Far East Journal of Mathematical Education
© 2013 Pushpa Publishing House, Allahabad, India
Published Online: September 2013
Available online at http://pphmj.com/journals/fjme.htm
Volume 11, Number 1, 2013, Pages 21-25

NOTES ON THE STRONG TRIANGLE INEQUALITY

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Abstract

An ultrametric space is a metric space whose distance function satisfies the strong triangle inequality. In an ultrametric space, many curious properties are valid. For example, every triangle is an isosceles with the unequal side (if any) being shortest. In this paper, we observe that many of the unusual properties in an ultrametric space are actually equivalent to the strong triangle inequality. On the other hand, a simple example shows that some properties valid in an ultrametric space, such as every open ball is closed, do not imply the strong triangle inequality.

1. Introduction

Let (M, d) be a metric space. That is, M is a set equipped with a distance function $d : M \times M \to \mathbb{R}_{\geq 0}$ satisfying the following properties: for all $x, y, z \in M$,

(a) $d(x, y) \geq 0$; equality holds if and only if $x = y$,

Received: March 15, 2013; Accepted: May 4, 2013
Keywords and phrases: strong-triangle inequality, ultrametric spaces.

(b) $d(x, y) = d(y, x)$, and

(c) the triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Here, $\mathbb{R}_{\geq 0}$ is the set of all non-negative real numbers. An ultrametric space is a metric space (M, d) in which the distance function *d* satisfies the strong triangle inequality: for all $x, y, z \in M$,

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

For example, the field of *p*-adic numbers is a well-known ultrametric space.

In an ultrametric space, many curious properties are valid (see [1, 2]). For example, every triangle is an isosceles with the unequal side (if any) being shortest. More precisely, for all $x, y, z \in M$, the largest and the second largest of the three numbers $d(x, y), d(y, z)$, and $d(x, z)$ are equal. Also, every point in a ball is a center of the ball, and given any two balls, either they are disjoint or one of them is contained in the other. It is interesting to recognize that those properties are equivalent to the strong triangle inequality. The proofs of equivalences, which will be given in Section 2, are very simple. Those properties are slight reformulations of the strong triangle inequality.

On the other hand, the phrase "with the unequal side (if any) being shortest" in "every triangle is an isosceles with the unequal side (if any) being shortest" is essential. The property "every triangle is an isosceles" alone does not imply the strong triangle inequality, as a simple example in Section 3 shows. The same example also shows that some properties valid in an ultrametric space, such as every open ball is closed, do not imply the strong triangle inequality.

2. Equivalent Properties

Let (M, d) be a metric space. For $a \in M$ and $r \in \mathbb{R}_{\geq 0}$, the open ball of radius r and center a is a set $B_{<r}(a) := \{x \in M : d(a, x) < r\}$, and the closed ball of radius r and center a is a set $B_{\le r}(a) := \{x \in M : d(a, x) \le r\}$. We have the following equivalences.

Proposition. *Let* (M, d) *be a metric space. Then the following properties are equivalent*:

(1) (M, d) *satisfies the strong triangle inequality. That is,* (M, d) *is an ultrametric space.*

(2) *Every triangle is an isosceles with the unequal side (if any) being shortest. That is, for all* $x, y, z \in M$, *the largest and the second largest of the three numbers* $d(x, y), d(y, z),$ *and* $d(x, z)$ *are equal.*

(3) *For any pair of open balls* B_1 *and* B_2, *if* $B_1 \cap B_2 \ne \emptyset$, *then* $B_1 \subseteq B_2$ *or* $B_2 \subseteq B_1$.

(4) *For any pair of closed balls* B_1 *and* B_2, *if* $B_1 \cap B_2 \ne \emptyset$, *then* $B_1 \subseteq B_2$ *or* $B_2 \subseteq B_1$.

(5) *Every point in an open ball is a center of the ball. That is, for all* $x, y \in M$ *and* $r \in \mathbb{R}_{\ge 0}$, *if* $y \in B_{<r}(x)$, *then* $B_{<r}(x) = B_{<r}(y)$.

(6) *Every point in a closed ball is a center of the ball. That is, for all* $x, y \in M$ *and* $r \in \mathbb{R}_{\ge 0}$, *if* $y \in B_{\le r}(x)$, *then* $B_{\le r}(x) = B_{\le r}(y)$.

Proof. It is well known that (1) implies the rest. See [1, 2].

(2) implies (1). Suppose that $\max\{d(x, y), d(y, z)\} < d(x, z)$ for some $x, y, z \in M$. Then, the largest and the second largest of the three numbers $d(x, y), d(y, z),$ and $d(x, z)$ cannot be equal.

(3) implies (1). Suppose that $\max\{d(x, y), d(y, z)\} < d(x, z)$ for some $x, y, z \in M$. Let $r = d(x, z)$. Then, $B_{<r}(x)$ and $B_{<r}(z)$ are not disjoint because y is in both sets. Thus, we have $B_{<r}(x) \subseteq B_{<r}(z)$ or $B_{<r}(z) \subseteq B_{<r}(x)$. In either case, we have $d(x, z) < r$. This is a contradiction because $r = d(x, z)$.

(4) implies (1). Let $r = \max\{d(x, y), d(y, z)\}$. Then, $B_{\le r}(x)$ and $B_{\le r}(z)$ are not disjoint because y is in both sets. Thus, we have $B_{\le r}(x) \subseteq B_{\le r}(z)$ or $B_{\le r}(z) \subseteq B_{\le r}(x)$. In either case, we have $d(x, z) \le r$. Hence, $d(x, z) \le \max\{d(x, y), d(y, z)\}$.

(5) implies (1). Suppose that $\max\{d(x, y), d(y, z)\} < d(x, z)$ for some $x, y, z \in M$. Let $r = d(x, z)$. Then, $B_{<r}(x) = B_{<r}(y)$ because $d(x, y) < r$ and hence $y \in B_{<r}(x)$. Then, $z \in B_{<r}(x)$ because $z \in B_{<r}(y)$. That is, $d(x, z) < r$. This is a contradiction because $r = d(x, z)$.

(6) implies (1). Let $r = \max\{d(x, y), d(y, z)\}$. Then, $B_{\le r}(x) = B_{\le r}(y)$ because $d(x, y) \le r$ and hence $y \in B_{\le r}(x)$. Then, $z \in B_{\le r}(x)$ because $z \in B_{\le r}(y)$. That is, $d(x, z) \le r$. Hence, $d(x, z) \le \max\{d(x, y), d(y, z)\}$. □

Note that the triangle inequality (c) in the defining conditions of a metric space stated in Section 1 is not used in the above proof. That is, the above proposition is valid for any (M, d) that satisfies (a) and (b) in the defining conditions of a metric space.

3. Non-equivalent Properties

In Section 2, we saw that the strong triangle inequality is equivalent to the property that every triangle is an isosceles with the unequal side (if any) being shortest. The phrase "with the unequal side (if any) being shortest" is essential. That is, the following property (7) alone does not imply the strong triangle inequality, as a simple example below shows.

(7) *Every triangle is an isosceles. That is, for all* $x, y, z \in M$, *at least two of the three numbers* $d(x, y)$, $d(y, z)$, *and* $d(x, z)$ *are equal.*

Example. Consider the metric space $(\mathbb{C}, d)$, where $\mathbb{C}$ is the set of complex numbers and d is the metric defined by the absolute value function on $\mathbb{C}$. Let M be the subset $\{0, 1, i\}$, where $i = \sqrt{-1}$. Consider the metric subspace (M, d). It satisfies the property (7). But, it does not satisfy the strong triangle inequality, because $\max\{d(1, 0), d(0, i)\} < d(1, i)$.

Moreover, in an ultrametric space, the following properties are valid:

(8) *Every open ball is closed*.

(9) *Every closed ball is open*.

However, the property (8) or (9) does not imply the strong triangle inequality, because in the above example, the topology on M is discrete and hence (M, d) satisfies the properties (8) and (9).

References

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