



CONTRIBUTION TO THE QUALITATIVE INVESTIGATION OF EQUATIONS OF CONSERVATION FOR TWO DIMENSIONAL INCOMPRESSIBLE FLOW PATTERNS VIA A COMPLEX ANALYSIS APPROACH

J. Venetis and E. Sideridis*

School of Civil Engineering

National Technical University of Athens

Greece

e-mail: john.venetis@yahoo.com

School of Applied Mathematics and Physical Sciences

Section of Mechanics

National Technical University of Athens

5 Heroes of Polytechnion Avenue GR – 157 73 Athens

Greece

e-mail: siderem@mail.ntua.gr

Abstract

The objective of this paper is to contribute towards the quality investigation of equations of conservation for two dimensional incompressible flow patterns via a Complex Analysis approach.

Specifically, the authors will utilize the aforementioned equations expressed in their known from literature equivalent special

Received: May 7, 2013; Accepted: June 29, 2013

2010 Mathematics Subject Classification: 76-XX, 32-XX.

Keywords and phrases: stream function, holomorphic function, analytic function, division ring, quaternion, Mittag-Leffler problem, holomorphic region.

*Corresponding author

formulation in terms of the stream function Ψ and its partial derivatives with respect to spatial coordinates.

In the sequel, by means of the Theory of Analytic Complex Functions, the authors will infer some quality information about the algebraic and geometrical properties for the family of functions which satisfy these equations for such flow patterns.

1. Introduction

The interface between Theoretical Fluid Dynamics and Complex Analysis is mainly illustrated in the implementation of conformal mapping methods. The application of these methods in such problems, premises indispensably a consideration of two dimensional flow patterns such that to be incompressible, inviscid, stationary and circulation free, where velocity components are apparently derivable from a potential.

On the other hand, the parabolic PDE which describes vorticity transfer inside viscous flow patterns belongs to the generic type of heat transfer by conduct.

Actually this PDE has been solved in a closed-form representation for two dimensions via Complex Analysis.

The major assumption is that the original dynamic problem reduces to a superimposition of quasi-stationary ones [4].

In the present paper, we will handle the system of equations of conservation expressed in terms of stream function and its partial derivatives with respect to spatial coordinates, incorporating concurrently the time-derivative terms as a time dependent non-homogeneity which occurs on the known from literature, equivalent to this system, alternative PDE.

2. Theoretical Remarks

The stream function $\Psi : R^3 \rightarrow R$ can be actually introduced for any incompressible flow patterns. Practically, its implementations mainly focused on two dimensional incompressible flux fields.

This function is a priori defined, such that to satisfy identically the equation of mass conservation.

In a two dimensional Cartesian Space, it is defined by means of the following differential representation:

$$V_x = \frac{\partial \Psi}{\partial y}$$

and

$$V_y = -\frac{\partial \Psi}{\partial x}. \quad (2.1)$$

On the other hand, this function also describes indeed in algebraic terms the geometry of circumstantial flux field, since the curves which are motivated by sequential constant rates of the stream function correspond injectively to the streamlines of the field.

The slope of an arbitrary curve with equation: $\Psi = ct$ (which corresponds to a streamline), at an arbitrary point: (x, y) is given by means of the following equation:

$$\left(\frac{dy}{dx}\right)_{\Psi} = \frac{V_y}{V_x}. \quad (2.2)$$

Besides, the difference of the corresponding rates of the stream function between two arbitrary points A and B of the field gives quantitatively and dimensionally the volumetric flow rate per unit of depth of fluid matter, between these points despite the occasional connecting trajectory:

$$\Psi_B - \Psi_A = \int_A^B (\vec{V} \cdot \vec{n}) ds. \quad (2.3)$$

Hence, when the rates of stream function occur increased along the positive direction of axis $y'y$, then the flow takes place along the positive direction of axis $x'x$ (from left to right).

3. Alternative Representation of the System of Governing PDEs

The stream function is by definition such that to satisfy identically the equation of mass conservation.

Therefore with the system of two equations of momentum conservation in hand, if we neglect gravitational forces and eliminate pressure terms, we obtain the following PDE [6]:

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) + \frac{\partial \Psi}{\partial y} \frac{\partial}{\partial x} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) - \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) \\ &= \frac{\mu}{\rho} \left(\frac{\partial^4 \Psi}{\partial x^4} + \frac{\partial^4 \Psi}{\partial y^4} + 2 \frac{\partial^4 \Psi}{\partial x^2 \partial y^2} \right) \end{aligned} \quad (3.1)$$

or equivalently in a more elegant representation:

$$\frac{\partial}{\partial t} (\nabla^2 \Psi) + \frac{\partial \Psi}{\partial y} \frac{\partial}{\partial x} (\nabla^2 \Psi) - \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y} (\nabla^2 \Psi) = \frac{\mu}{\rho} (\nabla^4 \Psi). \quad (3.2)$$

The latter expression holds for any kind of two dimensional incompressible unsteady flux fields, when the gravity influence is neglected.

Next, let us make the following substitutions:

$$\begin{aligned} x &= \frac{1}{2} (z + \bar{z}), \\ y &= \frac{1}{2i} (z - \bar{z}), \end{aligned} \quad \forall z \in C. \quad (3.3)$$

Then it follows:

$$\frac{\partial \Psi}{\partial x} = \frac{\partial \Psi}{\partial z} + \frac{\partial \Psi}{\partial \bar{z}}$$

and

$$\frac{\partial \Psi}{\partial y} = i \frac{\partial \Psi}{\partial z} - i \frac{\partial \Psi}{\partial \bar{z}}. \quad (3.4)$$

Apparently, it is valid:

$$\frac{\partial z}{\partial x} = 1, \quad \frac{\partial \bar{z}}{\partial x} = 1, \quad \frac{\partial z}{\partial y} = i, \quad \frac{\partial \bar{z}}{\partial y} = -i.$$

Thus, one can deduce:

$$\nabla^2 \Psi = 4 \frac{\partial^2 \Psi}{\partial z \partial \bar{z}}$$

and

$$\nabla^4 \Psi = 16 \frac{\partial^4 \Psi}{\partial z^2 \partial \bar{z}^2}. \quad (3.5)$$

Hence, equation (3.1) results to the following equivalent relationship with respect to the two conjugate complex variables:

$$\begin{aligned} & 4 \left(i \frac{\partial \Psi}{\partial z} - i \frac{\partial \Psi}{\partial \bar{z}} \right) \left(\frac{\partial^3 \Psi}{\partial z^2 \partial \bar{z}} + \frac{\partial^3 \Psi}{\partial z \partial \bar{z}^2} \right) - 4 \left(\frac{\partial \Psi}{\partial z} + \frac{\partial \Psi}{\partial \bar{z}} \right) \left(i \frac{\partial^3 \Psi}{\partial z^2 \partial \bar{z}} - i \frac{\partial^3 \Psi}{\partial z \partial \bar{z}^2} \right) \\ &= 16 \frac{\mu}{\rho} (\nabla^4 \Psi) - 2 \frac{\partial}{\partial t} (\nabla^2 \Psi). \end{aligned} \quad (3.6)$$

Then, after the essential hand calculation, one eventually obtains:

$$\frac{\partial \Psi}{\partial z} \cdot \frac{\partial^3 \Psi}{\partial z \partial \bar{z}^2} - \frac{\partial \Psi}{\partial \bar{z}} \cdot \frac{\partial^3 \Psi}{\partial z^2 \partial \bar{z}} = 2 \frac{\mu}{\rho} (\nabla^4 \Psi) - 2 \frac{\partial}{\partial t} (\nabla^2 \Psi). \quad (3.7)$$

We can also pinpoint here, that since the biharmonic term: $\nabla^4 \Psi$ can be also written out in the form: $\nabla^2(\nabla^2 \Psi)$, this latter PDE consists in a non-homogenous Cauchy-Riemann PDE, also known as $\bar{\partial}$ -equation [5].

The time dependent non-homogeneity here, is the quantity:

$$\frac{\partial \Psi}{\partial z} \cdot \frac{\partial^3 \Psi}{\partial z \partial \bar{z}^2} - \frac{\partial \Psi}{\partial \bar{z}} \cdot \frac{\partial^3 \Psi}{\partial z^2 \partial \bar{z}} + 2 \frac{\partial}{\partial t} (\nabla^2 \Psi).$$

Obviously, by solving $\bar{\partial}$ -equations one can generate holomorphic functions with known poles and isolated singularities as well, something that

consists in the substance of the known from literature Mittag-Leffler Problem [7].

However, the corresponding homogenous equation of equation (3.7) has a formal real solution in the following generic representation referred to as Goursat formula, obtained via the method of deduction [5]:

$$\Psi = 2 \operatorname{Re}[\bar{z}f_1(z) + k(z)], \quad (3.8)$$

where $f_1(z) = \int f_0(z)dz$.

Also, the auxiliary function f_0 is introduced as follows:

Since $\nabla^4 \Psi = 0$, it implies that

$$\nabla^2 \Psi = 2 \operatorname{Re} f(z) + c = 2 \operatorname{Re} f_0(z) = f_0(z) + \bar{f}_0(\bar{z}).$$

Hence, provided a partial solution of equation (3.7) one can superimpose after all the generic formal solution of this equation.

4. Investigation of Stream Function as a Complex Multi-valued One

Apparently, the right member of equation (3.7) is a real quantity. Hence, we can also infer that the left member of this equation, which constitutes a function in the form $f(\zeta): C^2 \rightarrow R: \zeta = \zeta(z_1, z_2), \forall (z_1, z_2) \in C^2: \bar{z}_1 \equiv z_2$ of the conjugate complex variables: z, \bar{z} , has indeed as range $R(f)$ a subset of R .

Nevertheless, we also know from advanced Complex Analysis that every ordered pair in the form: $(z_1, z_2) \in C^2$ consists in a quaternion according to Hamilton's definition and always belongs to a division ring, from modern Algebra stand point. Therefore, we can write out the following identity:

$$(z_1, z_2) \equiv ae + bi + cj + dk, \quad (4.1)$$

where the coefficients: a, b, c, d are real variables.

The following properties hold:

$$\begin{aligned}
 e &\equiv 1, \\
 i^2 &= j^2 = k^2 = -1, \\
 i \cdot j &= -j \cdot i = k, \\
 j \cdot k &= -k \cdot j = i, \\
 k \cdot i &= -i \cdot k = j.
 \end{aligned} \tag{4.2}$$

The term: ae consists in the scalar part of the quaternion.

The term: $bi + cj + dk$ consists in the vector part of the quaternion.

From geometrical stand point, the latter term can be considered equivalently as the geometrical summation of three vectors perpendicular to one another, drawn from the origin of a Cartesian coordinate system, moving or motionless.

On the other hand, the validity of Hartogs Theorem enables as to extend any holomorphic single-valued complex function with isolated singularities, to subsets of C^n which are actually referred to as *holomorphic regions*.

According to Cartan-Thullen Theorem, the following five statements are equivalent to one another:

- (i) A set called D is a *holomorphic region*.
- (ii) For any two open sets U, V such that $\emptyset \neq U \subset V \cap D \neq V$, there exists at least one function $f \in \mathfrak{U}(D)$ such that $f(\zeta) \neq \tilde{f}(\zeta)$, $\forall \tilde{f} \in \mathfrak{U}(V)$. We also denote that V consists in a connected set.

(We have symbolized by $\mathfrak{U}(V)$ the Space of every holomorphic function with domain of definition the connected set V .)

Besides:

$$\tilde{f}(\zeta) \equiv \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{z_1} dz_1 + \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\bar{z}_1} d\bar{z}_1, \quad \forall \zeta \in C^2, \tag{4.2A}$$

where the chain of the curves: $\gamma = \gamma_1 \cup \gamma_2$ has rotation index:

$$\delta(\zeta) \equiv \frac{1}{(2\pi i)^2} \int_{\gamma} \frac{d\zeta}{z_1 \bar{z}_1} dz_1 d\bar{z}_1.$$

(iii) $\forall K \subset\subset D \Rightarrow \text{dist}(K, \partial D) = \text{dist}(\bar{K}, \partial D)$ (which means that K consists in an open set such that to belong strongly to D and ∂D does not consists in a natural boundary).

$$(iv) \forall K \subset\subset D \Rightarrow \bar{K}_D \subset\subset D,$$

where the closure of the set K with respect to D is referred to as \bar{K}_D , which obviously means that:

$$\bar{K}_D = \{z \in D : |f(z)| \leq \sup |f(\zeta)|\}, \forall f \in \mathfrak{U}(V). \quad (4.2B)$$

$$(v) \forall X \subset D, \exists f \in \mathfrak{U}(D) \text{ such that } \sup |f(\zeta)| = \infty, \forall \zeta \in X.$$

Consider now, a multiple sequence $c_{k_n} : k_n \in N$ of complex variables and concurrently a sequence of the real variables $r_n : n \in N$ such that $r_n \geq 0$.

Next, let us formulate the following power series:

$$\sum_{k_1, \dots, k_n} |c_{k_n}| \cdot r_n^{k_n}.$$

Then according to Weierstrass Theorem, if the latter series converges with radius of convergence given by Cauchy-Hadamard formula:

$$\mathfrak{R} = \frac{1}{\limsup \sqrt[k]{|c_k|}}, \text{ then the following multi-valued complex function}$$

$f(\zeta)$ defined by the multiple power series:

$$f(\zeta) = \sum_{k_1, \dots, k_n} c_{k_1, \dots, k_n} \cdot (z_n - a_1)^{k_1} \dots (z_n - a_n)^{k_n} \quad (4.3)$$

is holomorphic

$$\forall \zeta = (z_1, z_2, \dots, z_n) : (z_1 - a_1) < r_1, (z_2 - a_2) < r_2, \dots, (z_n - a_n) < r_n.$$

It is also known from the theory of Analytic Functions that the function $f(\zeta)$ has derivatives of any order and actually its m -order derivative is given by the following formula:

$$\frac{\partial^{m_1+\dots+m_n}}{\partial^{m_1} \dots \partial^{m_n}} f(z) = \sum_{k \geq m} \frac{k!}{(k-m)!} c_k (z-a)^{k-m}, \quad (4.4)$$

where $z = z_1, z_2, \dots, z_n$.

Besides, the coefficients on equation (4.3) which are complex variables as well are also given by the next formula:

$$c_{k_1, \dots, k_n} = \frac{1}{k_1! \dots k_n!} \frac{\partial^{k_1+\dots+k_n}}{\partial^{k_1} \dots \partial^{k_n}} f(a). \quad (4.5)$$

Particularly for the original problem we study, we can write out:

$$\begin{aligned} \frac{\partial f}{\partial z_\xi}(\zeta) &= \sum_{k_1, \dots, k_n} k_\xi \cdot c_{k_1, \dots, k_n} (z_1 - a_1)^{k_1} \dots (z_\xi - a_\xi)^{k_\xi-1} \\ &\quad \dots (z_n - a_n)^{k_n}, \quad \xi = 1, 2. \end{aligned} \quad (4.6)$$

Then it implies:

$$\frac{\partial f}{\partial z_1}(\zeta) = \sum_{k_1, \dots, k_n} k_1 \cdot c_{k_1, \dots, k_n} (z_1 - a_1)^{k_1} \quad (4.7)$$

and

$$\frac{\partial f}{\partial z_2}(\zeta) = \sum_{k_1, \dots, k_n} k_2 \cdot c_{k_1, \dots, k_n} (z_1 - a_1)^{k_1} (z_2 - a_2)^{k_2-1}. \quad (4.8)$$

We remind that here, it is valid: $\bar{z}_1 \equiv z_2$.

Thus, the multi-valued complex function $f(\zeta)$ can be represented in general by Cauchy formula as an iterated integral with respect to the

variables z_1 and z_2 , but since this function is continuous on any domain of definition, it can be represented equivalently as a double integral in the form:

$$f(\zeta) = \frac{1}{(2\pi i)^2} \iint_{\zeta \in T} \frac{f(z_1, z_2)}{(z_1 - a_1)(z_2 - a_2)} dz_1 dz_2. \quad (4.9)$$

The region T is defined as follows:

$$T = \{(z_1, z_2) : \bar{z}_1 \equiv z_2 \wedge |z_1 - a_1| = \mathfrak{R}_1 \wedge |z_2 - a_2| = \mathfrak{R}_2\}, \quad (4.10)$$

where \mathfrak{R}_i denotes the circumstantial radius of convergence.

Apparently, the complex quantity $\frac{f(z_1, z_2)}{(z_1 - a_1)(z_2 - a_2)}$ can be expanded univocally in multiple analytic power series and since summation and integration are interchangeable operations in Complex Analysis, we can eventually deduce that the function $f(\zeta)$ is also analytic over any holomorphic region of its circumstantial domain $D(f) \subset C^2$.

Jordan Curve Theorem, also asserts us that for any closed, not self-intersecting and continuously contracted curve in the complex plane, (which can be the bound of a simply or multi connected region), its corresponding complementary set can be always represented as a union of two connected components [8].

Hence, by assuming that this curve consists in the solid bound of a two dimensional external viscous flux field, one can deduce that if a complex function is holomorphic on this bound (where it is valid that $\Psi = 0$), then at least one of these components of its complementary set, *a subset of it is actually the aforementioned flux field*, must consist in a holomorphic region.

On the other hand, since the conjugate complex variables z , \bar{z} are not functionally independent, there exists a function $G : C \rightarrow R$ such that:

$$G(z) \equiv \Psi(x, y) \equiv \Psi\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) \equiv f(z, \bar{z}) \equiv f(\zeta). \quad (4.11)$$

It is also known that in any region of the complex plane where function f is holomorphic, its anti-derivative with respect to z , if there exists, must be necessarily and sufficiently harmonic.

For any harmonic complex single-valued function $U(z)$, the following equivalence holds:

$$\nabla^2 U = 0 \Leftrightarrow U = g_1(z) + g_2(\bar{z}) \equiv 2 \operatorname{Re}(g_1(z)) + k, \quad k \in R. \quad (4.12)$$

Hence, we can investigate inceptively the existence of the anti-derivative of the function f .

Provided an open and simply connected set $D \subset C$, the function f has anti-derivative if and only if this aforementioned set is convex or generally astromorphic, even locally.

Then, the function $F = \int_{[\alpha, z]} f(\zeta) d\zeta$ consists in the set of the anti-derivatives of f , as long as the interval $[\alpha, z] \subset D, \forall z \in D$.

The following relationship is evident:

$$\int_{[\alpha, z]} f(\zeta) d\zeta = 2 \operatorname{Re}(g_1(z)) + k, \quad k \in R. \quad (4.13)$$

The solution of the above equation, with respect to the endpoints of the interval $[\alpha, z]$ enables us indeed to determine the holomorphic regions of the function f .

Therefore, we have also to exploit Green's Theorem in complex form which states that:

$$\int_{[\alpha, z]} f(\zeta) d\zeta = 2i \iint_{D_1} \frac{\partial f(z, \bar{z})}{\partial \bar{z}} dA, \quad (4.14)$$

where dA represents the element of the area $dx dy$ and D_1 denotes the corresponding disk to the above interval, with endpoints: a, z .

Thus, it implies:

$$2i \iint_{D_1} \frac{\partial f(z, \bar{z})}{\partial \bar{z}} dA = 2 \operatorname{Re}(g_1(z)) + k, \quad k \in \mathbb{R}. \quad (4.15)$$

The solution of the above equation with respect to the disk D_1 brings out the holomorphic regions of the function f .

Hence, the localization of the holomorphic regions in C^2 can be actualized via the combination of equations (4.14) and (4.15), provided beforehand that only isolated singularities there exist in the circumstantial domain of definition.

Moreover, we must also investigate the particular case, when a hole exists in the set $D \subset C$.

Suggestively, let us investigate the case when the set $D = C - \{0\}$, where obviously the function $\frac{1}{z}$ has not anti-derivatives. Then, the existence of anti-derivative in the above set is equivalent with the following statement:

$$\int_{C(0,1)} f(\zeta) d\zeta = 0. \quad (4.16)$$

On the other hand, in case this function is analytic the following relationship holds by definition:

$$\tilde{f}(\zeta) \equiv f(\zeta) - \left(\frac{c_0}{(\zeta - a)^m} + \dots + \frac{c_{m-1}}{(\zeta - a)} \right), \quad (4.16A)$$

where α is a non-essential singularity, isolated or not and also

$$c_j = \frac{1}{j} \frac{d^j}{dz^j} [(z - a)^m f(z)]|_{z=a}, \quad j = 0, 1, \dots, m-1.$$

Hence, by equalizing the right members of equations (4.2A) and (4.16A), one could determine the orientation of the chain of curves $\gamma = \gamma_1 \cup \gamma_2$, as well as the included area past non-essential singularities.

However, with equation (3.7) in hand, taking also into account that: $f(\zeta) : C^2 \rightarrow R : \zeta = \zeta(z_1, z_2), \forall (z_1, z_2) \in C^2 : \bar{z}_1 \equiv z_2$, we can infer:

$$\frac{\partial \Psi}{\partial z} \cdot \frac{\partial^3 \Psi}{\partial z \partial \bar{z}^2} - \frac{\partial \Psi}{\partial \bar{z}} \cdot \frac{\partial^3 \Psi}{\partial z^2 \partial \bar{z}} = Q(z, \bar{z}) \equiv \overline{Q(z, \bar{z})}, \quad (4.17)$$

where the function $Q : C^2 \rightarrow R$ consists in a complex-valued polynomial with generally complex coefficients. Besides, for the polynomial $Q(z, \bar{z})$, the following identity holds:

$$Q(z, \bar{z}) \equiv A_n(z) \bar{z}^n + \dots + A_1(z) \bar{z} + A_0(z), \quad (4.18)$$

where $A_0(z), A_1(z), \dots, A_n(z)$ are polynomials of the complex variable z having generally complex coefficients. Hence, we can also conclude:

$$\begin{aligned} \overline{Q(z, \bar{z})} &\equiv \overline{A_n(z) \bar{z}^n + \dots + A_1(z) \bar{z} + A_0(z)} \Leftrightarrow \\ \overline{Q(z, \bar{z})} &\equiv A_n(\bar{z}) z^n + \dots + A_1(\bar{z}) z + A_0(\bar{z}). \end{aligned} \quad (4.19)$$

Obviously, it comes along that:

$$A_0(\bar{z}) - A_0(z) = (A_n(z) \bar{z}^n + \dots + A_1(z) \bar{z}) - (A_n(\bar{z}) z^n + \dots + A_1(\bar{z}) z). \quad (4.20)$$

Then, by differentiating equation (4.18) and (4.19) with respect to \bar{z} and z sequentially, we are able to estimate the coefficients: $A_1(\bar{z}), A_1(z)$, since

$$\frac{\partial Q(z, \bar{z})}{\partial \bar{z}} = \frac{\partial \overline{Q(z, \bar{z})}}{\partial \bar{z}} \quad \text{and} \quad \frac{\partial Q(z, \bar{z})}{\partial z} = \frac{\partial \overline{Q(z, \bar{z})}}{\partial z}.$$

We have to denote here, that the evident derivation above was actually operated considering the variables z, \bar{z} as distinct to one another.

Hence, we can obtain:

$$\frac{\partial Q(z, \bar{z})}{\partial \bar{z}} = n A_n(z) \bar{z}^{n-1} + \dots + A_1(z), \quad (4.21)$$

$$\frac{\partial \overline{Q(z, \bar{z})}}{\partial \bar{z}} = \frac{d A_n(\bar{z})}{d \bar{z}} z^n + \dots + \frac{d A_1(\bar{z})}{d \bar{z}} z + \frac{d A_0(\bar{z})}{d \bar{z}} \quad (4.22)$$

and

$$\frac{\partial Q(z, \bar{z})}{\partial z} = \frac{dA_n(z)}{dz} \bar{z}^n + \dots + \frac{dA_1(z)}{dz} \bar{z} + \frac{dA_0(z)}{dz}, \quad (4.23)$$

$$\frac{\partial \overline{Q(z, \bar{z})}}{\partial z} = (n-1)A_n(\bar{z})z^{n-1} + \dots + A_1(\bar{z}). \quad (4.24)$$

In the sequel, the coefficients $A_1(z)$ and $A_1(\bar{z})$ are calculated by means of the following formulas:

$$\begin{aligned} A_1(z) = & ((n-1)A_n(z)\bar{z}^{n-1} + \dots + A_2(z)\bar{z}) \\ & - \left(\frac{dA_n(\bar{z})}{d\bar{z}} z^n + \dots + \frac{dA_1(\bar{z})}{d\bar{z}} z + \frac{dA_0(\bar{z})}{d\bar{z}} \right), \end{aligned} \quad (4.25)$$

$$\begin{aligned} A_1(\bar{z}) = & ((n-1)A_n(\bar{z})z^{n-1} + \dots + A_2(\bar{z})z) \\ & - \left(\frac{dA_n(z)}{dz} \bar{z}^n + \dots + \frac{dA_1(z)}{dz} \bar{z} + \frac{dA_0(z)}{dz} \right). \end{aligned} \quad (4.26)$$

Next, if we keep implementing this procedure above, we can also evaluate the whole number of the coefficients $A_i(z)$, $A_i(\bar{z})$, $i = 2, 3, \dots$, provided that the coefficient of zero order $A_0(z)$ is known and taking also into account that they are conjugate in couples.

Reverting now in equation (4.17), and operating sequential integrations with respect to the conjugate complex variables z , \bar{z} considering them as distinct to each other, one can eventually extract a quality but rigorous expression for the stream function Ψ .

On the other hand, since the rates of the polynomial $Q(z, \bar{z})$ are real variables, the following relationship holds:

$$|Q(z, \bar{z}) + i| = |Q(z, \bar{z}) - i| \quad (4.27)$$

and

$$|\Psi(z, \bar{z}) + i| = |\Psi(z, \bar{z}) - i|. \quad (4.28)$$

The geometrical interpretation of the above statements is that the rates of $Q(z, \bar{z})$ or $\Psi(z, \bar{z})$ are equidistant from the magnitudes $+1, -1$ located on the imaginary axis.

Consider next, an arbitrary positive parameterization of the curve $\Psi(z, \bar{z})$. Then, the angle between the tangent line and the axis of positive real numbers is given by the following relationship:

$$\frac{\Psi(z, \bar{z})}{|\Psi(z, \bar{z})|} = \text{EXP}\left(i \arg\left(\frac{d\Psi}{d\lambda}\right)\right) \quad (4.29)$$

or equivalently:

$$\left(\frac{dy}{dx}\right)_\Psi = \frac{V_y}{V_x}. \quad (4.30)$$

Besides, if $\Psi(z, \bar{z})$ has not global extremum, then for the real function $|\Psi(z, \bar{z})|^2$, the Principle of Maximum holds.

Thus, the above mapping turns all the resultant curves in the circumstantial stream lines network, keeping the angles constant in magnitude and in direction throughout. This property actually concerns any conformal mapping. One must also take into account here, that biharmonic functions do not in general remain biharmonic under a conformal transformation.

We can also remark here, that any two dimensional flux field can be imaged to complex plane O_{xy} (which is also referred to as Gauss plane), just like it is actually in the physical Euclidian Space, where the physical phenomenon evolves.

However, by means of an unknown mapping F , a one to one correspondence takes place from Gauss plane to axis O_Ψ , since $\Psi : C^2 \rightarrow R$.

According to Riemann Theorem, the mapping F is locally invertible.

However, any mapping in the form: $z \rightarrow \lambda z$ consists in a rotation with magnitude $\arg z$ and concurrently dilatation or contraction with ratio λ .

In fact, whenever this latter result is realistic from physical aspect, then all the algebraic properties of Analytic Functions can also be put into effect for a further modulation of equation (4.17).

5. Discussion

The intention of the present paper was the quality investigation of equations of conservation for two dimensional incompressible flow patterns via a Complex Analysis approach.

Actually we have exploited here a special form of momentum conservation PDEs, which the stream function Ψ occurs on, as the independent variable.

Next, by substituting the variables of position vector components by conjugate complex ones we obtained an alter-ego expression of the biharmonic representation of stream function as a complex multi-valued holomorphic function, whose range is still a subset of real numbers set. This means that this function is analytic having derivatives of any order.

Thus, it is able indeed to be represented in terms of Taylor or Laurent polynomial function past an arbitrary point of its domain provided that it does not consists in a singularity.

Hence, by sequential integrations with respect to the new conjugate complex variables, considering them as distinct to each other one can actually extract a quality but rigorous expression for the stream function, for the category of these physical problems.

References

- [1] M. Evgrafov, Analytic Functions, Dover, New York, 1978.
- [2] S. Goldstein, Lectures on Fluid Mechanics, Intersciences Publishers LTD, London, 1961.
- [3] E. Landau, Foundations of Analysis, Chelsea, New York, 1961.
- [4] O. Ladyzhenskaya, V. Solonnikov and N. Uraltseva, Linear and quasi-linear equations of parabolic type, American Math. Soc., 1968.

- [5] A. Markushevich, The Theory of Analytic Functions, A Second Course, MIR Publishers, Moscow, 1986.
- [6] A. Pandon, Incompressible Flow, John Wiley and Sons, New York, 1984.
- [7] R. Remmert, Classical Topics in Complex Function Theory, Springer, 1998.
- [8] H. Schaefer, Topological Vector Spaces, Springer-Verlag, 1970.
- [9] E. Titchmarsh, The Theory of Functions, 2nd ed., Oxford University Press, Oxford, 1975.
- [10] E. Whittaker and G. Watson, A Course of Modern Analysis, 4th ed., Cambridge University Press, London, 1963.
- [11] A. William and A. Vech, A Second Course in Complex Analysis, Benjamin, New York, 1967.
- [12] E. Zachmanoglou, Introduction to Partial Differential Equations with Applications, Williams and Wilkins, Baltimore, 1976.
- [13] E. Zauderer, Partial Differential Equations of Applied Mathematics, John Wiley and Sons, New York, 1989.