(x, v) \in \Gamma_+,$ the restriction $\varphi_{|\Gamma_+}$ of the solution of (2.2) can be written as:

$$\varphi_{|\Gamma_+} = \varphi_{|\Gamma_-} e^{-(\lambda+\sigma(v))\tau(x, v)} + \int_0^{\tau(t, v)} e^{-(\lambda+\sigma(v))s}\psi(x - sv, v)ds, \tag{2.3}$$

where

$$\tau(x, v) = t^+(x, v) + t^-(x, v) \quad \text{and} \quad (x, v) \in \Gamma_+, t^+(x, v).$$

To give an abstract formulation of (2.2) and (2.3), we define the following λ-operators:

$$M_\lambda : L^{p,-} \mapsto L^{p,+} \qquad u \mapsto M_\lambda u = ue^{-(\lambda+\sigma(v))\tau(x, v)}$$

$$B_\lambda : L^{p,-} \mapsto X_p \qquad u \mapsto B_\lambda u = ue^{-(\lambda+\sigma(v))t^-(x, v)}$$

$$G_\lambda : X_p \mapsto L^{p,+} \qquad \varphi \mapsto G_\lambda\varphi = \int_0^{\tau(x, v)} e^{-(\lambda+\sigma(v))s}\varphi(x - sv, v)ds$$

and

$$C_\lambda : X_p \mapsto X_p \qquad \varphi \mapsto G_\lambda\varphi = \int_0^{t^-(x, v)} e^{-(\lambda+\sigma(v))s}\varphi(x - sv, v)ds.$$

By the Hölder inequality, these operators are bounded such that

$$\| M_\lambda \| \leq 1,$$

$$\| B_\lambda \| \leq \frac{1}{[p(\operatorname{Re}\lambda + \lambda^*)]^{1/p}},$$

$$\| G_\lambda \| \leq \frac{1}{[q(\operatorname{Re}\lambda + \lambda^*)]^{1/q}},$$

$$\| C_\lambda \| \leq \frac{1}{\operatorname{Re}\lambda + \lambda^*}, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

With the help of these operators and the fact that φ will satisfy the boundary conditions, the equality (2.3) becomes:

$$\varphi_+ = M_\lambda H \varphi_+ + G_\lambda \psi.$$

Let $H(\lambda) = Id - M_\lambda H$. Then if $[H(\lambda)]^{-1}$ exists, then one obtains that

$$\varphi_+ = [H(\lambda)]^{-1} G_\lambda \psi. \tag{2.4}$$

In the other hand, the equality (2.2) can be written like:

$$\varphi = B_\lambda H \varphi_+ + C_\lambda \psi.$$

By substitution of (2.4) in the above equation, it appears that

$$\varphi = B_\lambda H [H(\lambda)]^{-1} G_\lambda \psi + C_\lambda \psi.$$

Consequently,

$$(\lambda.Id - T_H)^{-1} = B_\lambda H [H(\lambda)]^{-1} G_\lambda + C_\lambda. \tag{2.5}$$

Remark 2.2. We can prove that if $\| H \| \leq 1$ or one of the powers of the operator $M_\lambda H$ is compact, then the existence of the operator $(\lambda.Id - T_H)^{-1}$ is ensured for great values of $\operatorname{Re}\lambda$. □

3. Principal Results

We present here a principal result that allows us to generate a strongly continuous semigroup for the advection operator.

Theorem 3.1. *Let*

$$X_p = L^p(\,]-a, a[\, \times [-1, 1], dxd\xi), \quad 1 \leq p < \infty.$$

For all $H \in \mathcal{L}(L^{p,+}, L^{p,-})$, *the advection operator* T_H *generates a* C_0-*semigroup* $\{U_H(t);\, t \geq 0\}$ *in* X_p. *Furthermore,*

$$\| U_H(t) \| \leq \max\{1, \| H \|\} e^{t \max\left\{\frac{1}{2a}\| H \|, 0\right\}}, \quad t \geq 0. \tag{3.1}$$

Now, it well known that T_H generates or not a C_0-semigroup depends on the geometry of $\Omega \times V$. In fact, as illustrated by the following example, some models can be constructed such that T_H does not generate any strongly continuous semigroups.

Example. Consider a transport model in L^1 where $\Omega = \,]0, 1[$, $V = [0, +\infty[$. If we fix a Lebesgue measure $\nu(\cdot)$ on V, then we obtain

$$\Gamma_+ = \{1\} \times V, \quad \Gamma_- = \{0\} \times V \quad \text{and} \quad L^{1,\pm} = L^1([0, +\infty\,[, vdv)$$

and the boundary operator H verifies the identity:

$$H(\varphi(1, .)) = \varphi(0, .), \quad \varphi \in W_1.$$

We claim that T_H is not closed in L^1. For this end, pick $h \in L^1([0, +\infty\,[, dv)$ such that

$$\int_0^{+\infty} | h(v) | vdv = +\infty. \tag{3.2}$$

For $n \in \mathbb{N}$, let

$$\varphi_n(x, v) = \begin{cases} h(v) & \text{if } 0 < v < n \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\varphi_n \in W_1$. Since

$$\int_0^n |h(v)| v dv < +\infty, \quad \forall n \in \mathbb{N},$$

one obtains

$$\varphi_{n|_{\Gamma_\pm}} \in L^{1,\pm} \quad \text{and} \quad \varphi \in D(T_H), \ \ \forall n \in \mathbb{N}.$$

So $\lim_{n\to\infty} \varphi_n = \varphi$ and $\lim_{n\to\infty} T_H \varphi_n = 0$, where $\varphi(x, v) = h(v)$ for all $(x, v) \in \Omega \times V$, $\varphi \in X_1$. With the relation (3.2), we have:

$$\varphi_{|_{\Gamma_-}} = h \notin L^{1,\pm}.$$

Then $\varphi \notin D(T_H)$. The operator T_H is not closed and does not generate any C_0-semigroup in X_1. □

Suppose that the inverse Laplace transformation of the operator $(\lambda - T_H)^{-1}$ exists and noted as $S(t)$. Then the inverse Laplace transformation $M(t)$ of the operator $B_\lambda H[H(\lambda)]^{-1} G_\lambda$ exists and satisfies

$$M(t) = S(t) - U_0(t),$$

where $U_0(t)$ is a transport semigroup in the neutronic framework $(H = 0)$.

Theorem 3.2. *The advection operator T_H generates a C_0-semigroup $\{S(t)_{t\geq 0}\}$ if and only if the bounded operators $\{M(t)\}_{t\geq 0}$ verify*:

1. $M(0) = 0$,

2. *for all* $t_1, t_2 \geq 0$, $M(t_1 + t_2) = M(t_1)M(t_2) + U_0(t_1)M(t_2) + M(t_1)U_0(t_2)$,

3. *for all* $\varphi \in X_p$, *we have* $\lim_{t\to 0^+} \| M(t)\varphi \| = 0$.

Proof. We prove that these conditions are necessary and sufficient:

1. $M(0) = S(0) - U_0(0) = Id - Id = 0.$

2. Since

$$S(t_1) = M(t_1) + U_0(t_1)\rho S(t_2) = M(t_1) + U_0(t_2),$$

$$S(t_1 + t_2) = M(t_1 + t_2) + U_0(t_1 + t_2) = S(t_1)S(t_2)$$

and by identification, we obtain

$$M(t_1 + t_2) = M(t_1)M(t_2) + U_0(t_1)M(t_2) + M(t_1)U_0(t_2).$$

3. For all $\varphi \in X_p,$ we have

$$\lim_{t\to 0^+} \| S(t)\varphi - \varphi \| = 0,$$

then

$$\lim_{t\to 0^+} \| M(t)\varphi + U_0(t)\varphi - \varphi \| = 0.$$

Since

$$0 \le \lim_{t\to 0^+} | \| M(t)\varphi \| - \| U_0(t)\varphi - \varphi \| |$$

$$\le \lim_{t\to 0^+} \| M(t)\varphi + U_0(t)\varphi - \varphi \| = 0,$$

one obtains that $0 \le \lim_{t\to 0^+} \| M(t)\varphi \| \le 0$ so $\lim_{t\to 0^+} \| M(t)\varphi \| = 0.$ □

Remark 3.3. In the neutronic framework $H = 0,$ the family of operators $\{M(t)\}_{t\ge 0}$ is reduced to 0, and verifies trivially the conditions (1), (2) and (3), whereas in:

1. Periodic framework and σ even: $M(t)$ is written explicitly as:

$$e^{-\sigma(\xi)t}\left[\sum_{n>0} \varphi[(sgn\xi)2na + x - t\xi,\ \xi]\chi_{\left[\frac{(sgn\xi)x+(2n-1)a}{|\xi|},\ \frac{(sgn\xi)x+(2n+1)a}{|\xi|}\right]}\right].$$

2. Reflexive framework and σ even: $M(t)$ is written explicitly as

$$e^{-\sigma(\xi)t}\left[\sum_{n>0}\varphi[(sgn\xi)4na + x - \xi t,\ \xi]\chi_{\left[\frac{(sgn\xi)x+(4n-1)a}{|\xi|},\ \frac{(sgn\xi)x+(4n+1)a}{|\xi|}\right]}(t)\right]$$

$$+ e^{-\sigma(\xi)t}\left[\sum_{n>0}\varphi[-(sgn\xi)(4n+2)a - x + \xi t,\ \xi]\right.$$

$$\left.\cdot\ \chi_{\left[\frac{(sgn\xi)x+(4n+1)a}{|\xi|},\ \frac{(sgn\xi)x+(4n+3)a}{|\xi|}\right]}(t)\right].$$

□

Conjecture. We conjecture that

$$T_H \text{ generates a } C_0\text{-semigroup}$$

if and only if

$$\exists\lambda_0 \in \mathbb{R} \text{ such that }]\lambda_0, +\infty[\ \subseteq \rho(T_H).$$

4. The Compactness Properties

The transport operator A_H can be written as $A_H := T_H + K$, where K is the bounded operator defined on X_p as:

$$K : X_p \mapsto X_p$$

$$\psi \mapsto \int_V k(x, v, v')\psi(x, v')d\mu(v'),$$

where the nucleus collision $k : \Omega \times V \times V \mapsto \mathbb{R}$ is supposed to be measurable. We note, as we have mentioned earlier, that the operator K is locally in x. So, it will be seen as an application:

$$K(.) : x \in \Omega \mapsto K(x) \in \mathcal{L}(L^p(V;\, d\mu)).$$

Suppose that $K(.)$ is *strictly measurable*, this means that the application $x \in \Omega \mapsto K(x)f \in L^p(V; d\mu)$ is:

1. measurable for all $f \in L^p(V; d\mu)$,

2. bounded, in the sense: $\text{ess} - \sup_{x \in D} \| K(x) \|_{\mathcal{L}(L^p(V; d\mu))} < +\infty$.

We obtain a bounded operator $\varphi \in X_p \mapsto K(x)\varphi(x) \in X_p$ such that

$$\| K(x) \|_{\mathcal{L}(X_p)} \leq \text{ess} - \sup_{x \in D} \| K(x) \|_{\mathcal{L}(L^p(V; d\mu))}.$$

Now, we use the concept of regular collision operator defined first in (cf. [10, Chapter 4]).

Definition 4.1. Let $\mathcal{K}(L^p(V; d\mu))$ be the subspace of compact operators. Then the collision operator

$$K : x \in D \mapsto K(x) \in \mathcal{L}(L^p(V; d\mu))$$

is *regular* if:

1. $K(x) \in \mathcal{K}(L^p(V; d\mu))$ for almost x,

2. the operator $x \in \Omega \mapsto K(x) \in \mathcal{K}(L^p(V; d\mu))$ is measurable,

3. the set $\{K(x), x \in D\}$ is relatively compact in $\mathcal{L}(L^p(V; d\mu))$.

Let $\mathcal{R}(X_p)$ be the space of regular collision operators in X_p. Then:

Lemma 4.2. *A collision operator K can be approximated in the uniform topology, by a sequence* $(K_n)_n$ *of collision operators with nucleus*

$$\sum_{i \in I} \alpha_i(x) f_i(\xi) g_i(\xi'),$$

where

$\alpha_i \in L^\infty(D, dx)$, $f_i \in L^p(V; d\mu)$, $g_i \in L^q(V; d\mu)$, $q = \dfrac{p}{p-1}$ *and a finite set I.*

Proof. More details can be found in (cf. [10, Chapter 4]). □

Now, consider the following (H1)-hypothesis:

Let $\mathbb{S}^{n-1}$ be the unit sphere of $\mathbb{R}^n$ such that the hyperplanes, i.e.,

$$\{v \in \mathbb{R}^n / v.c = 0, \forall c \in \mathbb{S}^{n-1}\}$$

have zero μ -measure.

According to this, we have:

Theorem 4.3. *Let* $1 < p < +\infty$, Ω *be a bounded convex subset of* $\mathbb{R}^n$ *and that the* (H1)-*hypothesis is true. If* $\{H(\lambda)\}^{-1}$ *exists and* $K \in \mathcal{R}(X_p)$, *then, for all* $\lambda \in \mathbb{C}$ *such that* $Re\,\lambda > -\lambda^*$, *the operators* $K(\lambda.Id - T_H)^{-1}$ *and* $(\lambda.Id - T_H)^{-1}K$ *are compact subsets of* X_p.

Proof. Let $q = p/(p-1)$. Since K is a regular set and according to Lemma 4.2, we can establish the proof only for the following collision operators:

$$k(x, v, v') = \alpha(x) f(v) g(v'),$$

where $\alpha(.) \in L^{\infty}(D; dx)$, $f(.) \in L^p(V; d\mu(v))$ and $g(.) \in L^q(V; d\mu(v))$. Without losing of majority, we can suppose that *f* and *g* have a compact support. For some interpolation argument (cf. [10], Theorem 1.3), we content ourselves to the case $p = 2$. By the following duality:

$$K(\lambda.Id - T_H)^{-1} = [[(\lambda.Id - T_H)^{-1}]^* K^*]^*$$

and by the Schauder theorem, we can only use the operator $(\lambda.Id - T_H)^{-1}K$. In addition, the existence of the operator $[H(\lambda.Id)]^{-1}$ permits us to write $(\lambda.Id - T_H)^{-1}$ as:

$$(\lambda.Id - T_H)^{-1} = S_\lambda + (\lambda.Id - T_0)^{-1},$$

where

$$S_\lambda = \sum_{n\geq 0} B_\lambda H[H(\lambda)]^{-1} G_\lambda.$$

Then

$$(\lambda.Id - T_H)^{-1} K = \sum_{n\geq 0} B_\lambda H[H(\lambda)]^{-1} G_\lambda K + (\lambda.Id - T_0)^{-1} K.$$

Since the operator $(\lambda.Id - T_0)^{-1} K$ is compact (c.f. [9]), to prove that the operator $(\lambda.Id - T_H)^{-1} K$ is compact, one only needs to establish that the operator $G_\lambda K$ is compact but this is a consequence of the uniform convergence of the series $\sum_{n\geq 0} B_\lambda H[H(\lambda)]^{-1} G_\lambda K.$ For all $\varphi \in X_p,$ we have:

$$(G_\lambda K)(\varphi) = G_\lambda[K\varphi] = (C_\lambda K)(\varphi)_{|\Gamma_+}.$$

On the other hand, we can prove that the operator $G_\lambda : X_p \mapsto L^{p,+}$ can be written as

$$G_\lambda = T_r \circ C_\lambda,$$

where T_r is the trace operator $D(T_0) \to L^{p,+}$. Letting B_p be the unit ball of $X_p,$ we will prove that $(G_\lambda K)(B_p)$ is relatively compact in $L^{p,+}$. In fact,

$$(G_\lambda K)(B_p) = (C_\lambda K)(B_p)_{|\Gamma_+} = [(\lambda.Id - T_0)^{-1} K](B_p)_{|\Gamma_+}$$

$$= T_r[(\lambda.Id - T_0)^{-1} K](B_p).$$

But $(\lambda.Id - T_0)^{-1} K(B_p)$ is relatively compact in X_p and preserves also this property in $(D(T_0), \|\cdot\|)$. Since $T_r : (D(T_0), \|\cdot\|) \mapsto L^{p,+}$ is bounded, the range of the compact set $(D(T_0), \|\cdot\|)$ is compact in $L^{p,+}$, so

$(G_\lambda K)(B_p)$ is relatively compact in $L^{p,+}$. So, $K(\lambda.Id - T_H)^{-1}$ and $(\lambda.Id - T_H)^{-1}K$ are compact subsets of X_p. □

Remark 4.4. We note that the compactness of the operator $K(\lambda.Id - T_H)^{-1}$ (or $(\lambda.Id - T_H)^{-1}K$) is independent from the border operator *H*, in fact, we have used only the bounded property of *H*. □

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