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HYPERCYCLICITY AND SUPERCYCLICITY ON BANACH SEQUENCE SPACES

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Abstract

We investigate sufficient conditions for Hypercyclicity and Supercyclicity Criterion on Banach spaces $L^p(\beta)$.

Introduction

Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 \leq p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=-\infty}^{\infty}$ such that $\| f \|^p = \| f \|_\beta^p = \sum_{n=-\infty}^{\infty} | \hat{f}(n) |^p \beta(n)^p < \infty$.

The notation $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$ shall be used whether or not the series converges for any value of z. These are called *formal Laurent series*. Let $L^p(\beta)$ denote the space of such formal Laurent series. These are reflexive Banach spaces with the norm $\| \cdot \|_\beta$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_{k \in \mathbb{Z}}$ is a basis for $L^p(\beta)$ such that $\| f_k \| = \beta(k)$.

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Now consider M_z, the operator of multiplication by z on $L^p(\beta)$. Clearly M_z shifts the basis $\{f_k\}_k$. The operator M_z is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded. By the same method used in [8] we can see that $L^p(\beta)^* = L^q(\beta^{\frac{p}{q}})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Sources on formal power series include [7-11].

Let X be a complex Banach space and $B(X)$ be the set of bounded linear operators from X into itself. If $T \in B(X)$, then the orbit of a vector $x \in X$ is the set $Orb(T, x) = \{T^n x : n \in \mathbb{N} \cup \{0\}\}$. A vector $x \in X$ is called a *supercyclic vector* for an operator $T \in B(X)$ if the set $\{\lambda y : y \in Orb(T, x), \lambda \in \mathbb{C}\}$ is dense in X. An operator $T \in B(X)$ is supercyclic if it has a supercyclic vector. For several works see [1-6, 12-14].

Main Results

In this section we characterize some conditions under which the multiplication operator M_z holds in the hypothesis of the Hypercyclicity and Supercyclicity Criterion.

From now on we suppose that M_z is bounded on $L^p(\beta)$.

For the benefit of the reader, first we give a variation of the Hypercyclicity Criterion due independently to Kitai [4] and Gethner and Shapiro [3], and a version of the Supercyclicity Criterion due to Feldman et al. [2].

Hypercyclicity Criterion. Suppose that $T \in B(X)$. If there exist two dense sets Y and Z in X and a sequence $n_k \to \infty$ such that

(i) for every $x \in Y$, $T^{n_k} x \to 0$, and

(ii) there exists a function $S : Z \to Z$ such that $TSx = x$ for every $x \in Z$ and $S^{n_k} x \to 0$ for every $x \in Z$,

then T is hypercyclic.

Supercyclicity Criterion. Suppose that $T \in B(X)$. If there exist two dense sets Y and Z in X and a sequence $n_k \to \infty$ such that

(i) there exists a function $S : Z \to Z$ such that $TSx = x$ for every $x \in Z$, and

(ii) $\| T^{n_k} y \| \| S^{n_k} x \| \to 0$ for all $y \in Y$ and $x \in Z$,

then T is supercyclic.

Theorem 1. *The operator M_z is hypercyclic on $L^p(\beta)$ if and only if for all $\varepsilon > 0$ and $m \in \mathbb{N}$ there exists n large enough such that $\beta(j - n)\beta(j)^{-1} < \varepsilon$ and $\beta(j + n)\beta(j)^{-1} < \varepsilon$ for all $-m \leq j \leq m$.*

Proof. See Theorems 1 and 3 in [13].

Theorem 2. *The operator M_z is supercyclic on $L^p(\beta)$ if and only if for all $m \in \mathbb{N}$:*

$$\liminf_{n\to\infty} \max\left\{\frac{\beta(j - n)\beta(k + n)}{\beta(j)\beta(k)} : |j| \leq m, |k| \leq m\right\} = 0.$$

Proof. See Theorem 2 in [14].

In the following B is the operator defined by $Bf_j = f_{j-1}$ for all $j \in \mathbb{Z}$. Clearly B is bounded if and only if the sequence $\{\beta(k)/\beta(k+1)\}_k$ is bounded.

Theorem 3. *Suppose that B is bounded on $L^p(\beta)$. Then*

(i) *there exists a sequence of integers $n_k \to \infty$ such that $\lim_k \beta(n_k) = \lim_k \beta(-n_k) = 0$ if and only if M_z is hypercyclic on $L^p(\beta)$.*

(ii) *there exists a sequence $n_k \to \infty$ such that $\lim_k \beta(n_k)\beta(-n_k) = 0$ if and only if M_z is supercyclic on $L^p(\beta)$.*

Proof. (i) The necessity part follows easily from Theorem 1. So let $\lim_k \beta(n_k) = \lim_k \beta(-n_k) = 0$ for a sequence of integer $n_k \to \infty$. Let $\varepsilon > 0$

and $m \in \mathbb{N}$. By Theorem 1, it is sufficient to show that there exists k large enough such that $\beta(j \pm n_k)\beta(j)^{-1} < \varepsilon$ for all $|j| \leq m$. Put

$$\alpha_0 = \max\{\| M_z^j \| \beta(j)^{-1}, \| B^{-i} \| \beta(i)^{-1} : 0 \leq j \leq m, -m \leq i \leq 0\}.$$

Let δ be such that $0 < \delta\alpha_0 < \varepsilon$. Then for some natural number k, we have $\beta(n_k) < \delta$ and $\beta(-n_k) < \delta$. Note that

$$\frac{\beta(j \pm n_k)}{\beta(\pm n_{n_k})} \leq c_j = \begin{cases} \| M_z^j \| & j \geq 0 \\ \| B^{-j} \| & j < 0 \end{cases},$$

thus for all $|j| \leq m$ we get

$$\begin{aligned} \frac{\beta(j \pm n_k)}{\beta(j)} &= \frac{\beta(j \pm n_k)}{\beta(\pm n_k)} \beta(j)^{-1} \beta(\pm n_k) \\ &\leq \delta\alpha_0 < \varepsilon. \end{aligned}$$

So M_z is hypercyclic on $L^p(\beta)$.

(ii) By Theorem 2 and the same method used in the proof of part (i), it is clear.

Theorem 4. *Let M_z be invertible on $L^p(\beta)$ and suppose that there exists a sequence of integers $n_k \to \infty$. Then we have*

(i) *if $\lim_k \beta(n_k) = \lim_k \beta(-n_k) = 0$, then M_z holds in the Hypercyclicity Criterion.*

(ii) *if $\lim_k \beta(n_k)\beta(-n_k) = 0$, then M_z holds in the Supercyclicity Criterion.*

Proof. Let $Y = Z$ be the linear span of $\{f_j\}_{j \in \mathbb{Z}}$. Then Y and Z are dense in $L^p(\beta)$.

(i) For all $j \in \mathbb{Z}$, we have

$$\begin{aligned} \| M_z^{n_k} f_j \| &= \| M_z^j M_z^{n_k} f_0 \| \leq \| M_z^j \| \| M_z^{n_k} f_0 \| \\ &= \| M_z^j \| \| f_{n_k} \| = \| M_z^j \| \beta(n_k) \end{aligned}$$

and

$$\| M_z^{-n_k} f_j \| = \| M_z^j M_z^{-n_k} f_0 \| \leq \| M_z^j \| \| M_z^{-n_k} f_0 \|$$

$$= \| M_z^j \| \| f_{-k_n} \| = \| M_z^j \| \beta(-n_k).$$

So we get $\lim_k M_z^{n_k} f_j = \lim_k M_z^{-n_k} f_j = 0$ for all $j \in \mathbb{Z}$ and this implies that $\lim_k M_z^{-n_k} f = \lim_k M_z^{n_k} f = 0$ for all f in Y and Z. Thus M_z holds in the hypothesis of the Hypercyclicity Criterion.

(ii) For all i and j in $\mathbb{Z}$ we have

$$\| M_z^{-n_k} f_j \| \| M_z^{n_k} f_i \| \leq \| M_z^i \| \| M_z^j \| \beta(-n_k) \beta(n_k).$$

Therefore $\lim_k \| M_z^{n_k} f \| \| M_z^{-n_k} g \| = 0$ for all f in Y and g in Z. Now we can see that M_z holds in the hypothesis of the Supercyclicity Criterion.

Corollary 5. *Let M_z be invertible. Then M_z is hypercyclic* (*supercyclic*) *if and only if M_z satisfies the hypothesis of the Hypercyclicity* (*Supercyclicity*) *Criterion.*

Proof. Note that by our convention, M_z is bounded on $L^p(\beta)$. So, in Theorem 3, the boundedness of B on $L^p(\beta)$ implies that indeed M_z is invertible on $L^p(\beta)$. Now by Theorems 1, 2, 3 and 4, the proof is clear.

Note that in the following theorem, M_z is not necessarily invertible.

Theorem 6. *Suppose that there exist an $\alpha > 0$ and a sequence of integers $n_k \to \infty$ such that $\beta(n_k) \geq \alpha^r \beta(n_k - r)$ for all positive integers k and r. Then we have*

(i) *if $\lim_k \beta(n_k) = \lim_k \beta(-n_k) = 0$, then M_z is hypercyclic on $L^p(\beta)$.*

(ii) *if $\lim_k \beta(n_k)\beta(-n_k) = 0$, then M_z is supercyclic on $L^p(\beta)$.*

Proof. Let $\varepsilon > 0$ and $m \in \mathbb{N}$. Put

$$\alpha_0 = \max\{\alpha^{j-m}\beta(j)^{-1}, \| M_z^{i+m} \| \beta(i)^{-1} : |i| \le m, |j| \le m\}.$$

(i) By Theorem 1, it is sufficient to show that $\beta(j+n)\beta(j)^{-1} < \varepsilon$ and $\beta(j-n)\beta(j)^{-1} < \varepsilon$ for all $|j| \le m$ and n sufficiently large. Suppose that $0 < \delta < \frac{\varepsilon}{\alpha_0}$. Then there exists k large enough such that $\beta(n_k) < \delta$ and $\beta(-n_k) < \delta$. For $j \in \mathbb{Z}$ with $|j| \le m$ and $n = n_k - m$ we have

$$\frac{\beta(j+n)}{\beta(j)} = \frac{\beta(j-m+n_k)}{\beta(n_k)} \beta(j)^{-1}\beta(n_k)$$

$$\le \alpha^{j-m}\beta(j)^{-1}\beta(n_k) \le \alpha_0\delta < \varepsilon.$$

Also, with the same choice of n and for $|i| \le m$ we get

$$\frac{\beta(i-n)}{\beta(i)} = \frac{\beta(i+m-n_k)}{\beta(-n_k)} \beta(i)^{-1}\beta(-n_k)$$

$$\le \| M_z^{i+m} \| \beta(i)^{-1}\delta$$

$$\le \alpha_0\delta < \varepsilon.$$

So M_z is hypercyclic on $L^p(\beta)$.

(ii) We will see that the condition of Theorem 2 is satisfied. Let δ be such that $0 < \delta\alpha_0^2 < \varepsilon$. By the hypothesis, there exists an integer k large enough such that $\beta(n_k)\beta(-n_k) < \delta$. Now for $n = n_k - m, |i| \le m$ and $|j| \le m$ we have

$$\frac{\beta(j+n)\beta(i-n)}{\beta(j)\beta(i)} = \left(\frac{\beta(j-m+n_k)}{\beta(n_k)}\beta(j)^{-1}\right)\beta(n_k)\left(\frac{\beta(i+m-n_k)}{\beta(-n_k)}\beta(i)^{-1}\right)\beta(-n_k)$$

$$\le (\alpha^{j-m}\beta(j)^{-1})(\| M_z^{i+m} \| \beta(i)^{-1})\delta \le \alpha_0^2\delta < \varepsilon.$$

Hence M_z is supercyclic on $L^p(\beta)$.

Remark 7. (i) Theorem 4 is an immediate consequence of the above theorem. Because if M_z is invertible, then it is bounded below, i.e., there exists an $\alpha > 0$ such that $\| M_z f \| \geq \alpha \| f \|$ for all f in $L^p(\beta)$. This implies that $\beta(i+1) = \| f_{i+1} \| \geq \alpha \| f_i \| = \alpha\beta(i)$ for all $i \in \mathbb{Z}$ and so $\beta(i) \geq \alpha^k \beta(i-k)$ for all $i \in \mathbb{Z}$ and for all $k \in \mathbb{Z}^+$.

(ii) By Theorems 1 and 2, clearly the converse of Theorem 6 is also true.

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