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INT-SOFT FILTERS IN *CI*-ALGEBRAS

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Abstract

The notion of int-soft filters of a *CI*-algebra is introduced, and related properties are investigated. Characterization of an int-soft filter is discussed, and conditions for a soft set to be an int-soft filter are provided. New int-soft filter from old one is established.

1. Introduction

In 1966, Imai and Iséki [2] and Iséki [3] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. As a generalization of a BCK-algebra, Kim and Kim [5] introduced the notion of a *BE*-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in *BE*-algebras. They gave several descriptions of ideals in *BE*-algebras. The notion of *CI*-algebras is introduced by Meng [6] as a generalization of *BE*-algebras. Filter theory and properties in *CI*-algebras are studied by Kim [4], Meng [7] and Piekart and Walendziak [9]. Molodtsov [8] introduced the concept of soft set as a new mathematical

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tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets.

In this paper, we introduce the notion of int-soft filter of a *CI*-algebra, and investigate their properties. We consider characterization of an int-soft filter, and provide conditions for a soft set to be an int-soft filter. We make a new int-soft filter from old one.

2. Preliminaries

An algebra $(X; *, 1)$ of type $(2, 0)$ is called a *CI-algebra* if it satisfies the following properties:

(CI1) $x * x = 1$,

(CI2) $1 * x = x$,

(CI3) $x * (y * z) = y * (x * z)$,

for all $x, y, z \in X$.

Let $(X; *, 1)$ be a *CI*-algebra. Then a subset *F* of *X* is called a *filter* (see [5]) of *X* if

(F1) $1 \in F$;

(F2) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

A soft set theory is introduced by Molodtsov [8]. In what follows, let *U* be an initial universe set and *X* be a set of parameters. Let $\mathscr{P}(U)$ denote the power set of *U* and $A, B, C, \ldots \subseteq X$.

A *soft set* (see [8]) $\mathscr{F}_A$ of *X* over *U* is defined to be the set of ordered pairs

$$\mathscr{F}_A := \{(x, f_A(x)) : x \in X, f_A(x) \in \mathscr{P}(U)\},$$

where $f_A : X \to \mathscr{P}(U)$ such that $f_A(x) = \emptyset$ if $x \in X \setminus A$.

For a soft set $\mathscr{F}_X$ of X over U and a subset γ of U, the γ-*inclusive set* of $\mathscr{F}_X$, denoted by $i_X(f_X;\gamma)$, is defined to be the set

$$i_X(f_X;\gamma) := \{x \in X \mid \gamma \subseteq f_X(x)\}.$$

3. Int-soft Filters

In what follows, we take a *CI*-algebra X, as a set of parameters unless otherwise specified.

Definition 3.1. A soft set $\mathscr{F}_X$ of X over U is called an *int-soft filter* of X if it satisfies:

$$(\forall x \in X)(f_X(1) \supseteq f_X(x)), \tag{3.1}$$

$$(\forall x, y \in X)(f_X(x * y) \cap f_X(x) \subseteq f_X(y)). \tag{3.2}$$

Example 3.2. Let $E = X$ be the set of parameters, where $X = \{1, a, b, c\}$ is a *CI*-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	c
c	c	c	c	1

Let $\mathscr{F}_X$ be a soft set of X over U defined as follows:

$$f_X : X \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \gamma_1, & \text{if } x \in \{1, a, b\}, \\ \gamma_2, & \text{if } x = c, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \supsetneq \gamma_2$. It is easy to check that $\mathscr{F}_X$ is an int-soft filter of X.

Proposition 3.3. *Every int-soft filter $\mathscr{F}_X$ of X over U satisfies*:

(1) $(\forall x, y \in X)(x * y = 1 \Rightarrow f_X(x) \subseteq f_X(y))$,

(2) $(\forall a,\ b,\ x \in X)(a * (b * x) = 1 \Rightarrow f_X(x) \supseteq f_X(a) \cap f_X(b)),$

(3) $(\forall x,\ y,\ z \in X)(f_X(x * z) \supseteq f_X(x * (y * z)) \cap f_X(y)),$

(4) $(\forall x,\ y \in X)(f_X(x) \subseteq f_X((x * y) * y)),$

(5) $(\forall x,\ a,\ b \in X)(f_X((a * (b * x)) * x) \supseteq f_X(a) \cap f_X(b)).$

Proof. (1) Assume that $x * y = 1$ for all $x,\ y \in X.$ Then

$$f_X(x) = f_X(1) \cap f_X(x) = f_X(x * y) \cap f_X(x) \subseteq f_X(y)$$

for all $x,\ y \in X$ by using (3.1) and (3.2).

(2) Let $a,\ b,\ x \in X$ be such that $a * (b * x) = 1.$ Using (3.1) and (3.2), we have

$$\begin{aligned} f_X(x) &\supseteq f_X(b * x) \cap f_X(b) \\ &\supseteq f_X(a * (b * x)) \cap f_X(a) \cap f_X(b) \\ &= f_X(1) \cap f_X(a) \cap f_X(b) \\ &= f_X(a) \cap f_X(b). \end{aligned}$$

(3) Using (3.2) and (CI3), we have

$$f_X(x * z) \supseteq f_X(y * (x * z)) \cap f_X(y) = f_X(x * (y * z)) \cap f_X(y)$$

for all $x,\ y,\ z \in X.$

(4) If we take $y = (x * y) * y$ in (3.2), then

$$\begin{aligned} f_X((x * y) * y) &\supseteq f_X(x * ((x * y) * y)) \cap f_X(x) \\ &= f_X((x * y) * (x * y)) \cap f_X(x) \\ &= f_X(1) \cap f_X(x) = f_X(x) \end{aligned}$$

by using (CI3), (CI1) and (3.1).

(5) Using (3) and (4), we have

$$f_X((a * (b * x)) * x) \supseteq f_X((a * (b * x)) * (b * x)) \cap f_X(b)$$

$$\supseteq f_X(a) \cap f_X(b)$$

for all $a, b, x \in X$. □

As a generalization of Proposition 3.3(2), we have the following result.

Proposition 3.4. *If a soft set* $\mathscr{F}_X$ *of X over U is an int-soft filter of X, then*

$$\prod_{i=1}^{n} a_i * x = 1 \Rightarrow f_X(x) \supseteq \bigcap_{i=1}^{n} f_X(a_i) \tag{3.3}$$

for all $x, a_1, ..., a_n \in X$, *where*

$$\prod_{i=1}^{n} a_i * x = a_n * (a_{n-1} * (\cdots (a_1 * x) \cdots)).$$

Proof. The proof is by induction on *n*. Let $\mathscr{F}_X$ be an int-soft filter of *X* over *U*. By (1) and (2) of Proposition 3.3, we know that the condition (3.3) is valid for $n = 1, 2$. Assume that $\mathscr{F}_X$ satisfies the condition (3.3) for $n = k$, that is,

$$\prod_{i=1}^{k} a_i * x = 1 \Rightarrow f_X(x) \supseteq \bigcap_{i=1}^{k} f_X(a_i)$$

for all $x, a_1, ..., a_k \in X$. Suppose that $\prod_{i=1}^{k+1} a_i * x = 1$ for all $x, a_1, ..., a_k$, $a_{k+1} \in X$. Then

$$f_X(a_1 * x) \supseteq \bigcap_{i=2}^{k+1} f_X(a_i).$$

Since $\mathscr{F}_X$ is an int-soft filter of *X*, it follows from (3.2) that

$$f_X(x) \supseteq f_X(a_1 * x) \cap f_X(a_1) \supseteq \left(\bigcap_{i=2}^{k+1} f_X(a_i) \right) \cap f_X(a_1) = \bigcap_{i=1}^{k+1} f_X(a_i).$$

This completes the proof. □

We provide conditions for a soft set to be an int-soft filter.

Theorem 3.5. *If a soft set* $\mathscr{F}_X$ *of X over U satisfies two conditions* (3.1) *and Proposition* 3.3(2), *then* $\mathscr{F}_X$ *is an int-soft filter of X.*

Proof. Since $x * ((x * y) * y) = 1$ for all $x, y \in X$ (see [6, Proposition 3.2]), it follows from (3.1) and Proposition 3.3(2) that $f_X(y) \supseteq f_X(x * y) \cap f_X(x)$ for all $x, y \in X$. Hence $\mathscr{F}_X$ is an int-soft filter of *X*. □

Theorem 3.6. *If a soft set* $\mathscr{F}_X$ *of X over U satisfies two conditions* (3.1) *and Proposition* 3.3(3), *then* $\mathscr{F}_X$ *is an int-soft filter of X.*

Proof. If we take $x = 1$ in Proposition 3.3(3) and use (CI2), then

$$f_X(z) = f_X(1 * z) \supseteq f_X(1 * (y * z)) \cap f_X(y) = f_X(y * z) \cap f_X(y)$$

for all $y, z \in X$. Therefore, $\mathscr{F}_X$ is an int-soft filter of *X*. □

Theorem 3.7. *If a soft set* $\mathscr{F}_X$ *of X over U satisfies the condition Proposition* 3.3(5) *and*

$$(\forall x, y \in X)(f_X(y * x) \supseteq f_X(x)), \tag{3.4}$$

then $\mathscr{F}_X$ *is an int-soft filter of X.*

Proof. Using (CI1), (CI2) and Proposition 3.3(5), we have

$$\begin{aligned} f_X(y) &= f_X(1 * y) \\ &= f_X(((x * y) * (x * y)) * y) \\ &\supseteq f_X(x * y) \cap f_X(x) \end{aligned}$$

for all $x, y \in X$. If we take $y = x$ in (3.4), then $f_X(1) = f_X(x * x) \supseteq f_X(x)$ for all $x \in X$. Therefore, $\mathscr{F}_X$ is an int-soft filter of *X*. □

Theorem 3.8. *A soft set* $\mathscr{F}_X$ *of X over U is an int-soft filter of X if and only if the* γ*-inclusive set* $i_X(f_X; \gamma)$ *is a filter of X for all* $\gamma \in \mathscr{P}(U)$ *with* $i_X(f_X; \gamma) \neq \varnothing$.

The filter $i_X(f_X; \gamma)$ in Theorem 3.8 is called the *inclusive filter* of *X*.

Proof. Assume that $\mathscr{F}_X$ is an int-soft filter of *X*. Let $x, y \in X$ and $\gamma \in \mathscr{P}(U)$ be such that $x * y \in i_X(f_X; \gamma)$ and $x \in i_X(f_X; \gamma)$. Then $\gamma \subseteq f_X(x)$ and $\gamma \subseteq f_X(x * y)$. It follows from (3.1) and (3.2) that $\gamma \subseteq f_X(x) \subseteq f_X(1)$ and $\gamma \subseteq f_X(x * y) \cap f_X(x) \subseteq f_X(y)$ for all $x, y \in X$. Hence $1 \in i_X(f_X; \gamma)$ and $y \in i_X(f_X; \gamma)$. Thus, $i_X(f_X; \gamma)$ is a filter of *X*.

Conversely, suppose that $i_X(f_X; \gamma)$ is a filter of *X* for all $\gamma \in \mathscr{P}(U)$ with $i_X(f_X; \gamma) \neq \varnothing$. For any $x \in X$, let $f_X(x) = \gamma$. Then $x \in i_X(f_X; \gamma)$. Since $i_X(f_X; \gamma)$ is a filter of *X*, we have $1 \in i_X(f_X; \gamma)$ and so $f_X(x) = \gamma \subseteq f_X(1)$. For any $x, y \in X$, let $f_X(x * y) = \gamma_{x*y}$ and $f_X(x) = \gamma_x$. Take $\gamma = \gamma_{x*y} \cap \gamma_x$. Then $x * y \in i_X(f_X; \gamma)$ and $x \in i_X(f_X; \gamma)$ which imply that $y \in i_X(f_X; \gamma)$. Hence

$$f_X(y) \supseteq \gamma = \gamma_{x*y} \cap \gamma_x = f_X(x * y) \cap f_X(x).$$

Thus, $\mathscr{F}_X$ is an int-soft filter of *X*. □

We make a new int-soft filter from old one.

Theorem 3.9. *Let* $\mathscr{F}_X$ *be a soft set of X over U and define a soft set* $\mathscr{F}_X^*$ *of X over U by*

$$f_X^* : X \to \mathscr{P}(U), \quad x \mapsto \begin{cases} f_X(x), & \text{if } x \in i_X(f_X; \gamma), \\ \delta, & \text{otherwise}, \end{cases}$$

where γ *is any subset of* U *and* δ *is a subset of* U *satisfying* $\delta \subsetneq \bigcap_{x \notin i_X(f_X;\gamma)} f_X(x)$. *If* $\mathscr{F}_X$ *is an int-soft filter of* X, *then so is* $\mathscr{F}_X^*$.

Proof. Assume that $\mathscr{F}_X$ is an int-soft filter of X. Then $i_X(f_X;\gamma)(\neq \emptyset)$ is a filter of X for all $\gamma \subseteq U$ by Theorem 3.8. Hence $1 \in i_X(f_X;\gamma)$, and so $f_X^*(1) = f_X(1) \supseteq f_X(x) \supseteq f_X^*(x)$ for all $x \in X$. Let $x, y \in X$. If $x * y \in i_X(f_X;\gamma)$ and $x \in i_X(f_X;\gamma)$, then $y \in i_X(f_X;\gamma)$. Hence

$$f_X^*(y) = f_X(y) \supseteq f_X(x * y) \cap f_X(x) = f_X^*(x * y) \cap f_X^*(x).$$

If $x * y \notin i_X(f_X;\gamma)$ or $x \notin i_X(f_X;\gamma)$, then $f_X^*(x * y) = \delta$ or $f_X^*(x) = \delta$. Thus

$$f_X^*(y) \supseteq \delta = f_X^*(x * y) \cap f_X^*(x).$$

Therefore, $\mathscr{F}_X^*$ is an int-soft filter of X. □

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