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A NOTE ON RIGHT SIMPLE ELEMENTS ORDERED SEMIGROUPS

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Abstract

An element a of an ordered semigroup $(S, \cdot, \leq)$ is said to be right simple if $(aS] = S$ (x is an element of $(aS]$ if $x \leq as$ for some $s \in S$). The purpose of this note is to study right simple element ordered semigroups: ordered semigroups containing right simple elements.

1. Preliminaries

An element a of a semigroup S is said to be a *right simple element* of S if $aS = S$. If every element of S is right simple, then S is called a *right simple semigroup*. A *right simple element semigroup* is defined as a semigroup S containing right simple elements.

In [3], Grimble (see also in [2, p. 40, Exercise 7]; and in [9]) proved the following theorem.

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Theorem 1.1. *Let S be a right simple element semigroup, and let R denote the set of all right simple elements of S. Then the following conditions hold*:

(i) *R is a subsemigroup of S*;

(ii) $S \backslash R$, *if it is nonempty, is the maximum right ideal of S and is prime*.

The purpose of this note is to extend the result based on ordered semigroups.

A semigroup $(S, \cdot)$ together with a partial order $\leq$ (on S) that is *compatible* with the semigroup operation, meaning that for $x, y, z \in S$,

$$x \leq y \Rightarrow zx \leq zy, \; xz \leq yz,$$

is called an *ordered semigroup* [1].

Let $(S, \cdot, \leq)$ be an ordered semigroup. If A, B are nonempty subsets of S, we let

$$AB = \{xy \mid x \in A, \; y \in B\}.$$

For $x \in S$, we write Ax and xA instead of $A\{x\}$ and $\{x\}A$, respectively. A nonempty subset A of S is called a *subsemigroup* of S if $AA \subseteq A$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left* (respectively, *right*) *ideal* [8] of S if

(i) $SA \subseteq A$ (respectively, $AS \subseteq A$);

(ii) for $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a (two-sided) *ideal* of S. The *maximum left* (respectively, *right*, *two-sided*) *ideal* of S is defined as the usual way.

A left (respectively, right, two-sided) ideal A of an ordered semigroup $(S, \cdot, \leq)$ is said to be *prime* [4] if for $x, y \in A$, $xy \in A$ implies $x \in A$ or $y \in A$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. For a nonempty subset A of S, let

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

If $a \in S$, then we write $(\{a\}]$ as $(a]$.

For nonempty subsets A, B of an ordered semigroup $(S, \cdot, \leq)$, the following conditions hold (see [8]):

(1) $A \subseteq (A]$;

(2) $A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $(A](B] \subseteq (AB]$;

(4) $((A](B]] = (AB]$;

(5) $(A \cup B] = (A] \cup (B]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. An element a of S is said to be *right simple* if $(aS] = S$. If every element of S is right simple, then S is called a *right simple ordered semigroup* [5]. We call S the *right simple element ordered semigroup* if S contains a right simple element.

2. Main Results

We begin with the following proposition considered the direct product of two right simple element ordered semigroups.

Proposition 2.1. *If* $(S, \cdot, \leq)$ *and* $(T, \circ, \preceq)$ *are right simple element ordered semigroups, then* $S \times T$ *is a right simple element ordered semigroup. Moreover, if* R *and* R' *are the sets of all right simple elements of* S *and* T*, respectively, then* $R \times R'$ *is the set of all right simple elements of* $S \times T$.

Proof. The proof is straightforward. □

The next result has been done on semigroups by Grimble [3] and it is also appears in ([2, p. 40, Exercise 7]; and in [9]).

Theorem 2.2. *Let* $(S, \cdot, \leq)$ *be a right simple element ordered semigroup, and let R denote the set of all right simple elements of S. Then the following conditions hold*:

(i) *R is a subsemigroup of S*;

(ii) $S \backslash R,$ *if it is nonempty, is the maximum right ideal of S and is prime.*

Proof. If *S* is right simple, then the claim is clear. We suppose that *S* is not right simple.

(i) If $a, b \in R,$ then $(aS] = S$ and $(bS] = S,$ and hence

$$S = (aS] = (a(bS]] \subseteq ((a](bS]] = (abS].$$

Thus $ab \in R.$

(ii) Assume that $S \backslash R$ is nonempty. Let $x \in S$ and $a \in S \backslash R.$ If $ax \in R,$ then $S = (axS] \subseteq (aS],$ and so $a \in R.$ This is a contradiction. Hence $ax \in S \backslash R.$ Let $x \in S$ and $a \in S \backslash R$ such that $x \leq a.$ If $x \in R,$ then $S = (xS] \subset (aS],$ and so $a \in R.$ This is a contradiction. Thus $x \in S \backslash R.$ This proves that $S \backslash R$ is a right ideal of *S*.

To show that $S \backslash R$ is the maximum right ideal of *S*, let *I* be a right ideal of *S* such that $(S \backslash R) \subset I.$ Then there is $a \in I \backslash (S \backslash R).$ Since $S = (aS] \subseteq I,$ so $S = I.$

It follows by (i) that $S \backslash R$ is prime and the proof is complete. □

The converse of Theorem 2.2 is as follows:

Theorem 2.3. *If an ordered semigroup* $(S, \cdot, \leq)$ *has the unique proper maximal right ideal A such that* $S \backslash A \neq (b]$ *for all* $b \in S \backslash A,$ *then the set of right simple elements of S is* $S \backslash A.$

Proof. Let *R* denote the set of all right simple elements of *S*. Since *A* is a proper right ideal of *S*, every elements of *A* is not right simple. Otherwise, if $a \in A$ is right simple, then $S = (aS] \subseteq A.$ This is a contradiction. Thus

$R \subseteq S \backslash A$. Let $b \in S \backslash A$. We have $(bS]$ is a right ideal of S. Suppose that $(bS] \subset S$. By assumption, $(bS] \subseteq A$. Thus, $(A \cup b]$ is a right ideal of S such that $A \subset (A \cup b]$, and thus $(A \cup b] = S$. This implies that $S \backslash A = (b]$ which is a contradiction. Hence $(bS] = S$. □

Let $(S, \cdot, \leq)$ be an ordered semigroup. For $a, b \in S$, the Green's relation $\mathcal{R}$ [6] on S is defined by

$$a\mathcal{R}b \text{ if and only if } (a \cup aS] = (b \cup aS].$$

An element a of S is said to be *right regular* [7] if $a \in (a^2S]$.

Theorem 2.4. *Let $(S, \cdot, \leq)$ be a right simple element ordered semigroup. Then the set of right simple elements of S, denoted by R, is an $\mathcal{R}$-class of S. Moreover, each right simple element is right regular.*

Proof. Let $a, b \in R$. Then $(aS] = S$ and $(bS] = S$. We have $(a \cup aS] = (b \cup bS]$, and hence $a\mathcal{R}b$. Let $x \in S$ be such that $x\mathcal{R}a$ for some $a \in \mathcal{R}$. This means that

$$(x \cup xS] = (a \cup aS] = S.$$

If $N = \emptyset$, then $x \in R$. If $N \neq \emptyset$ and $x \in N$, then $S = N$. This is a contradiction. Therefore, $x \notin N$, that is, $x \in R$.

If $a \in R$, then $a \in (aS] \subseteq (a(aS]] \subseteq (a^2S]$, and thus a is right regular. □

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