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EXISTENCE OF NONOSCILLATORY SOLUTIONS OF DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract

The author presents sufficient conditions for the existence of nonoscillatory solutions for the equation

$$\Delta(a_n \Delta(x_n - c_n x_{n-k})) + f(n, x_{n-\ell}) - g(n, x_{n-m}) = 0.$$

An example is given to illustrate the main results.

1. Introduction

In this paper, we consider the nonlinear neutral delay difference equation of the form

$$\Delta(a_n \Delta(x_n - c_n x_{n-k})) + f(n, x_{n-\ell}) - g(n, x_{n-m}) = 0, \quad n \in \mathbb{N}(n_0), \tag{1.1}$$

where $\mathbb{N}(n_0) = \{n_0, n_0 + 1, ...\}$, n_0 is a nonnegative integer, $\{a_n\}$ is a

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positive real sequence such that $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$, $\{c_n\}$ is a real sequence, $f, g : \mathbb{N}(n_0) \times \mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing in the second variable and $uf(n, u) > 0$, $ug(n, u) > 0$ for $u \neq 0$, and k, ℓ and m are nonnegative integers.

Let $\theta = \max\{k, \ell, m\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ which is defined for $n \geq n_0 - \theta$ and satisfies equation (1.1) for all $n \in \mathbb{N}(n_0)$. A solution of equation (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

The oscillatory and nonoscillatory behavior of solutions of particular form of equation (1.1) has been investigated by several authors, see [1]. In fact, most of the results established for equation (1.1) for the case either $c_n \equiv 0$ or $f(n, u) \equiv 0$ or $g(n, u) \equiv 0$. Very few results are available in the literature dealing with oscillatory properties of solutions of neutral difference equations with positive and negative coefficients, see for example [2, 3, 4, 5] and the references cited therein.

Motivated by the above observation in this paper we establish some sufficient conditions for the existence of nonoscillatory solutions of equation (1.1). The results obtained in this paper generalize those obtained in [5].

2. Main Results

In this section, we establish sufficient conditions for the existence of nonoscillatory solution of equation (1.1) subject to the following condition:

(H) *f* and *g* satisfy the Lipschitz condition of the form

$$|f(n, u) - f(n, v)| \leq p_n |u - v|$$

$$|g(n, u) - g(n, v)| \leq q_n |u - v|, \quad u, v \in [a, b],$$

where $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences for all $n \in \mathbb{N}(n_0)$ and

$\sum_{n=n_0}^{\infty} A_n p_n < \infty$, and $\sum_{n=n_0}^{\infty} A_n q_n < \infty$, where $A_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s}$, and $A(s, n) = \sum_{j=n}^{s-1} \frac{1}{a_j}$.

We begin with the following theorem.

Theorem 2.1. *With respect to the difference equation* (1.1), *assume condition* (H) *holds*. *If one of the following conditions holds*:

(i) $0 \le c_n \le c < 1$;

(ii) $1 < c \le c_n \le d < \infty$;

(iii) $-1 < -c \le c_n < 0$;

(iv) $-\infty < -c \le c_n \le -d < -1$.

Then equation (1.1) *has a bounded nonoscillatory solution*.

Proof. The proof of the theorem will be divided into four cases, depending on the four different conditions of $\{c_n\}$.

Case 1. $0 \le c_n \le c < 1$. Let B be the set of all bounded real sequences $x = \{x_n\}$, $n \in \mathbb{N}(n_0)$ with the supnorm $\| x \| = \sup_{n \ge n_0} | x_n |$. Set

$$S = \{x \in B : M_1 \le x_n \le M_2,\ n \in \mathbb{N}(n_0)\},$$

where M_1 and M_2 are positive constants, and $M_1 < (1 - c)M_2$, $M_2 \le b$. It is clear that S is a bounded, closed and convex subset of B. Choose α, $M > 0$ such that $M_1 < \alpha < (1 - c)M_2$, and

$$M = \min\left\{\frac{1 - c}{3}, \frac{\alpha - M_1}{M_2}, \frac{(1 - c)M_2 - \alpha}{M_2}\right\}. \tag{2.1}$$

Choose $N \in \mathbb{N}(n_0)$ sufficiently large so that

$$\sum_{n=N}^{\infty} A_n p_n \le M, \qquad \sum_{n=N}^{\infty} A_n q_n \le M. \tag{2.2}$$

Define a mapping $T : S \to B$ as follows:

$$(Tx)(n) = \begin{cases} \alpha + c_n x_{n-k} & \\ \quad + \sum_{s=n}^{\infty} A(s, n)[g(s, x_{s-n}) - f(s, x_{s-\ell})], & n \geq N, \\ (Tx)(N), & n_0 \leq n < N. \end{cases}$$

Clearly, Tx is continuous. For every $x \in S$ and $n \in \mathbb{N}(N)$, we have, in view of (2.1) and (2.2),

$$(Tx)(n) \geq \alpha - MM_2 \geq M_1,$$

and

$$(Tx)(n) \leq \alpha + cM_2 + MM_2 \leq M_2.$$

Thus $TS \subset S$. For any $x, y \in S$ and $n \geq N$, we have

$$|(Tx)(n) - (Ty)(n)| \leq \left(c + \sum_{s=n}^{\infty} A(s, n) p_s + \sum_{s=n}^{\infty} A(s, n) q_s \right) \| x - y \|$$

$$\leq (c + 2M) \| x - y \|.$$

Hence $\| Tx - Ty \| \leq (c + 2M) \| x - y \|$. In view of $M \leq \frac{1-c}{3} < \frac{1-c}{2}$, we have $0 < c + 2M < 1$, which implies that T is a contraction mapping. By Banach contraction mapping principle, there exists a unique $x \in S$, such that $Tx = x$. It is easy to see that $\{x_n\}$ is a bounded positive solution of equation (1.1).

Case 2. $1 < c \leq c_n \leq d < \infty$. Let B be the set of all bounded real sequences defined for all $n \in \mathbb{N}(n_0)$ with the supnorm. Set

$$S = \{x \in B : M_3 \leq x_n \leq M_4, n \in \mathbb{N}(n_0)\},$$

where M_3 and M_4 are positive constants, and $(d-1)M_3 < (c-1)M_4$, $M_4 = b$.

Define a mapping $T : S \to B$ as follows:

$$(Tx)(n) = \begin{cases} \dfrac{\alpha}{c_{n+k}} + \dfrac{x_{n+k}}{c_{n+k}} + \dfrac{1}{c_{n+k}} \sum_{s=n+k}^{\infty} A(s,\, n+k) \\ \qquad\qquad [f(s,\, x_{s-\ell}) - g(s,\, x_{s-m})], & n \geq N, \\ (Tx)(N), & n_0 \leq n < N. \end{cases}$$

The remaining part of the proof is similar to that of Case 1 and hence the details are omitted.

Case 3. $-1 < -c \leq c_n \leq 0$. Let B be the set of all bounded real sequences defined for all $n \in \mathbb{N}(n_0)$ with the supnorm. Set

$$S = \{x \in B : D_1 \leq x_n \leq D_2,\, n \in \mathbb{N}(n_0)\},$$

where D_1 and D_2 are positive constants, and $D_1 < (1-c)D_2$, $D_2 \leq b$.

Define a mapping $T : S \to B$ as follows:

$$(Tx)(n) = \begin{cases} c + c_n x_{n-k} + \sum_{s=n}^{\infty} A(s,\, n) \\ \qquad\qquad [g(s,\, x_{s-m}) - f(s,\, x_{s-\ell})], & n \geq N, \\ (Tx)(N), & n_0 \leq n < N. \end{cases}$$

The remaining part of the proof is similar to that of Case 1 and hence the details are omitted.

Case 4. $-\infty < -c \leq c_n \leq -d < -1$. Let B be the set of all bounded real sequences defined for all $n \in \mathbb{N}(n_0)$ with the supnorm. Set

$$S = \{x \in B : D_3 \leq x_n \leq D_4,\, n \in \mathbb{N}(n_0)\},$$

where D_3 and D_4 are positive constants, and $\left(d - \dfrac{c}{d}\right)D_4 > \left(c - \dfrac{d}{c}\right)D_3$, $D_4 \leq b$.

Define a mapping $T : S \to B$ as follows:

$$(Tx)(n) = \begin{cases} \dfrac{-\alpha}{c_{n+k}} + \dfrac{x_{n+k}}{c_{n+k}} + \dfrac{1}{c_{n+k}} \sum_{s=n+k}^{\infty} A(s,\, n+k) \\ \qquad\qquad [f(s,\, x_{s-\ell}) - g(s,\, x_{s-m})], & n \geq N, \\ (Tx)(N), & n_0 \leq n < N. \end{cases}$$

The rest of the proof is similar to Case 1 and hence the details are omitted. The proof of the theorem is complete. □

We conclude this paper with the following example.

Example 2.1. Consider the difference equation

$$\Delta^2\left(x_n - \frac{1}{(n+1)} x_{n-1}\right) + \frac{6(n-1)^3 x_{n-2}^3}{(2n-3)^3 (n+1)(n+2)(n+3)}$$

$$- \frac{6(n-2)^{3/2}}{(2n-3)^{3/2} n(n+1)(n+2)(n+3)} x_{n-3}^{3/2} = 0, \quad n \geq 3. \qquad (2.3)$$

It is easy to see that all conditions of Case 1 of Theorem 2.1 are satisfied. Therefore equation (2.3) has bounded nonoscillatory solution. In fact, $\{x_n\} = \left\{2 - \dfrac{1}{n+1}\right\}$ is one such solution of equation (2.3).

Remark 2.1. Similar to Example 2.1, one can construct examples for the other cases and the details are left to the reader.

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