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AN ANALOGUE OF THE PROTH-GILBREATH CONJECTURE

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Abstract

Let $\mathrm{gpf}(x)$ be the greatest prime factor of the integer $x > 1$. We formulate and prove an analogue of the Proth-Gilbreath's conjecture that uses the function $(x, y) \mapsto \mathrm{gpf}(x + y)$ instead of $(x, y) \mapsto |y - x|$ in the definition of the recursion. Thus, the new type of recursion is induced by the mapping that associates to any infinite

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prime vector $(q_1, q_2, q_3, \ldots)$ the vector $(\mathrm{gpf}(q_1 + q_2), \mathrm{gpf}(q_2 + q_3), \mathrm{gpf}(q_3 + q_4), \ldots)$. If we start the recursion from the initial prime vector $(p_1, p_2, p_3, \ldots)$, where p_i represents the ith prime, then we show that the first component in any subsequent infinite vector is always an element of the special set $A = \{2, 3, 5, 7\}$. A comparative analysis of the convergence speed in the classical Proth-Gilbreath recursion versus the one in its GPF-analogue is presented. The analysis shows that the components of the iterates in the GPF-analogue of the Gilbreath recursion are rapidly decreasing and quickly become elements of A, whereas the components of the iterates in the classical Gilbreath are decreasing at a slower rate.

1. Introduction

A conjecture usually attributed to Norman L. Gilbreath (although its origins can be traced back to François Proth in the 19th century) [9, 10, 13] involves the iteration of the mapping

$$\phi(x_1, x_2, x_3, \ldots) = (|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_4|, \ldots) \tag{1}$$

starting from the initial vector $P := (2, 3, 5, 7, 11, \ldots)$ representing the ordered sequence of primes:

The Proth-Gilbreath conjecture. The first component in the kth iterate of P is 1 for every positive k.

Computational evidence has been provided (in 1993, Odlyzko [9] verified it for all $k \leq 3.4 \cdot 10^{11}$), however, the conjecture remains unproven.

Let $\mathrm{gpf}(x)$ denote the greatest prime factor of the integer $x > 1$. Extensive research has been done on the behavior of the greatest prime factor function in various contexts: Fermat numbers [11], quadratic polynomials [7], cubic polynomials [8], integers in an interval [12], the asymptotic behavior of $\sum_{n \leq x} 1/\mathrm{gpf}(n)$ [6], etc.

An interesting class of sequences ("*greatest prime factor sequences*" [1, 2, 4]) consists of the sequences of primes $(q_n)_n$ satisfying a recurrence relation of the form

$$q_n = \operatorname{gpf}(a_1 q_{n-1} + a_2 q_{n-2} + \cdots + a_r q_{n-r} + a_0), \tag{2}$$

where $a_0, \ldots, a_r$ are nonnegative integers, not all zero. Computational evidence suggests a most intriguing property of the greatest prime factor sequences, a property expressed by the following "*GPF conjecture*" [1]:

The GPF conjecture. Any prime sequence $(q_n)_n$ satisfying a recurrence relation (2) is ultimately periodic.

Note that the particular case in which $r = 1$ and a_1 divides a_0, has been proved [2]. Also, a multidimensional analogue has been formulated and investigated for sequences of prime vectors satisfying a similar recursion type [5]. Moreover, a study of the non-associative commutative magma $\mathbb{P}$ consisting of the set of primes under the operation $(x, y) \mapsto \operatorname{gpf}(x + y)$ revealed an interesting property of the special set $A := \{2, 3, 5, 7\}$, to the effect that A is included in every submagma of $\mathbb{P}$ with more than one element [3]. The set A will play an important role in what follows.

2. The GPF Analogue of the Proth-Gilbreath Conjecture

Our contribution is in an area of interface between the two conjectures mentioned in the introduction. In the present paper, we formulate and prove an analogue of the Proth-Gilbreath's conjecture that uses the function $(x, y) \mapsto \operatorname{gpf}(x + y)$ instead of $(x, y) \mapsto |y - x|$ in the definition of the recursion. Thus, we will investigate the iterations of the mapping γ that associates to any infinite prime vector $(q_1, q_2, q_3, \ldots)$ the vector

$$\gamma(q_1, q_2, q_3, \ldots) := (\operatorname{gpf}(q_1 + q_2), \operatorname{gpf}(q_2 + q_3), \operatorname{gpf}(q_3 + q_4), \ldots). \tag{3}$$

An analysis of the repeated iterates of P under the action of (3) reveals that the first component in any subsequent vector is an element of the set A.

For a formal statement, let us use the notation $p_{0,n} := p_n$ (the nth prime) for $n \geq 1$, $p_{1,n} := \operatorname{gpf}(p_n + p_{n+1})$ for $n \geq 1$ and, inductively, $p_{k,n} = \operatorname{gpf}(p_{k-1,n} + p_{k-1,n+1})$ for $n \geq 1$. Thus, the vector $(p_{k,n})_{n \geq 1}$ represents $\gamma^{(k)}(P)$, the kth iterate of $P = (p_1, p_2, p_3, \ldots)$ under γ. The "GPF analogue" of the Proth-Gilbreath conjecture is the following:

Theorem 1 (The GPF version of the Proth-Gilbreath conjecture)**.** *For any* $k \geq 1$, *we have* $p_{k,1} \in A$.

To illustrate it, we include below a portion of the infinite array in which P constitutes the first row, while $(p_{k,n})_{n \geq 1}$ constitutes the $k+1$-th row for any $k \geq 1$:

$$\begin{array}{cccccccccccc}
2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 & 23 & 29 & 31 & \cdots \\
5 & 2 & 3 & 3 & 3 & 5 & 3 & 7 & 13 & 5 & 17 & \cdots \\
7 & 5 & 3 & 3 & 2 & 2 & 5 & 5 & 3 & 11 & 5 & \cdots \\
3 & 2 & 3 & 5 & 2 & 7 & 5 & 2 & 7 & 2 & 5 & \cdots \\
5 & 5 & 2 & 7 & 3 & 3 & 7 & 3 & 3 & 7 & 7 & \cdots \\
5 & 7 & 3 & 5 & 3 & 5 & 5 & 3 & 5 & 7 & 3 & \cdots \\
3 & 5 & 2 & 2 & 2 & 5 & 2 & 2 & 3 & 5 & 5 & \cdots \\
2 & 7 & 2 & 2 & 7 & 7 & 2 & 5 & 2 & 5 & 2 & \cdots \\
3 & 3 & 2 & 3 & 7 & 3 & 7 & 7 & 7 & 7 & 7 & \cdots \\
3 & 5 & 5 & 5 & 5 & 5 & 7 & 7 & 7 & 7 & 3 & \cdots \\
2 & 5 & 5 & 5 & 5 & 3 & 7 & 7 & 7 & 5 & 2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}. \tag{4}$$

Note that if Q_k $(k \geq 0)$ is the transpose of $\gamma^{(k)}(P)$, i.e., Q_k is the transpose of the $(k+1)$th row of (4), then the vector sequence $(Q_k)_{k \geq 0}$ satisfies the recursion $Q_k = \operatorname{gpf}(CQ_{k-1})$, where C is the infinite circulant matrix with first row $(1, 1, 0, 0, \ldots)$, the greatest prime factor function applies componentwise, and Q_0 is the transpose of P. Thus, the transposes of the rows of the array (4) may be seen as the terms of an infinite-dimensional instance of the multidimensional GPF sequences discussed in [5]. A comparison among the convergence speeds in the classical Proth-Gilbreath problem versus its GPF-analogue will be discussed in the last section.

3. Proof of the Main Result

With the notation introduced in the previous section, if $k \geq 1$ let us agree to call $p_{k-1,n}$ the *left parent*, and $p_{k-1,n+1}$ the *right parent*, of the "successor" entry $p_{k,n}$.

Let q be a prime number in the nth row for some $n \geq 1$. By *a path of q*, we understand a finite sequence $q_1, q_2, \ldots, q_n$ of array elements, where $q = q_1$, q_j is a parent of q_{j-1} for $j \geq 2$, and q_n lies on the first row. By an *admissible path of q*, we will understand a path $q_1, q_2, \ldots, q_n$ of q such that for each $j \geq 2$, either q_j is the larger of the parents of q_{j-1} or q_j is the left parent of q_{j-1} when q_{j-1} has equal parents.

Let q_1 be *on the first column and the nth row of the array*. The admissible path of q_1 will be of the form $q_1, q_2, \ldots, q_n$. Let $q_n = p_k$, the kth prime number for some $k \geq 1$. Note that such a path turns right $k - 1$ times.

Lemma 2 (Turns in admissible paths)**.** *In an admissible path, if* $1 \leq j \leq n-1$, *then we have* $q_{j+1} \geq q_j$ *unless* q_j *and* $q_{j+1} = q_j - 2$ *form a twin pair*.

Proof. Let q_1 be on the first column and the nth row of the array. The admissible path of q_1 will be of the form $q_1, q_2, \ldots, q_n$. Let $q_n = p_k$, the kth prime number for some $k \geq 1$. Then the path turned right $k - 1$ times.

If q_{j+1} is the right parent of q_j, let p be the left parent of q_j. Then $q_{j+1} > p$, which implies that q_{j+1} is odd. Since $\operatorname{gpf}(q_{j+1} + p) = q_j$ and $q_{j+1} > p$, the elements q_{j+1}, q_j and p must be relatively prime. Suppose $q_{j+1} < q_j$. If p is an odd prime, then $q_{j+1} < q_j \leq \dfrac{q_{j+1} + p}{2} < q_{j+1}$, a contradiction. Therefore, $p = 2$, in which case, we must have $q_j = 2 + q_{j+1}$.

Thus, we have shown that $q_{j+1} \geq q_j$ unless q_j and $q_{j+1} = q_j - 2$ form a twin prime pair.

If q_{j+1} is q_j's left parent, let p be q_j's right parent. Then $q_{j+1} \geq p$. If $q_{j+1} = p$, then $q_j = q_{j+1} = p$. If $q_{j+1} > p$, then arguing as before, we obtain, again, that $q_{j+1} \geq q_j$ unless q_j and $q_{j+1} = q_j - 2$ form a twin pair. □

Lemma 3 (Structure of admissible paths)**.** *Let us assume* $j \leq n - 2$ *satisfies* $q_j \geq 11$. *Then in the admissible path through* q_j, *either* $q_{j+1} \geq q_j$, *or there exists* $k > j$ *such that* $q_l = q_j - 2$ *for all* $j < l < k$, *while* $q_l > q_j$ *for all* $l \geq k$. *Moreover, if* $q_{j+1} > q_j$, *then* $q_{j+1} > q_j + 2$.

Proof. Indeed, if $q_{j+1} < q_j$, then $q_{j+1} = q_j - 2$, therefore q_{j+2} cannot be less than q_{j+1} (otherwise $q_{j+2} = q_{j+1} - 2 = q_j - 4 \geq 7$ is composite, and since the only arithmetic progression of primes with difference 2 is 3, 5, 7, it would follow $q_j = 3$, a contradiction).

Thus, either $q_{j+2} = q_{j+1} = q_j - 2$ or $q_{j+2} > q_{j+1} = q_j - 2$.

We will show that if $q_{j+2} > q_{j+1} = q_j - 2$, then $q_{j+2} \neq q_j$ (so that either $q_{j+2} = q_j - 2$ or $q_{j+2} > q_j$). Indeed, let a be the other parent of q_{j+1}. If $q_{j+2} = q_j$, then $a \leq q_j - 2$ (from the definition of an admissible path $a \leq q_j$, however, the equality $a = q_j$ is ruled out since in that case $q_{j+1} = q_j$). If $a = q_j - 2$ (necessarily a prime), then the recurrence relation $\operatorname{gpf}(a + q_{j+2}) = q_{j+1}$ translates into $\operatorname{gpf}((q_j - 2) + q_j) = q_j - 2$, a contradiction. Therefore, $a < q_j - 2$ and, since $q_j - 4$ is not a prime (as argued before), it follows $a < q_j - 4$. But a must be odd (indeed, if $a = 2$, then $a + q_{j+2} = 2 + q_j$ must be composite and odd, in which case, $q_{j+1} \leq$

$\frac{2 + q_j}{3} < q_j - 2)$. Since a is odd, $a + q_{j+2} = a + q_j$ is even, and therefore

$$q_j - 2 = q_{j+1} = \mathrm{gpf}(a + q_{j+2}) \leq \frac{a + q_{j+2}}{2} < \frac{q_j - 4 + q_j}{2} = q_j - 2,$$

which is a contradiction. We conclude that in the case $q_{j+2} > q_{j+1} = q_j - 2$, then $q_{j+2} \neq q_j$, which means that either $q_{j+2} = q_j - 2$ or $q_{j+2} > q_j$.

Carrying on with a similar argument, we find that either $q_{j+1} \geq q_j$ or

$$q_{j+1} = q_{j+2} = \cdots = q_{k-1} = q_j - 2$$

for some $k > j$, and $q_k > q_j$, in which case, $q_k > q_j + 2$ since $q_j + 2$ is composite (otherwise $q_j - 2, q_j, q_j + 2$ would be an arithmetic progression of primes, necessarily implying $q_j = 5$, a contradiction). Then $q_k \geq q_j + 4$, and starting from q_k, every subsequent term of the odd prime sequence $(q_l)_{l \geq k}$ either increases, remains the same, or drops precisely by 2. Therefore, the terms of $(q_l)_{l \geq k}$ must remain to the right of $q_j + 2$ (composite), and thus $q_l > q_j + 2 > q_j$ for all $l \geq k$.

Finally, assume $q_{j+1} > q_j = q \geq 11$, say. Let us assume it is not the case that $q_{j+1} > q_j + 2$. Then $q_{j+1} = q + 2$. Let a be the other parent of q_j. Then $a \leq q + 2$ and $\mathrm{gpf}(a + q + 2) = q$. Thus, a is a prime of the form $mq - 2$, and since $a \leq q + 2$, we get that $q - 2$, q and $q + 2$ will be an arithmetic progression of primes with difference 2 and $q \geq 11$, a contradiction. This concludes the proof. □

As a consequence of Lemma 3, any increase by d in an admissible path starting from a term $q_j \geq 11$ must satisfy $d > 2$, consequently it cannot be "overturned" by a subsequent decrease. Also, note that we cannot have any two consecutive decreases by 2 following from a prime that is 11 or more. As a direct consequence, if an admissible part starts from a term $q_1 \geq 11,$ then all its subsequent terms satisfy $q_j \geq 11.$

Before moving to the final part of the proof, let us summarize the general behavior of an admissible path L consisting of the sequence of primes $q_1, q_2, ..., q_n$:

The current prime q_j *in L may be followed by a further equal term* $(q_{j+1} = q_j)$ *only in a left turn, and may change* $(q_{j+1} \neq q_j)$ *in both left and right turns. If* $q_{j+1} \neq q_j \geq 11,$ *then either* $q_{j+1} > q_j + 2$ *or* $q_{j+1} = q_j - 2$ *in which case, after a possible constant segment (consisting of left turns), it will lead to a further increase that will lead to subsequent terms that are all greater than* q_j. *Any increase from a term* $q_j \geq 11$ *cannot be overturned by a subsequent decrease (necessarily by* 2). *Moreover, if* $q_1 \geq 11,$ *then* $q_j \geq 11$ *for all* $1 \leq j \leq n.$

Proof of Theorem 1. We proceed by contradiction. Assume $q_1 \geq 11 = p_5,$ and recall that $q_n = p_k,$ so that there are $k - 1$ right turns in the admissible path of q_1. Note that $q_j \geq 11$ for all $j = 1, 2, ..., n.$ In any right turn from an element q_j along this path, the next prime either increases (leading to a terminal segment consisting solely of entries greater than q_j - from Lemma 3) or decreases by 2 (from Lemma 2), in which case 2 is strictly smaller than the amount of the last increase that occurred previously, if such an increase occurred (from the proof of Lemma 3), it follows that each one out of the $k - 1$ right turns generates a novel prime, distinct from the previous ones. Thus, there are at least k distinct primes in the admissible path that starts from $q_1 \geq p_5,$ and the last path element, q_n is either the largest of the q_j's or 2 less than the largest of the q_j's. Consequently, $p_k = q_n \geq$

$p_{5+(k-1)-1} = p_{k+3}$, a contradiction. Thus, $q_1 < 11$, which proves that for any $k \geq 1$, we have $p_{k,1} \in \{2, 3, 5, 7\} = A$. □

4. Rate of Convergence: A Computational Analysis

The computational analysis (necessarily finitistic) of the convergence speed starts with a seed of the form

$$S_1 := (p_1, p_2, \ldots, p_M)$$

consisting of the first M primes, for a large enough M (in our analysis, we used $M = 200000$). Due to the fact the available information is limited by the finite vector lengths, under both types of iteration, GPF as well as Proth-Gilbreath, the length of the subsequent string must decrease by 1 each time we apply the corresponding transformation. Let us denote the GPF iterates of S_1 by S_k^{GPF}, and the Proth-Gilbreath iterates of S_1 by S_k^{PG}, where $k \geq 2$. For example,

$$S_2^{GPF} = (\mathrm{gpf}(p_1 + p_2), \mathrm{gpf}(p_2 + p_3), \ldots, \mathrm{gpf}(p_{M-1} + p_M))$$

and

$$S_2^{PG} = (| p_1 - p_2 |, | p_2 - p_3 |, \ldots, | p_{M-1} - p_M |).$$

Then S_k^{GPF} and S_k^{PG} are both vectors of length $M - k + 1$ for $k = 2, 3, \ldots$. Let X_k and Y_k be their maximum entries (sup norms), respectively. We notice that in each case, the iteration eventually enters a minimal, "stable state". The GPF-stable state is characterized by $X_k \leq 7$. Indeed, since $A = \{2, 3, 5, 7\}$ is closed under the operation $(x, y) \mapsto \mathrm{gpf}(x + y)$, whenever $X_k \leq 7$, it follows $X_t \leq 7$ for $t \geq k$. The PG-stable state is characterized by $Y_k \leq 2$: we find that the corresponding vectors have the first component 1 and all other 0 or 2 (note that whenever $Y_k \leq 2$, it follows $Y_t \leq 2$ for $t \geq k$).

The rates of convergence for the iterates we will measured by the rates of decay of the maximum components X_k and Y_k of S_k^{GPF} and S_k^{PG}, respectively, in the processes that lead to the corresponding stable states. Our analysis provided computational evidence to the effect that the GPF iterates approach *much faster* the GPF-stable state than the PG iterates approach the PG-stable state.

Thus, we found that even if we start with a fairly large M, after relatively very few iterates the vectors S_k^{GPF} will have the maximum component $X_k \leq 7$, that is, all $M - k + 1$ components of S_k^{GPF} are in the special set $A = \{2, 3, 5, 7\}$. For example, if the seed S_1 has $M = 200000$ components (which are, therefore, the first 200000 primes, the maximum being 2750159), then the vector S_{17}^{GPF} already has $X_{17} = 7$, that is all of its 199984 components of S_{17}^{GPF} are in A. To get a sense of the rate of convergence in this particular instance, the sequence of the maximum components X_k is 2750159, 1374617, 1127051, 530197, 212579, 34673, 10853, 4969, 757, 211, 193, 61, 43, 23, 13, 13, 7. On the other hand, the classical Proth-Gilbreath iteration generates a sequence of vectors for which the maximum components decrease (after the first iteration) much slower. With the same seed S_1, the maximum components Y_k of S_k^{PG} $(k \geq 2)$ exhibit a slow, roughly linear decrease: the first couple of terms Y_k following $Y_1 = 2750159$ are 148, 142, 142, 142, 130, 130, 126, 124, 118, 112, 108, 108, 104, 104, 104, 104, 104, 102, ... through $Y_{100} = 14$, until they reach the value $2 = Y_{114}$.

Comparing the logarithmic plots displayed in Figures 1 and 2, it is clearly noticeable that the rate of decrease to the minimal state is higher in the case of the GPF iterates than in the case of the classical Proth-Gilbreath process.

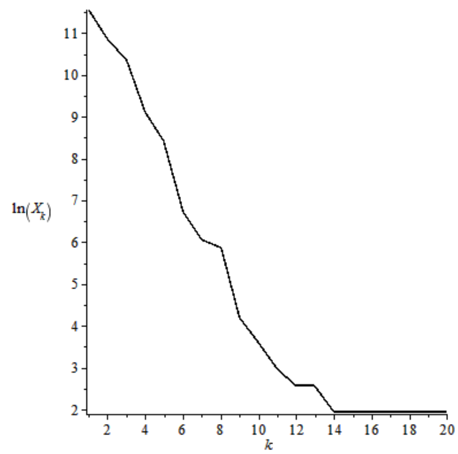

Figure 1. Logarithmic plot of the maximum components X_k of S_k^{GPF} with $2 \leq k \leq 20$.

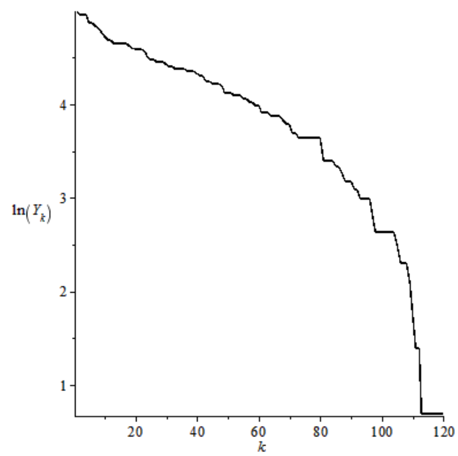

Figure 2. Logarithmic plot of the maximum components Y_k of S_k^{PG} with $2 \leq k \leq 120$.

Our computer experiment provides supporting evidence for a more general result than the one proved in the present paper. This refers to the behavior of the set of initial r components of the GPF-iterates produced from the initial infinite prime vector $P = (p_1, p_2, p_3, \ldots)$.

Conjecture. Let $r \geq 1$ be a fixed integer. Then for all sufficiently large k, we have $p_{k,i} \in A$ for all $i = 1, 2, \ldots, r$.

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