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BOUNDEDNESS IN THE PERTURBED NONLINEAR DIFFERENTIAL SYSTEMS

Yoon Hoe Goo

Department of Mathematics
Hanseo University
Seosan Chungnam 356-706
Republic of Korea
e-mail: yhgoo@hanseo.ac.kr

Abstract

In this paper, we investigate bounds for solutions of the perturbed nonlinear differential systems.

1. Introduction

The behavior of solutions of a perturbed system is determined in terms of the behavior of solutions of an unperturbed system. There are three useful methods for showing the qualitative behavior of the solutions of perturbed nonlinear system: Lyapunov's second method, the method of variation of constants formula, and the use of inequalities.

Pinto [12] introduced the notion of h-stability with the intention of investigating results about stability for a weakly stable system under some perturbations. Using this notion, we shall give some results on the boundedness of solutions of perturbed nonlinear systems. Gonzalez and Pinto

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[8] proved theorems which relate the asymptotic behavior and boundedness of the solutions of nonlinear differential systems. Also, Choi and Ryu [3, 5] investigated bounds of solutions for nonlinear perturbed systems and nonlinear functional differential systems.

In this paper, we obtain some results on boundedness of solutions of the nonlinear perturbed differential systems under suitable conditions on perturbed term. To do this we need some integral inequalities.

2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. For $x \in \mathbb{R}^n$, let $|x| = \left(\sum_{j=1}^n x_j^2\right)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x| \le 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0 \tag{2.2}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0. \tag{2.3}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.2).

We recall some notions of h-stability [12].

Definition 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c|x_0|h(t)h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough.

Definition 2.2. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called *h-stable* (*hS*) if there exists $\delta > 0$ such that (2.1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [6].

Definition 2.3. A matrix $A(t) \in \mathcal{M}$ is t_∞-*similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{2.4}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [6].

We give some related properties that we need in the sequel.

Lemma 2.4 [13]**.** *The linear system*

$$x' = A(t)x, \qquad x(t_0) = x_0, \tag{2.5}$$

where $A(t)$ *is an* $n \times n$ *continuous matrix, is an h-system* (*respectively h-stable*) *if and only if there exist* $c \geq 1$ *and a positive continuous* (*respectively bounded*) *function h defined on* $\mathbb{R}^+$ *such that*

$$|\phi(t, t_0)| \leq ch(t)h(t_0)^{-1} \tag{2.6}$$

for $t \geq t_0 \geq 0$, *where* $\phi(t, t_0)$ *is a fundamental matrix of* (2.5).

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \qquad y(t_0) = y_0, \tag{2.7}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.7) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.5. *If* $y_0 \in \mathbb{R}^n$, *then for all t such that* $x(t, t_0, y_0) \in \mathbb{R}^n$,

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))g(s, y(s))ds.$$

Theorem 2.6 [3]**.** *If the zero solution of* (2.1) *is hS, then the zero solution of* (2.2) *is hS.*

Theorem 2.7 [4]**.** *Suppose that* $f_x(t, 0)$ *is* t_∞*-similar to* $f_x(t, x(t, t_0, x_0))$ *for* $t \geq t_0 \geq 0$ *and* $|x_0| \leq \delta$ *for some constant* $\delta > 0$. *If the solution* $v = 0$ *of* (2.2) *is hS, then the solution* $z = 0$ *of* (2.3) *is hS.*

Lemma 2.8 (Bihari-type inequality)**.** *Let* $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in u. Suppose that, for some* $c > 0$,

$$u(t) \le c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \qquad t \ge t_0 \ge 0.$$

Then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^{t} \lambda(s) ds \right], \qquad t_0 \le t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$ *and*

$$b_1 = \sup\left\{ t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \operatorname{dom} W^{-1} \right\}.$$

Lemma 2.9 [5]. *Let* $u, \lambda_1, \lambda_2, w \in C(\mathbb{R}^+)$, $w(u)$ *be nondecreasing in u and* $\frac{1}{v} w(u) \le w\left(\frac{u}{v}\right)$ *for some* $v > 0$. *If, for some* $c > 0$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_1(s) \left\{ \int_{t_0}^{s} \lambda_2(\tau) w(u(\tau)) d\tau \right\} ds, \quad t \ge t_0 \ge 0,$$

then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^{t} \lambda_2(s) ds \right] \exp\left(\int_{t_0}^{t} \lambda_1(s) ds \right), \quad t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma* 2.8 *and*

$$b_1 = \sup\left\{ t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda_2(s) ds \in \operatorname{dom} W^{-1} \right\}.$$

Lemma 2.10 [3]. *Let* $u, \lambda_1, \lambda_2, \lambda_2, r, w \in C(\mathbb{R}^+)$, $w(u)$ *be nondecreasing in* u, *and* r*-submultiplicative, i.e.,* $w(pu) \le r(p) w(u)$ *for* $p > 0$, $u \ge 0$. *If the inequality*

$$u(t) \le u(t_0) + \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) \left\{ \int_{t_0}^{s} \lambda_3(\tau) w(u(\tau)) d\tau \right\} ds,$$

for $t \geq t_0 \geq 0,$ *holds, then*

$$u(t) \leq \exp\left\{\int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s))ds\right\}W^{-1}\Big[W(u(t_0))+\int_{t_0}^{t}\lambda_3(\tau)$$

$$\cdot \exp\left\{-\int_{t_0}^{\tau}(\lambda_1(s)+\lambda_2(s))ds\right\}r\left(\exp\left(\int_{t_0}^{\tau}(\lambda_1(s)+\lambda_2(s))ds\right)\right)d\tau\Big]$$

for any $t \in [t_0, T),$ *where W,* W^{-1} *are the same functions as in Lemma* 2.8,

$$\int_{t_0}^{T}\lambda(s)ds \leq \int_{t_0}^{\infty}\frac{ds}{w(s)}$$

and

$$\hat{\lambda}(t) = \lambda_3(t)\exp\left(-\int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s))\right)r\left(\exp\left(\int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s))ds\right)\right).$$

3. Main Results

In this section, we investigate bounds for the nonlinear differential systems.

Theorem 3.1. *Let* $u, a, k, w \in C(\mathbb{R}^+),$ $w(u)$ *be nondecreasing in u and* $\frac{1}{v}w(u) \leq w\left(\frac{u}{v}\right)$ *for some* $v > 0.$ *Suppose that the solution* $x = 0$ *of* (2.1) *is hS with a nondecreasing function h and the perturbed term g in* (2.7) *satisfies*

$$|\Phi(t, s, z)g(t, z)| \leq a(s)\left(|z| + \int_{t_0}^{s}k(\tau)w(|z|)d\tau\right), \quad t \geq t_0 \geq 0,$$

where $\int_{t_0}^{\infty}a(s)ds < \infty$ *and* $\int_{t_0}^{\infty}k(s)ds < \infty.$ *Then any solution* $y(t) = y(t, t_0, y_0)$ *of* (2.7) *is bounded on* $[t_0, b_1)$ *and it satisfies*

$$|y(t)| \leq h(t)W^{-1}\left[W(c)+\int_{t_0}^{t}k(s)ds\right]\exp\left(\int_{t_0}^{t}a(s)ds\right), \quad t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.8 *and*

$$b_1 = \sup\left\{t \geq t_0 : W(c) + \int_{t_0}^{t} k(s)ds \in domW^{-1}\right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.7), respectively. By Lemma 2.5, we obtain

$$| y(t) | \leq | x(t) | + \int_{t_0}^{t} | \Phi(t, s, y(s)) g(s, y(s)) | ds$$

$$\leq c_1 | y_0 | h(t) h(t_0)^{-1} + \int_{t_0}^{t} a(s)\left[| y(s) | + \int_{t_0}^{s} k(\tau) w(| y(\tau) |) d\tau \right] ds$$

$$\leq c_1 | y_0 | h(t) h(t_0)^{-1} + \int_{t_0}^{t} a(s) h(t) \frac{| y(s) |}{h(s)} ds$$

$$+ \int_{t_0}^{t} a(s) \int_{t_0}^{s} h(t) k(\tau) w\left(\frac{| y(\tau) |}{h(\tau)} \right) d\tau ds,$$

since h is nondecreasing. Set $u(t) = | y(t) | h(t)^{-1}$. Then, by Lemma 2.9, we have

$$| y(t) | \leq h(t) W^{-1}\left[W(c) + \int_{t_0}^{t} k(s) \right] \exp\left(\int_{t_0}^{t} a(s) ds \right), \quad t_0 \leq t < b_1,$$

where $c = c_1 | y_0 | h(t_0)^{-1}$. Therefore, we obtain the result. □

Remark 3.2. Letting $k(t) = 0$ in Theorem 3.1, we obtain the same result as that of Corollary 3.2 in [10].

Also, we examine the bounded property for the perturbed system.

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s)) ds, \; y(t_0) = y_0, \tag{3.1}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$.

Theorem 3.3. *Let* $u, a, k, r, w \in C(\mathbb{R}^+)$, $w(u)$ *be nondecreasing in* u *and* r*-submultiplicative, i.e.,* $w(pu) \leq r(p)w(u)$ *for* $p > 0$, $u \geq 0$. *Suppose that* $f_x(t, 0)$ *is* t_∞*-similar to* $f_x(t, x(t, t_0, x_0))$ *for* $t \geq t_0 \geq 0$ *and* $|x_0| \leq \delta$ *for some constant* $\delta > 0$, *the solution* $x = 0$ *of* (2.1) *is hS and g in* (3.1) *satisfies*

$$\left| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \right| \leq a(s)\left(|y(s)| + \int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau \right).$$

Then any solution $y(t) = y(t, t_0, y_0)$ *of* (3.1) *is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \leq |y_0| h(t)h(t_0)^{-1} \exp\left(\int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s))ds \right)$$

$$\times W^{-1}\left[W(c) + \int_{t_0}^{t} \hat{\lambda}(s)ds \right],$$

where $\hat{\lambda}(t) \in L_1(\mathbb{R}^+)$ *is defined in Lemma* 2.10, $\lambda_1(t) = c_1 a(t)$, $\lambda_2(t) = c_1 |y_0|^{-1} h(t_0)h(t)^{-1}$, *and* $\lambda_3(t) = k(t)r(|y_0| h(t_0)^{-1} h(t))$.

Proof. Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$ be solutions of (2.1) and (3.1), respectively. By Theorem 2.6, since the solution $x = 0$ of (2.1) is *hS*, the solution $v = 0$ of (2.2) is *hS*. Therefore, by Theorem 2.7, the solution $z = 0$ of (2.3) is *hS*. By Lemma 2.4 and Lemma 2.5, we have

$$|y(t)| \leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau))d\tau \right| ds$$

$$\leq c|y_0| h(t)h(t_0)^{-1}$$

$$+ \int_{t_0}^{t} c_1 h(t)h(s)^{-1} a(s)\left(|y(s)| + \int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau \right) ds.$$

Set $u(t) = |y(t)| h(t)^{-1} h(t_0) |y_0|^{-1}$. Then, we obtain

$$|u(t)| \le c + \int_{t_0}^{t} c_1 a(s) u(s) ds$$

$$+ \int_{t_0}^{t} c_1 |y_0|^{-1} h(t_0) h(s)^{-1} a(s)$$

$$\times \left\{ \int_{t_0}^{s} k(\tau) r(|y_0| h(t_0)^{-1} h(\tau)) w(u(\tau)) d\tau \right\} ds.$$

In view of Lemma 2.10, we obtain

$$u(t) \le \exp\left\{ \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s)) ds \right\} W^{-1}\left[W(c) + \int_{t_0}^{t} \hat{\lambda}(s) ds \right].$$

Hence, the proof is complete. □

Remark 3.4. Letting $k(t) = 0$ in Theorem 3.3, we obtain the same result as that of Theorem 3.3 in [10].

We need the lemma to prove the following theorem.

Lemma 3.5. *Let* $u, p, q, w, r \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that for some* $c \ge 0$,

$$u(t) \le c + \int_{t_0}^{t} \left(p(s) \int_{t_0}^{s} \left(q(\tau) w(u(\tau)) + v(\tau) \int_{t_0}^{\tau} r(a) w(u(a)) da \right) d\tau \right) ds,$$

$$t \ge t_0. \tag{3.2}$$

Then

$$u(t) \le W^{-1}\left[W(c) + \int_{t_0}^{t} \left(p(s) \int_{t_0}^{s} \left(q(\tau) + v(\tau) \int_{t_0}^{\tau} r(a) da \right) d\tau \right) ds \right],$$

$$t_0 \le t < b_1, \tag{3.3}$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$ *and*

$$b_1 = \sup\left\{t \geq t_0 : W(c) + \int_{t_0}^{t}\left(p(s)\right.\right.$$

$$\left.\left.\times \int_{t_0}^{s}\left(q(\tau) + v(\tau)\int_{t_0}^{\tau} r(a)da\right)d\tau\right)ds \in \operatorname{dom} W^{-1}\right\}.$$

Proof. Setting

$$z(t) = c + \int_{t_0}^{t}\left(p(s)\int_{t_0}^{s}\left(q(\tau)w(u(\tau)) + v(\tau)\int_{t_0}^{\tau} r(a)w(u(a))da\right)d\tau\right)ds,$$

we have $z(t_0) = c$ and

$$\begin{aligned} z'(t) &= p(t)\int_{t_0}^{t}\left(q(\tau)w(u(\tau)) + v(\tau)\int_{t_0}^{\tau} r(a)w(u(a))da\right)d\tau \\ &\leq p(t)\int_{t_0}^{t}\left(q(\tau) + v(\tau)\int_{t_0}^{\tau} r(a)da\right)w(u(\tau))d\tau \\ &\leq \left[p(t)\int_{t_0}^{t}\left(q(\tau) + v(\tau)\int_{t_0}^{\tau} r(a)da\right)d\tau\right]w(z(\tau)), \quad t \leq t_0, \end{aligned} \tag{3.4}$$

since $z(t)$ and $w(u)$ are nondecreasing and $u(t) \leq z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

$$z(t) \leq c + \int_{t_0}^{t}\left(p(s)\int_{t_0}^{s}\left(q(\tau) + v(\tau)\int_{t_0}^{\tau} r(a)da\right)d\tau w(z(s))\right)ds. \tag{3.5}$$

It follows from Lemma 2.8 that (3.5) yields the estimate (3.3). □

Theorem 3.6. *Let* $u, w \in C(\mathbb{R}^+)$, $w(u)$ *be nondeacreasing in* u *and* $\frac{1}{v}w(u) \leq w\left(\frac{u}{v}\right)$ *for some* $v > 0$. *Suppose that* $f_x(t, 0)$ *is* t_∞*-similar to* $f_x(t, x(t, t_0, x_0))$ *for* $t \geq t_0 \geq 0$ *and* $|x_0| \leq \delta$ *for some constant* $\delta > 0$. *If the solution* $x = 0$ *of* (2.1) *is an h-system with a positive continuous function h and g in* (3.1) *satisfies*

$$|g(t,\ y)| \le a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds, \quad t \ge t_0, \quad y \in \mathbb{R}^n$$

where $a,\ b,\ k : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous with*

$$\int_{t_0}^{\infty} \frac{1}{h(s)} \int_{t_0}^{s} \left(h(\tau)a(\tau) + b(\tau)\int_{t_0}^{\tau} h(r)k(r)dr \right) d\tau ds < \infty, \tag{3.6}$$

for all $t_0 \ge 0$, *then any solution* $y(t) = y(t,\ t_0,\ y_0)$ *of* (3.1) *satisfies*

$$|y(t)| \le h(t)W^{-1}\Bigg[W(c) + \int_{t_0}^{t} \frac{c_2}{h(s)} \times \int_{t_0}^{s} \left(h(\tau)a(\tau) + b(\tau)\int_{t_0}^{\tau} h(r)c(r)dr \right) d\tau ds \Bigg],$$

$t_0 \le t < b_1$, *where* $W,\ W^{-1}$ *are the same functions as in Lemma* 2.8 *and*

$$b_1 = \sup\Bigg\{ t \ge t_0 : W(c) + \int_{t_0}^{t} \frac{c_2}{h(s)} \times \int_{t_0}^{s} \left(h(\tau)a(\tau) + b(\tau)\int_{t_0}^{\tau} h(r)c(r)dr \right) d\tau ds \in \operatorname{dom} W^{-1} \Bigg\}.$$

Proof. Let $x(t) = x(t,\ t_0,\ x_0)$ and $y(t) = y(t,\ t_0,\ x_0)$ be solutions of (2.1) and (3.1), respectively. By Theorem 2.6, since the solution $x = 0$ of (2.1) is an h-system, the solution $v = 0$ of (2.2) is an h-system. Therefore, by Theorem 2.7, the solution $z = 0$ of (2.3) is an h-system. By Lemma 2.5, we have

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,\ s,\ y(s))| \int_{t_0}^{s} |g(\tau,\ y(\tau))| d\tau ds \\ &\le c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t} c_2 \frac{h(t)}{h(s)} \int_{t_0}^{s} h(\tau)a(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right) d\tau ds \\ &\quad + \int_{t_0}^{t} c_2 \frac{h(t)}{h(s)} \int_{t_0}^{s} b(\tau)\int_{t_0}^{\tau} h(r)c(r)w\left(\frac{|y(r)|}{h(r)}\right) dr d\tau ds. \end{aligned}$$

Setting $u(t) = |y(t)|h(t)^{-1}$ and using Lemma 3.5, we obtain

$$|y(t)| \leq h(t)W^{-1}\Bigg[W(c) + \int_{t_0}^{t} \frac{c_2}{h(s)}$$

$$\times \int_{t_0}^{s} \left(h(\tau)a(\tau) + b(\tau)\int_{t_0}^{\tau} h(r)c(r)dr \right) d\tau ds\Bigg],$$

$t_0 \leq t < b_1$, where $c = c_1 |y_0| h(t_0)^{-1}$. Hence, the proof is complete. □

Remark 3.7. Letting $k(s) = 0$ in Theorem 3.6, we obtain the same result as that of Theorem 3.5 in [9].

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