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A WAY FOR PRODUCING INTEGRABLE SYSTEMS

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Abstract

It may be a new way for generating new integrable systems by using Lie algebras of infinitesimal generators of some evolution equations. In the paper, we adopt the Lie algebra of the infinitesimal generators of the heat equation to introduce an isospectral Lax pair, for which a new integrable system is obtained.

1. Introduction

It is an important topic to search for new integrable systems. Zhang and his co-workers [1-3] once constructed some interesting Lie algebras and obtained some resulting integrable hierarchies and corresponding Hamiltonian structures. It follows that Zhao [4-6] applied some Lie algebras to deduce a few integrable systems and some of their properties. Recently, we find the Lie algebras of infinitesimal generators of some equations of evolution type which can be used to generate new integrable systems under the zero curvature equation. In the paper, we want to make use of the Lie algebra of the infinitesimal generators of the heat equation to introduce an isospectral Lax pair for which a new integrable system is obtained via using zero curvature equations.

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2. A New Integrable System

In [7] the Lie algebra of the infinitesimal generators of the heat equation, denoted by $G = X_1, \ldots, X_6,$ presents that:

$$X_1 = \frac{\partial}{\partial x_1},\ X_2 = \frac{\partial}{\partial x_2},\ X_3 = x_1\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2},$$

$$X_4 = x_1x_2\frac{\partial}{\partial x_1} + x_2^2\frac{\partial}{\partial x_2} - \left[\frac{x_1^2}{4} + \frac{x_2}{2}\right]u\frac{\partial}{\partial u},$$

$$X_5 = x_2\frac{\partial}{\partial x_1} - \frac{1}{2}x_1u\frac{\partial}{\partial u},\ X_6 = u\frac{\partial}{\partial u},$$

where x_1, x_2 are independent variables, while u is a dependent variable.

Define the Lie bracket as follow:

$$[X_i, X_j] = X_iX_j - X_jX_i,$$

one infers that

$$[X_1, X_3] = X_1, [X_1, X_4] = X_5, [X_1, X_5] = -\frac{1}{2}X_6, [X_2, X_3] = 2X_2,$$

$$[X_2, X_4] = X_3 - \frac{1}{2}X_6, [X_2, X_5] = X_1, [X_3, X_4] = 2X_4, [X_3, X_5] = X_5,$$

other commutative relations are all zero.

Let us define a loop algebra of the Lie algebra G:

$$\tilde{G} = \{X_1(n), \ldots, X_6(n)\},\quad X_i(n) = X_i\lambda^n,\quad i = 1, \ldots, 6,$$

$$[X_i(n), X_j(m)] = [X_i, X_j]\lambda^{m+n},\quad 1 \le i, j \le 6.$$

Using the loop algebra, $\tilde{G}$ introduces the isospectral problems:

$$\phi_x = U\phi, U = X_1(1) + u_1X_1(0) + u_2X_5(0), \tag{1}$$

$$\phi_t = V\phi, V = V_{1,m}X_1(-m) + V_{2,m}X_2(-m) + V_{4,m}X_4(-m) + V_{5,m}X_5(-m). \tag{2}$$

The stationary equation of the compatibility condition of equations (1) and (2) admits that

$$\begin{cases}(V_{1,m})_x = -u_2V_{2,m},\\ (V_{5,m})_x = V_{4,m+1} + u_1V_{4,m},\\ (V_{6,m})_x = -\frac{1}{2}V_{5,m+1} - \frac{1}{2}u_1V_{5,m} + \frac{1}{2}u_2V_{1,m}.\end{cases} \tag{3}$$

It is easy to see that equation (3) is local, where the term $V_{1,m}$ is free.

Set

$$V_+^{(n)} = V_{1,m}X_1(n-m) + V_{2,m}X_2(n-m)$$
$$+ V_{4,m}X_4(n-m) + V_{5,m}X_5(n-m) = \lambda^n V - V_-^{(n)}.$$

A direct calculation yields that

$$-V_{+,x}^{(n)} + [U, V_+^{(n)}] = -V_{4,n+1}X_5(0) + \frac{1}{2}V_{5,n+1}X_6(0).$$

Take $V^{(n)} = V_+^{(n)} - \frac{V_{5,n+1}}{u_2}X_1(0)$, we have that

$$-V_x^{(n)} + [U, V^{(n)}] = \left(\frac{V_{5,n+1}}{u_2}\right)_x X_1(0) - V_{4,n+1}X_5(0).$$

Therefore, the compatibility condition of the following Lax pair

$$\phi_x = U\phi,\ \phi_t = V^{(n)}\phi.$$

gives rise to a Lax integrable hierarchy

$$\begin{cases}u_{1,t} = -\left(\frac{V_{5,n+1}}{u_2}\right)_x,\\ u_{2,t} = V_{4,n+1},\end{cases} \tag{4}$$

by the recurrence relations (3) we can get some explicit evolution equations via reducing equation (4). As for as the Hamiltonian structure of equation (4) is concerned, we shall discuss it in the forthcoming days.

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