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MORE ON THE POINTWISE SEMI-OPEN INTERSECTION PROPERTY

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Abstract

Within this paper, questions concerning subspaces of spaces with the pointwise semi-open intersection property are resolved, equality of some of the topologies associated with the pointwise semi-open intersection property is investigated, and additional topological characterizations of finite sets are given.

1. Introduction

A question often arising for a given property of topological spaces concerns subspaces: "Does the space have the property iff each subspace of the space has the property?", i.e., is the property a subspace property? In a recent paper [1], it was noted that the proof of the converse statement in a subspace theorem is always quick and easy simply citing the space is a subspace of itself and thus, in no way, utilizes the property itself. In response, proper subspace inherited properties were defined and investigated

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[1], giving the properties themselves a new, central role in the consideration of subspace questions.

Definition 1.1. Let (X, T) be a space and *P* be a property of topological spaces. If every proper subspace of (X, T) has property *P* implies (X, T) has property *P*, then property *P* is called a *proper subspace inherited property* (*psip*) [1].

The work in this paper began with questions concerning subspaces of spaces with the pointwise semi-open intersection property and grew into the work below. Definitions and results used in this paper are given below.

Semi-open sets were introduced in 1963 [5].

Definition 1.2. Let (X, T) be a space and let $A \subseteq X$. Then *A* is semi-open, denoted by $A \in SO(X, T)$, iff there exists an $O \in T$ such that $O \subseteq A \subseteq Cl(O)$ [5].

Recently [9], semi-open sets were used to define the pointwise semi-open intersection property.

Definition 1.3. Let (X, T) be a space. Then (X, T) has the pointwise semi-open intersection property iff for each $x \in X$, there exist semi-open sets *U* and *V* such that $\{x\} = U \cap V$ [9].

In 1984 [8], the semi-open set generated topology was introduced and used to characterize semi-compact spaces.

Definition 1.4. A space (X, T) is semi-compact iff each cover of *X* by semi-open sets has a finite subcover [2].

Definition 1.5. Let (X, T) be a space. Then the topology $TSO(X, T)$ on *X* with subbase $SO(X, T)$ is called the *semi-open set topology* on *X* generated by (X, T) [8].

Theorem 1.1. *A space* (X, T) *is semi-compact iff* $(X, TSO(X, T))$ *is compact* [8].

Further work on the pointwise semi-open intersection property led to the results below.

Theorem 1.2. *Let* (X, T) *be a space*. *Then the following are equivalent*: (a) (X, T) *has the pointwise semi-open intersection property*, (b) $X = \{x \mid \{x\} \in T\} \cup \{x \mid$ *there exist disjoint open sets U and V such that* $x \in Cl(U) \cap Cl(V)\}$, *and* (c) $TSO(X, T)$ *is the discrete topology on X* [3].

In the work below, questions concerning subspaces of spaces with the pointwise semi-open intersection property are resolved, conditions on a space (X, T) for which $SO(X, T)$ is a topology and for which $T = TSO(X, T)$ are given, and additional topological characterizations of nonempty finite sets are given.

2. Subspaces of Spaces with the Pointwise Semi-open Intersection Property

As in the previous work on proper subspace inherited properties, only spaces with three or more elements are considered. The examples below show subspaces of spaces with the pointwise semi-open intersection property need not have the pointwise semi-open intersection property.

Example 2.1. Let *X* be an infinite set, let $Y = \{x_i \mid i \in \mathbf{N}\}$, where **N** denotes the set of natural numbers, be a subset of *X*, let $n \in \mathbf{N}$; $n \geq 2$, and let T_n be the topology on *X* with subbase $\{\{x_i\} \mid i = 1, ..., n\} \cup \{X\}$. Then, for each *n*, (X, T_n) has the pointwise semi-open intersection property and for $Z = X \setminus \{x_i \mid i = 1, ..., n\}$, (Z, T_Z) does not have the pointwise semi-open intersection property, where T_Z is the subspace topology on *Z*.

Thus, for a set, there can be many topologies on the set having the pointwise semi-open intersection property.

Regular open sets were introduced in 1937 [10].

Definition 2.1. Let (X, T) be a space and let $O \subseteq X$. If $O =$

$Int(Cl(O))$, then O is said to be *regularly open* in (X, T), denoted by $O \in RO(X, T)$ [10].

In the 1937 paper [10], it was proven that for a space (X, T), $RO(X, T)$ is a base for a topology T_S on X, and (X, T_S) was called the *semiregularization space* of (X, T).

Theorem 2.1. *Let* (X, T) *be a space. Then* (a) (X, T) *has the pointwise semi-open intersection property if and only if* (b) (X, T_S) *has the pointwise semi-open intersection property.*

Proof. (a) implies (b): Let $x \in X$. Consider the case that $\{x\} \in T$. Suppose $\{x\} \notin T_S$. Then $O = Int_T(Cl(\{x\})) \neq \{x\}$. Let $y \in O \setminus \{x\}$. Then $\{y\} \notin T$. Let U and V be disjoint T-open sets such that $y \in Cl_T(U) \cap Cl_T(V)$. Then $x \notin U$ or $x \notin V$, say $x \notin U$. Since U and $\{x\}$ are disjoint T-open sets, $U \cap Cl_T(\{x\}) = \emptyset$, $U \cap Int_T(Cl_T(\{x\})) = \emptyset$, and $Cl_T(U) \cap Int_T(Cl(\{x\})) = \emptyset$, which is a contradiction. Hence $\{x\} \in T_S$. Thus, consider the case that $\{x\} \notin T$. Let Z and W be disjoint T-open sets such that $x \in Cl_T(Z) \cap Cl_T(W)$. Since for each $Y \in T$, $Cl_T(Y) = Cl_T(Int_T(Cl_T(Y)))$, $A = Int_T(Cl_T(Z))$ and $B = Int_T(Cl_T(W))$ are disjoint T_S-open sets. Let $C \in T_S$ be such that $x \in C$. Then $C \in T$ and $C \cap Cl_T(Z) = C \cap Cl_T(Int_T(Cl_T(Z))) \neq \emptyset$. Thus, $C \cap Cl_{T_S}(A) \neq \emptyset$. Similarly, $C \cap Cl_{T_S}(B) \neq \emptyset$ and A and B are disjoint T_S-open sets such that $x \in Cl_{T_S}(A) \cap Cl_{T_S}(B)$. Hence $X = \{x \mid \{x\} \in T_S\} \cup \{x \mid$ there exist disjoint T_S-open sets A and B such that $x \in Cl_{T_S}(A) \cap Cl_{T_S}(B)\}$ and (X, T_S) has the pointwise semi-open intersection property.

(b) implies (a): Let $x \in X$. Since $T_S \subseteq T$, if $\{x\} \in T_S$, then $\{x\} \in T$. Thus, consider the case that $\{x\} \notin T_S$. Let A and B be disjoint T_S-open sets such that $x \in Int_T(Cl_{T_S}(A)) \cap Int_T(Cl_{T_S}(B))$. Then A and B are disjoint T-open sets. Let $O \in T$ be such that $x \in O$. Then $x \in Int_T(Cl_T(O)) \in T_s \subseteq T$ and $\emptyset \neq A \cap Int_T(Cl_T(O))$. Let $z \in A \cap Int_T(Cl_T(O))$. Thus, $z \in Cl_T(O)$ and $O \cap A = O \cap (A \cap Int_T(Cl_T(O))) \neq \emptyset$ and $x \in Cl_T(A)$. Similarly, $x \in Cl_T(B)$. Hence $X = \{x \mid \{x\} \in T\} \cup \{x \mid$ there exist disjoint T-open sets U and V such that $x \in Cl_T(U) \cap Cl_T(V)\}$ and (X, T) has the pointwise semi-open intersection property.

Based on the discussion above, what would happen if every proper subspace of a space has the pointwise semi-open intersection property?

Theorem 2.2. *Let (X, T) be a space such that every proper subspace of (X, T) has the pointwise semi-open intersection property. Then (X, T) is T_1 and has the pointwise semi-open intersection property.*

Proof. Let x and y be distinct elements of X and let $Y = \{x, y\}$. Since X has three or more elements, Y is a proper subset of X and (Y, T_Y) has the pointwise semi-open intersection property. Since there do not exist disjoint T_Y-open sets whose closures intersect, $\{x\}, \{y\} \in T_Y$ and there exist T-open sets C and D such that $x \in C$, $y \notin C$, $y \in D$, and $x \notin D$. Hence (X, T) is T_1.

Let $u \in X$. If $\{u\} \in T$, then $\{u\} \in TSO(X, T)$. Thus, consider the case that $\{u\} \notin T$. Let $v \in X$; $v \neq u$. Then $\{v\}$ is T-closed, $Z = X \setminus \{v\} \in T$, and (Z, T_Z) has the pointwise semi-open intersection property. Since $\{u\} \notin T$, $\{u\} \notin T_Z$. Let U and V be disjoint T_Z-open sets such that $u \in Cl_{T_Z}(U) \cap Cl_{T_Z}(V)$. Since $Z \in T$, U and V are disjoint T-open sets, $u \in Cl_{T_Z}(U) \cap Cl_{T_Z}(V) \subseteq Cl_T(U) \cap Cl_T(V)$, and $\{u\} \in TSO(X, T)$. Thus, (X, T) has the pointwise semi-open intersection property.

Corollary 2.1. *The pointwise semi-open intersection property is a proper subspace inherited property.*

Could the pointwise semi-open intersection property be strengthened, where the new property would be both a subspace property and a proper subspace inherited property?

Definition 2.2. Let (X, T) be a space. If every subspace of (X, T) has the pointwise semi-open intersection property, then (X, T) is said to have the hereditary pointwise semi-open intersection property.

Theorem 2.3. *Let* (X, T) *be a space. Then the following are equivalent*: (a) (X, T) *has the hereditary pointwise semi-open intersection property*, (b) (X, T) *has the pointwise semi-open intersection property and, in* (X, T), *the pointwise semi-open intersection property and the hereditary pointwise semi-open intersection property are equivalent*, (c) *for each* $x \in X$ *and* $Y = X \setminus \{x\}$, (Y, T_Y) *has the hereditary pointwise semi-open intersection property*, (d) *every proper subspace of* (X, T) *has the pointwise semi-open intersection property*, *and* (e) *every proper subspace of* (X, T) *has the hereditary pointwise semi-open intersection property*.

Proof. Clearly, (a) implies (b).

(b) implies (c): Let $x \in X$ and let $Y = X \setminus \{x\}$. Since (X, T) has the hereditary pointwise semi-open intersection property, (Y, T_Y) has the pointwise semi-open intersection property and, since every subspace of (Y, T_Y) is a subspace of (X, T), every subspace of (Y, T_Y) has the pointwise semi-open intersection property. Hence (Y, T_Y) has the hereditary pointwise semi-open intersection property.

(c) implies (d): Let Z be a proper subset of X. Let $x \in X \setminus Z$ and let $Y = X \setminus \{x\}$. Since (Y, T_Y) has the hereditary pointwise semi-open intersection property, $(Z, T_{YZ}) = (Z, T_Z)$ has the pointwise semi-open intersection property.

(d) implies (e): Let Z be a proper subset of X. Then (Z, T_Z) has the pointwise semi-open intersection property and every subspace of (Z, T_Z) has the pointwise semi-open intersection property, which implies (Z, T_Z) has the hereditary pointwise semi-open intersection property.

(e) implies (a): Since every proper subspace of (X, T) has the hereditary pointwise semi-open intersection property, every proper subspace of (X, T) has the pointwise semi-open intersection property and, by Theorem 2.2, (X, T) has the pointwise semi-open intersection property and (X, T) has the hereditary pointwise semi-open intersection property.

Corollary 2.2. *The hereditary pointwise semi-open intersection property is both a subspace property and a psip.*

The examples in Example 2.1, which are not T_1, show that the hereditary pointwise semi-open intersection property is stronger than the pointwise semi-open intersection property. Must spaces with the pointwise semi-open intersection property have a separation property?

In 1975 [6], T_i; $i = 0, 1, 2,$ was generalized to semi-T_i by replacing open in the definition of T_i by semi-open; $i = 0, 1, 2,$ respectively.

Theorem 2.4. *Let (X, T) have the pointwise semi-open intersection property. Then (X, T) is semi-T_2.*

Proof. Since $TSO(X, T)$ is the discrete topology on X, $(X, TSO(X, T))$ is T_2 and since $(X, TSO(X, Y))$ is T_2 iff (X, T) is semi-T_2 [4], (X, T) is semi-T_2.

Does the hereditary pointwise semi-open intersection property behave in the same manner as the pointwise semi-open intersection property with respect to semiregularization spaces as given in Theorem 2.1?

Example 2.2. Let T be the topology on $\mathbf{N}$ with base $\{\{n\} \mid n \geq 3\} \cup \{O \mid 1 \in O$ and $\mathbf{N} \setminus O$ is finite$\} \cup \{O \mid 2 \in O$ and $\mathbf{N} \setminus O$ is finite$\}$. Then (X, T) has the hereditary pointwise semi-open intersection property, but since every T_s-open set containing 1 contains 2, and vice-versa, $(\mathbf{N}, T_s)$ does not have the hereditary pointwise semi-open intersection property.

Theorem 2.5. *Let (X, T) be a space. Then the following are equivalent*:

(a) (X, T) *has the hereditary pointwise semi-open intersection property*, (b) *for each subset Y of X*, $(Y, (T_Y)_s)$ *has the pointwise semi-open intersection property*, *and* (c) *for each proper subset Y of X*, $(Y, (T_Y)_s)$ *has the pointwise semi-open intersection property*.

Proof. (a) implies (b): Let Y be a subset of X. If $Y = X$, then, by Theorem 2.1, (X, T_s) has the pointwise semi-open intersection property. Thus, consider the case that Y is a proper subset of X. Then, by Theorem 2.3, (Y, T_Y) has the pointwise semi-open intersection property and, by Theorem 2.1, (Y, T_Y) has the pointwise semi-open intersection property.

Clearly, (b) implies (c).

(c) implies (a): Let Y be a proper subset of X. Since $(Y, (T_Y)_s)$ has the pointwise semi-open intersection property, by Theorem 2.1, (Y, T_Y) has the pointwise semi-open intersection property. Hence, by Theorem 2.3, (X, T) has the hereditary pointwise semi-open intersection property.

As proven in Theorem 2.2, if (X, T) has the hereditary semi-open intersection property, then (X, T) is T_1 and has the pointwise semi-open intersection property. What about the converse statement?

Example 2.3. Let X be the positive real number and let $Y = X \setminus \mathbf{N}$. Let T be the topology on X with base $\{\{n\} \mid n \in \mathbf{N}\} \cup \{O \mid X \setminus O \text{ is finite}\}$. Then (X, T) is T_1 and has the pointwise semi-open intersection property, but (Y, T_Y) does not have the pointwise semi-open intersection property.

3. Equality of Topologies and New Topological Characterizations of Nonempty Finite Sets

Results in a 1965 paper [7] include the following result.

Theorem 3.1. *Let* (X, T) *be a space*. *Then* $SO(X, T)$ *is a topology on X iff* (X, T) *is extremally disconnected* [7].

Definition 3.1. A space (X, T) is extremally disconnected iff for each $O \in T$, $Cl(O) \in T$ [11].

The result above is used in the next two results.

Corollary 3.1. *Let* (X, T) *be a space. Then the following are equivalent*: (a) $SO(X, T)$ *is a topology on* X, (b) (X, T) *is extremally disconnected*, *and* (c) $SO(X, T) = TSO(X, T)$.

Theorem 3.2. *Let* (X, T) *be a space. Then* $T = TSO(X, T)$ *iff* $T = SO(X, T)$.

Proof. Suppose $T = TSO(X, T)$. Since $T \subseteq SO(X, T) \subseteq TSO(X, T)$, $T = SO(X, T)$.

Conversely, suppose $T = SO(X, T)$. Then $SO(X, T)$ is a topology on X, $SO(X, T) = TSO(X, T)$, and $T = TSO(X, T)$.

Theorem 3.3. *Let* (X, T) *be a space and* $C(T)$ *be the family of closed sets in* (X, T). *Then the following are equivalent*: (a) T *is the discrete topology on* X, (b) $T = C(T)$ *and* (X, T) *has the pointwise semi-open intersection property*, (c) $T = TSO(X, T)$ *and* (X, T) *has the pointwise semi-open intersection property*, *and* (d) (X, T) *is extremally disconnected and has the pointwise semi-open intersection property*.

Proof. (a) implies (b): Since T is the discrete topology on X, every subset of X is both open and closed and $T = C(T)$. Since for each $x \in X$, $\{x\} \in T$ iff $\{x\} \in SO(X, T)$ [3], then $SO(X, T)$ is the discrete topology on X and (X, T) is extremally disconnected.

(b) implies (c): Suppose there exists $x \in X$ such that $\{x\} \notin T$. Let U and V be disjoint open sets such that $x \in Cl(U) \cap Cl(V)$. Then $Cl(U) = U$ and $Cl(V) = V$, which is a contradiction. Thus, T is the discrete topology on X and by the arguments above $T = SO(X, T) = TSO(X, T)$.

(c) implies (d): By Theorem 3.2, $SO(X, T)$ is a topology on X and (X, T) is extremally disconnected.

(d) implies (a): Suppose there exists an $x \in X$ such that $\{x\} \notin T$. Let U and V be disjoint open sets such that $x \in Cl(U) \cap Cl(V)$. Since U and V are disjoint open sets and (X, T) is extremally disconnected, $Cl(U)$ and $Cl(V)$ are disjoint sets, which is a contradiction. Hence T is the discrete topology on X.

Theorem 3.4. *Let X be a nonempty set. Then X is finite iff for each topology T on X for which (X, T) has the pointwise semi-open intersection property, (X, T) is semi-compact.*

Proof. If X is finite, then for any topology T on X, (X, T) is semi-compact. Thus, consider the case that for each topology T on X for which (X, T) has the pointwise semi-open intersection property, (X, T) is semi-compact. Suppose X is infinite and let T be the discrete topology on X. By Theorem 3.3, (X, T) has the pointwise semi-open intersection property and $T = TSO(X, T)$. Then $\{\{x\} \mid x \in X\}$ is a cover of X by $TSO(X, Y)$-open sets with no finite subcover and $(X, TSO(X, T))$ is not compact, which contradicts (X, T) is semi-compact. Hence X is finite.

Theorem 3.5. *Let X be a nonempty set. Then X is finite iff for a topology T on X, (X, T) is semi-T_1 iff (X, T) is semi-T_2.*

Proof. Suppose X is finite. Let T be a topology on X for which (X, T) is semi-T_1. Then X is finite and $(X, TSO(X, T))$ is T_1, which implies $TSO(X, T)$ is the discrete topology on X. Thus, $(X, TSO(X, T))$ is T_2, which implies (X, T) is semi-T_2. Semi-T_2 always implies semi-T_1.

Conversely, suppose that for a topology T on X, (X, T) is semi-T_1 iff (X, T) is semi-T_2. Suppose X is infinite. Let T be the finite complement topology on X. Then (X, T) is T_1 and thus, semi-T_1, but (X, T) is not semi-T_2. Hence X is finite.

References

[1] C. Dorsett, Proper subspace inherited properties, new characterizations of classical topological properties, and related new properties, Question and Answers in General Topology, accepted.

[2] C. Dorsett, Semi compact R_1 and product spaces, Bull. Malaysian Math. Soc. 2(3) (1980), 15-19.

[3] C. Dorsett, The pointwise semi-open intersection property, semi-strongly continuous, and irresolute functions, Texas J. Sci., accepted.

[4] C. Dorsett, New characterizations of semi-T_2, semi- T_1, and semi- T_0 spaces, Ultra Scientist 21(2) M (2009), 491-494.

[5] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963), 36-41.

[6] S. Maheshwari and R. Prasad, Some new separation axioms, Ann. Soc. Sci. Bruxelles Ser. I 89(3) (1975), 395-402.

[7] O. Njastad, On some classes of nearly open sets, Pacific J. Math. 15 (1965), 961-970.

[8] I. L. Reilly and M. K. Vamanamurthy, On semicompact spaces, Bull. Malaysian Math. Soc. (2) 7 (1984), 61-67.

[9] P. Roberson and J. Scheers, Semi-strong continuity, Texas J. Sci., accepted.

[10] M. Stone, Application of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. 41 (1937), 374-481.

[11] S. Willard, General Topology, Addison-Wesley, 1970.