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OSCILLATION CRITERIA FOR THIRD ORDER NONLINEAR DELAY DIFFERENCE EQUATIONS

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Abstract

In this paper, we discuss the oscillation criteria for third order nonlinear delay difference equation of the form

$$\Delta(a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta}) + q_n f(x_{n-\tau}) = 0, \quad n \in \mathbb{N}_0.$$

An example is given to illustrate the main result.

1. Introduction

In this paper, we are concerned with the oscillation of third order nonlinear delay difference equation of the form

$$\Delta(a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta}) + q_n f(x_{n-\tau}) = 0, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and

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$\mathbb{N}_0 = \{n_0, n_0 + 1, n_0 + 2, ...\}$ and n_0 is a nonnegative integer subject to the following conditions:

(C_1) $\{a_n\}$, $\{b_n\}$ and $\{q_n\}$ are positive real sequences;

(C_2) α and β are ratios of odd positive integers;

(C_3) τ is a nonnegative integer;

(C_4) $f : \mathbb{R} \to \mathbb{R}$ is continuous functions such that $uf(u) > 0$ for $u \neq 0$ and $-f(-uv) \geq f(uv) \geq f(u)f(v)$ for $uv > 0$.

By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ and satisfying equation (1.1) for all $n \in \mathbb{N}_0$. We consider only that solution $\{x_n\}$ of equation (1.1) which satisfies $\sup\{|x_n| : n \geq N\} > 0$ for all $n \in \mathbb{N}_0$. A solution of equation (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In recent years, there is a great interest in studying the oscillatory behavior of third order nonlinear delay difference equations, see for example [1-5, 7-9, 11, 12] and the references cited therein. Motivated by this observation, in this paper, we obtain some sufficient conditions for the oscillation of all solutions of equation (1.1).

In Section 2, we establish some sufficient conditions for the oscillation of all solutions of equation (1.1) and an example is given to illustrate the main result. The result established in this paper is discrete analogue of that in [6].

2. Main Results

In this section, we establish some new oscillation theorem for equation (1.1). Throughout this paper, we use the following notation without further mention:

$$\delta_{n, n_0} = \sum_{s=n_0}^{n-1} b_s^{-1/\alpha},$$

$$\delta_n = \sum_{s=n_0}^{n} a_s^{-1/\beta}$$

and

$$\bar{\delta}_n = \sum_{s=n_0}^{n} b_s^{-1/\beta}.$$

We begin with the following lemma:

Lemma 2.1. *Assume that for all sufficiently large* $N_1 \in \mathbb{N}_0$, *there is a* $N > N_1$ *such that* $n - \tau > N_1$ *for* $n \geq N$ *and*

(H_1) *either*

$$\sum_{s=n_0}^{\infty} \frac{1}{a_s^{1/\beta}} = \infty \tag{2.1}$$

or

$$\sum_{n=N}^{\infty} a_n^{-1/\beta} \left[\sum_{s=N}^{n} q_s f(\delta_{s-\tau}^{1/\alpha}) f(\delta_{s-\tau, N}) \right]^{1/\beta} = \infty; \tag{2.2}$$

(H_2) *either*

$$\sum_{s=n_0}^{\infty} \frac{1}{b_s^{1/\alpha}} = \infty \tag{2.3}$$

or

$$\sum_{n=n_0}^{\infty} b_n^{-1/\alpha} \left[\sum_{s=n_0}^{n} a_s^{-1/\beta} \left[\sum_{t=s_0}^{s} q_t f(\bar{\delta}_{s-\tau}) \right]^{1/\beta} \right]^{1/\alpha} = \infty \tag{2.4}$$

hold. Let $\{x_n\}$ *be an eventually positive solution of equation* (1.1). *Then one of the following two cases holds*:

(i) $\Delta x_n > 0,\ \Delta(b_n(\Delta x_n)^{\alpha}) > 0$ *for all* $n \geq N$;

(ii) $\Delta x_n < 0,\ \Delta(b_n(\Delta x_n)^{\alpha}) > 0$ *for all* $n \geq N$.

Proof. Let $\{x_n\}$ be a positive solution, from equation (1.1), we have

$$\Delta(a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta}) = -q_n f(x_{n-\tau}) < 0 \text{ for } n \geq n_1.$$

Consequently, $(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta}$ is strictly decreasing and then Δx_n and $\Delta(b_n(\Delta x_n)^{\alpha})$ are eventually of one sign. We claim that $\Delta(b_n(\Delta x_n)^{\alpha}) > 0.$ If not, then we have two cases:

Case (i). There exists $n_2 \geq n_1,$ sufficiently large such that

$$\Delta x_n > 0, \text{ and } \Delta(b_n(\Delta x_n)^{\alpha}) < 0 \text{ for } n \geq n_2.$$

Case (ii). There exists $n_2 \geq n_1,$ sufficiently large such that

$$\Delta x_n < 0, \text{ and } \Delta(b_n(\Delta x_n)^{\alpha}) < 0 \text{ for } n \geq n_2.$$

For Case (i), we have $b_n(\Delta x_n)^{\alpha}$ is strictly decreasing and there exists a negative constant M such that

$$a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta} < M \text{ for all } n \geq n_2$$

or

$$\Delta(b_n(\Delta x_n)^{\alpha}) < \frac{M^{1/\beta}}{a_n^{1/\beta}}.$$

Summing from n_2 to $n-1,$ we get

$$b_n(\Delta x_n)^{\alpha} \leq b_{n_2}(\Delta x_{n_2})^{\alpha} + M^{1/\beta}\sum_{s=n_2}^{n-1}\frac{1}{a_s^{1/\beta}}.$$

Letting $n \to \infty$ and using (2.1) then $b_n(\Delta x_n)^{\alpha} \to -\infty,$ which contradicts that $\Delta x_n > 0.$ Hence (2.2) is satisfied, we have

$$x_n - x_{n_3} = \sum_{s=n_3}^{n-1} \Delta x_s$$

$$= \sum_{s=n_3}^{n-1} b_s^{-1/\alpha}((\Delta x_s)^{\alpha})^{1/\alpha} b_s^{1/\alpha}$$

$$\geq (b_n(\Delta x_n)^{\alpha})^{1/\alpha} \sum_{s=n_3}^{n-1} b_s^{-1/\alpha} \text{ for all } n \geq n_3$$

and hence

$$x_n \geq (b_n(\Delta x_n)^{\alpha})^{1/\alpha} \sum_{s=n_3}^{n-1} b_s^{-1/\alpha} \text{ for all } n \geq n_3$$

$$\geq (b_n(\Delta x_n)^{\alpha})^{1/\alpha} \delta_{n, n_3} \text{ for all } n \geq n_3.$$

There exists a $n_4 \geq n_3$ with $n - \tau \geq n_3$ for all $n \geq n_4$ such that

$$x_{n-\tau} \geq (b_{n-\tau}(\Delta x_{n-\tau})^{\alpha})^{1/\alpha} \delta_{n-\tau, n_3} \text{ for all } n \geq n_4.$$

From equation (1.1),

$$0 \geq \Delta(a_n(\Delta y_n)^{\beta}) + q_n f(y_{n-\tau}^{1/\alpha}) f(\delta_{n-\tau, n_3}) \text{ for all } n \geq n_4, \tag{2.5}$$

where $y_n = b_n(\Delta x_n)^{\alpha}.$ It is clear that $y_n > 0$ and $\Delta y_n < 0.$ It follows that

$$\Delta(a_n(\Delta y_n)^{\beta}) \leq 0 \text{ for } n \geq n_4.$$

Summing from $n-1$ to $n_4,$ we get

$$a_n(\Delta y_n)^{\beta} - a_{n_4}(\Delta y_{n_4})^{\beta} \leq 0$$

or

$$-a_n(\Delta y_n)^{\beta} \geq -a_{n_4}(\Delta y_{n_4})^{\beta}$$

or

$$\Delta y_n \geq \frac{-a_{n_4}^{1/\beta}(\Delta y_{n_4})}{a_n^{1/\beta}} \text{ for all } n \geq n_4.$$

Summing the last inequality from n to ∞, we obtain

$$-[y_\infty - y_n] \geq -\sum_{s=n}^{\infty} \frac{a_{s_4}^{1/\beta}(\Delta y_{s_4})}{a_s^{1/\beta}}$$

or

$$y_n \geq -a_{n_4}^{1/\beta}(\Delta y_{n_4}) \sum_{s=n}^{\infty} a_s^{-1/\beta}$$

hence

$$y_n \geq -a_{n_4}^{1/\beta}(\Delta y_{n_4})\delta_n$$

or

$$y_n \geq k_1 \delta_n \text{ for all } n \geq n_5,$$

where $k_1 = -a_{n_4}^{1/\beta}(\Delta y_{n_4}) > 0.$ There exists a $n_5 \geq n_4$ with $n - \tau \geq n_4$ for all $n \geq n_5$ such that

$$y_{n-\tau} \geq k_1 \delta_{n-\tau} \text{ for all } n \geq n_5.$$

Summing (2.5) from n_5 to $n-1$ and using the above inequality, we get

$$\sum_{s=n_5}^{n-1} q_s f(y_{n-\tau}^{1/\alpha}) f(\delta_{n-\tau, n_3}) \leq a_{n_5} \Delta(y_{n_5})^\beta - a_n \Delta(y_n)^\beta$$

or

$$\sum_{s=n_5}^{n-1} q_s f(k_1^{1/\alpha} \delta_{n-\tau}^{1/\alpha}) f(\delta_{n-\tau, n_3}) \leq -a_n \Delta(y_n)^\beta.$$

Now using condition (C_4), we have

$$\sum_{s=n_5}^{n-1} q_s f(k_1^{1/\alpha}) f(\delta_{n-\tau}^{1/\alpha}) f(\delta_{n-\tau, n_3}) \leq -a_n \Delta(y_n)^\beta$$

or

$$\left[\frac{f(k_1^{1/\alpha})}{a_n} \sum_{s=n_5}^{n-1} q_s f(\delta_{n-\tau}^{1/\alpha}) f(\delta_{n-\tau, n_3}) \right]^{1/\beta} \leq -\Delta(y_n).$$

Summing the above inequality from n_5 to ∞, we get

$$(f(k_1^{1/\alpha}))^{1/\beta} \sum_{s=n_5}^{\infty} \frac{1}{a_s^{1/\beta}} \left[\sum_{t=s_5}^{s-1} q_t f(\delta_{t-\tau}^{1/\alpha}) f(\delta_{t-\tau, t_3}) \right]^{1/\beta} \leq y_{n_5} < \infty$$

which contradicts the condition (2.2).

For Case (ii), we have

$$b_n(\Delta x_n)^\alpha \leq b_{n_2}(\Delta x_{n_2})^\alpha = c < 0$$

or

$$\Delta x_n \leq \frac{c^{1/\alpha}}{b_n^{1/\alpha}}.$$

Summing the last inequality from n_2 to $n-1$, we get

$$x_n \leq x_{n_2} + c^{1/\alpha} \sum_{s=n_2}^{n-1} b_n^{-1/\alpha}.$$

Letting $n \to \infty$, then (2.3) yields $x_n \to -\infty$. This contradicts that $x_n > 0$. Otherwise, if (2.4) is satisfied. One can choose $n_3 \geq n_2$ with $n - \tau \geq n_2$ for all $n \geq n_3$ such that

$$x_n \geq -(b_n(\Delta x_n)^\alpha)^{1/\alpha} \sum_{s=n_3}^{n-1} b_s^{-1/\alpha}$$

or

$$x_{n-\tau} \geq -(b_{n-\tau}(\Delta x_{n-\tau})^{\alpha})^{1/\alpha} \sum_{s=n_3}^{n-\tau-1} b_s^{-1/\alpha}$$

hence

$$x_{n-\tau} \geq k_2 \overline{\delta}_{n-\tau} \text{ for } n \geq n_3,$$

where $k_2 = -(b_{n-\tau}(\Delta x_{n-\tau})^{\alpha})^{1/\alpha}$. Then equation (1.1) and (C_4) yield

$$\begin{aligned}\Delta(a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta}) &= -q_n f(x_{n-\tau}) \\ &\leq -q_n f(k_2 \overline{\delta}_{n-\tau}) \\ &\leq -q_n f(k_2) f(\overline{\delta}_{n-\tau})\end{aligned}$$

or

$$\Delta(a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta}) \leq q_n L f(\overline{\delta}_{n-\tau}),$$

where $L = -f(k_2)$, summing the above inequality from n_3 to $n-1$, we get

$$a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta} \leq L \sum_{s=n_5}^{n-1} q_s f(\overline{\delta}_{s-\tau})$$

or

$$\Delta(b_n(\Delta x_n)^{\alpha}) \leq \frac{L^{1/\beta}}{a_n^{1/\beta}} \left[\sum_{s=n_3}^{n-1} q_s f(\overline{\delta}_{s-\tau}) \right]^{1/\beta}.$$

Summing the last inequality from n_3 to $n-1$, we have

$$b_n(\Delta x_n)^{\alpha} \leq L^{1/\beta} \sum_{s=n_3}^{n-1} a_n^{-1/\beta} \left[\sum_{t=s_3}^{s-1} q_t f(\overline{\delta}_{t-\tau}) \right]^{1/\beta}$$

or

$$\Delta x_n \leq \frac{L^{1/\alpha\beta}}{b_n^{1/\alpha}}\left[\sum_{s=n_3}^{n-1} a_n^{-1/\beta}\left[\sum_{t=s_3}^{s-1} q_t f(\bar{\delta}_{t-\tau})\right]^{1/\beta}\right]^{1/\alpha}.$$

Again summing the last inequality from n_3 to $n-1$, we have

$$x_n \leq L^{1/\alpha\beta}\sum_{s=n_3}^{n-1} b_s^{-1/\alpha}\left[\sum_{t=s_3}^{s-1} a_t^{-1/\beta}\left[\sum_{j=t_3}^{t-1} q_t f(\bar{\delta}_{j-\tau})\right]^{1/\beta}\right]^{1/\alpha}.$$

From condition (2.4), we get $x_n \to -\infty$ as $n \to \infty$ which contradicts that x_n is a positive solution of (1.1). Then we have $\Delta(b_n(\Delta x_n)^{\alpha}) > 0$ for $n \geq n_1$ and of one sign thus either $\Delta x_n > 0$ or $\Delta x_n < 0$. The proof is now complete. □

Lemma 2.2. *Assume that* (H_1) *and* (H_2) *hold. Let* $\{x_n\}$ *be an eventually positive solution of equation* (1.1) *for all* $n \in \mathbb{N}_0$ *and suppose that Case* (ii) *of Lemma* 2.1 *holds. If*

$$\sum_{n=n_0}^{\infty} b_n^{-1/\alpha}\left[\sum_{s=n}^{\infty} a_n^{-1/\beta}\left(\sum_{t=s}^{\infty} q_t\right)^{1/\beta}\right]^{1/\alpha} = \infty, \tag{2.6}$$

then $\lim_{n\to\infty} x_n = 0.$

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1). Then there exists $\ell \geq 0$ such that $\lim_{n\to\infty} x_n = \ell$. Assume $\ell > 0$, then we have $x_{n-\tau} \geq \ell$ for $n \geq n_2 \geq n_1$. Summing equation (1.1) from n to ∞, we have

$$a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta} \geq \sum_{s=n}^{\infty} q_s f(x_{s-\tau}) \geq f(x_{n-\tau})\sum_{s=n}^{\infty} q_s$$

or

$$\Delta(b_n(\Delta x_n)^{\alpha}) \geq \left(\frac{f(\ell)}{a_n}\right)^{1/\beta}\left(\sum_{s=n}^{\infty} q_s\right)^{1/\beta}.$$

Summing the last inequality from n to ∞, we have

$$-b_n(\Delta x_n)^{\alpha} \geq (f(\ell))^{1/\beta}\sum_{s=n}^{\infty} a_s^{-1/\beta}\left(\sum_{t=s}^{\infty} q_t\right)^{1/\beta}$$

or

$$-\Delta x_n \geq \frac{(f(\ell))^{1/\alpha\beta}}{b_n^{1/\alpha}}\left[\sum_{s=n}^{\infty} a_s^{-1/\beta}\left(\sum_{t=s}^{\infty} q_t\right)^{1/\beta}\right]^{1/\alpha}.$$

Again summing the last inequality from n_2 to ∞, we get

$$x_{n_2} \geq (f(\ell))^{1/\alpha\beta}\sum_{n=n_2}^{\infty} b_n^{-1/\alpha}\left[\sum_{s=n}^{\infty} a_s^{-1/\beta}\left(\sum_{t=s}^{\infty} q_t\right)^{1/\beta}\right]^{1/\alpha}.$$

This contradicts to the condition (2.6). The proof is complete. □

Theorem 2.1. *Let* (H_1) *and* (H_2) *hold and there exists an integer* σ *such that*

$$\sigma > \tau. \tag{2.7}$$

If both first order delay equations

$$\Delta y_n + q_n f(y_{n-\tau}^{1/\alpha\beta}) f\left(\sum_{s=n_2}^{n-\tau-1} b_s^{-1/\alpha}\left(\sum_{t=n_2}^{s} a_t^{-1/\beta}\right)^{1/\alpha}\right) = 0 \tag{2.8}$$

and

$$\Delta x_n + (f(x_{n+2\sigma-\tau}))^{1/\alpha\beta} b_n^{-1/\alpha}\left[\sum_{s=n}^{n+\sigma} a_s^{-1/\beta}\left[\sum_{t=s}^{s+\sigma} q_t\right]^{-1/\beta}\right]^{1/\alpha} = 0 \tag{2.9}$$

are oscillatory, then equation (1.1) *is oscillatory*.

Proof. Assume that equation (1.1) has a nonoscillatory solution. Without loss of generality, there is a $n_1 \geq n_0$ sufficiently large such that $x_n > 0$ and $x_{n-\tau} > 0$ for all $n \geq n_1$. From equation (1.1), we have

$$\Delta(a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta}) = -q_n f(x_{n-\tau}) < 0 \text{ for all } n \geq n_1.$$

Thus, $a_n\Delta(b_n(\Delta x_n)^{\alpha})$ is strictly decreasing, then $\Delta(b_n(\Delta x_n)^{\alpha})$ and Δx_n are eventually of one sign. Then from Lemma 2.1, we have the following cases for sufficiently large $n_2 \geq n_1$:

(i) $\Delta x_n > 0$, $\Delta(b_n(\Delta x_n)^{\alpha}) > 0$,

(ii) $\Delta x_n < 0$, $\Delta(b_n(\Delta x_n)^{\alpha}) > 0$.

Case (i). Let $a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta} = y_n$. Then we have

$$\Delta(b_n(\Delta x_n)^{\alpha}) = \frac{y_n^{1/\beta}}{a_n^{1/\beta}}.$$

Summing the last inequality from n_2 to $n-1$, we have

$$b_n(\Delta x_n)^{\alpha} = b_{n_2}(\Delta x_{n_2})^{\alpha} + \sum_{s=n_2}^{n-1} a_s^{-1/\beta} y_s^{1/\beta} \geq y_s^{1/\beta} \sum_{s=n_2}^{n-1} a_s^{-1/\beta}$$

or

$$\Delta x_n \geq y_n^{1/\alpha\beta} \frac{1}{b_n^{1/\alpha}} \left(\sum_{s=n_2}^{n-1} a_s^{-1/\beta} \right)^{1/\alpha}.$$

Summing the last inequality from n_2 to $n-1$, we get

$$\begin{aligned} x_n &\geq x_{n_2} + \sum_{s=n_2}^{n-1} y_s^{1/\alpha\beta} b_s^{-1/\alpha} \left(\sum_{t=n_2}^{s-1} a_t^{-1/\beta} \right)^{1/\alpha} \\ &\geq y_n^{1/\alpha\beta} \sum_{s=n_2}^{n-1} b_s^{-1/\alpha} \left(\sum_{t=n_2}^{s-1} a_t^{-1/\beta} \right)^{1/\alpha}. \end{aligned}$$

There exists $n_3 \geq n_2$ such that $n - \tau \geq n_3$ for all $n \geq n_3$. Then

$$x_{n-\tau} \geq y_{n-\tau}^{1/\alpha\beta} \sum_{s=n_2}^{n-\tau-1} b_s^{-1/\alpha} \left(\sum_{t=n_2}^{s-1} a_t^{-1/\beta} \right)^{1/\alpha} \quad \text{for all } n \geq n_3.$$

This and equation (1.1), (C_4) yield for all $n \geq n_3$,

$$-\Delta y_n = q_n f(x_{n-\tau}) \geq q_n f(y_{n-\tau}^{1/\alpha\beta})\, f\left(\sum_{s=n_2}^{n-\tau-1} b_s^{1/\alpha} \left(\sum_{t=n_2}^{s-1} a_t^{-1/\beta} \right)^{1/\alpha} \right).$$

Summing the last inequality from n to ∞, we get

$$y_n \geq \sum_{s=n}^{\infty} q_s f(y_{s-\tau}^{1/\alpha\beta})\, f\left(\sum_{t=n_2}^{s-\tau-1} b_t^{1/\alpha} \left(\sum_{j=n_2}^{t-1} a_j^{-1/\beta} \right)^{1/\alpha} \right).$$

The function y_n is strictly decreasing, and by Theorem 6.19.3 [1], there exists a positive solution of equation (2.8) which tends to zero this contradicts that equation (2.8) is oscillatory.

Case (ii). Summing equation (1.1) from n to $n + \sigma$, we have

$$a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta} \geq \sum_{s=n}^{n+\sigma} q_s f(x_{s-\tau})$$

or

$$\Delta(b_n(\Delta x_n)^{\alpha}) \geq \left(\frac{f(x_{n+\sigma-\tau})}{a_n} \right)^{1/\beta} \left(\sum_{s=n}^{n+\sigma} q_s \right)^{1/\beta}.$$

Summing the above inequality from n to $n + \sigma$, we obtain

$$-b_n(\Delta x_n)^{\alpha} \geq \sum_{s=n}^{n+\sigma} \left(\frac{f(x_{s+\sigma-\tau})}{a_s} \right)^{1/\beta} \left(\sum_{t=s}^{s+\sigma} q_t \right)^{1/\beta}$$

or

$$-(\Delta x_n)^{\alpha} \geq \frac{(f(x_{n+2\sigma-\tau}))^{1/\beta}}{b_n} \sum_{s=n}^{n+\sigma} a_s^{-1/\beta} \left(\sum_{t=s}^{s+\sigma} q_t \right)^{1/\beta}$$

or

$$-\Delta x_n \geq \frac{(f(x_{n+2\sigma-\tau}))^{1/\alpha\beta}}{b_n^{1/\alpha}} \left[\sum_{s=n}^{n+\sigma} a_s^{-1/\beta} \left(\sum_{t=s}^{s+\sigma} q_t \right)^{1/\beta} \right]^{1/\alpha}.$$

Summing the last inequality from n to ∞, we get

$$x_n \geq (f(x_{n+2\sigma-\tau}))^{1/\alpha\beta} \sum_{s=n}^{\infty} b_s^{-1/\alpha} \left[\sum_{t=s}^{s+\sigma} a_t^{-1/\beta} \left(\sum_{j=t}^{t+\sigma} q_j \right)^{1/\beta} \right]^{1/\alpha}.$$

Since by Lemmas 2.1 and 2.2, there exists a positive solution of equation (2.9) which tends to zero this contradicts that equation (2.9) is oscillatory. The proof is complete. □

Theorem 2.2. *Let* (H_1), (H_2) *and* (2.6) *hold. Assume that the first order delay equation* (2.8) *is oscillatory, then every solution* $\{x_n\}$ *of equation* (1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof. The proof follows from Case (i) of Theorem 2.1 and Lemma 2.2 and hence the details are omitted. □

We conclude this paper with the following example.

Example 2.1. Consider the difference equations

$$\Delta\left(n \left(\Delta \left(\frac{1}{n^2} (\Delta x_n)^{1/3} \right) \right)^3 \right) + \frac{1}{n} x_{n-2} = 0, \quad n \geq 1. \tag{2.10}$$

Here $f(u) = u$, $q_n = \frac{1}{n}$, $a_n = n$, $b_n = \frac{1}{n^2}$, $\tau = 2$, $\alpha = \frac{1}{3}$ and $\beta = 3$.

Further $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \infty$, $\sum_{n=1}^{\infty} n^6 = \infty$. It is easy to see that condition (2.6) holds. Further equation (2.8) reduces to

$$\Delta y_n + \frac{1}{n} \sum_{s=1}^{n-3} n^6 \left(\sum_{t=1}^{s} \frac{1}{t^{1/3}} \right)^3 y_{n-2} = 0. \tag{2.11}$$

Then by Theorem 7.5.1 [10], equation (2.11) is oscillatory, since

$$\liminf_{n\to\infty} \sum_{s=n-2}^{n-1} \frac{1}{s} \left(\sum_{t=1}^{s-1} t^6 \left(\sum_{j=1}^{t} \frac{1}{j^{1/3}} \right)^3 \right) = \infty > \left(\frac{2}{3}\right)^3.$$

Hence by Theorem 2.2, every nonoscillatory solution of equation (2.10) tends to zero as $n \to \infty$.

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