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BORCHERDS-KAC-MOODY LIE ALGEBRAS, SPECIAL PRIMITIVE WEIGHTS AND LIE ALGEBRA HOMOLOGY

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Abstract

Let $\mathfrak{g}$ be a Borcherds-Kac-Moody Lie algebra, and let Λ be a dominant integral weight arising from $\mathfrak{g}$. We also let $L(\Lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight Λ. In this paper, we study relationships between non-vanishing terms of the character of $L(\Lambda)$ parameterized by primitive weights and irreducible components of Kostant's homology space.

1. Introduction

Borcherds-Kac-Moody (denoted by BKM for short) Lie algebras were first introduced and studied by Borcherds [1]. They are Lie algebras generalizing the class of Kac-Moody Lie algebras. In the representation theory of BKM Lie algebras, highest weight representations and their character formulas have led to various important results and applications

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such as the proof the Moonshine theorem and the link with automorphic forms.

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, and let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. We also let V be a highest weight $\mathfrak{g}$-module with highest weight $\Lambda \in \mathfrak{h}^*$. Then the character $ch(V)$ can be parameterized by primitive weights and has the following form:

$$ch(V) = \frac{1}{\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{mult(\alpha)}} \sum_{\lambda\leq\Lambda,\, |\lambda+\rho|^2=|\Lambda+\rho|^2} c(\lambda)e^{\lambda}, \quad (1.1)$$

where Δ_+ is the set of positive roots, $mult(\alpha)$ are the root multiplicities and $c(\lambda)$ are certain coefficients satisfying $c(\Lambda) = 1$ (see [4, Chapter 9] for more details). As is well-known, the primitive weights λ satisfy the condition $|\lambda + \rho|^2 = |\Lambda + \rho|^2$.

The concept of primitive weights is first introduced by Kac to streamline the proof of Weyl character formula [4, 5]. In fact, the Weyl-Kac character formula is obtained from (1.1) by expressing the summation in terms of the elements of the Weyl group of $\mathfrak{g}$ (see [4, Theorem 10.4]).

Formula (1.1) is also true for the more general context of BKM Lie algebras. Thus, the primitive weights still play important roles in the representation theory of BKM Lie algebras.

On the other hand, the primitive weights are also involved in the theory of Lie algebra homology. In more details, let $\mathfrak{g}$ be a symmetrizable BKM Lie algebra, $\mathfrak{n}^-$ be a direct sum of negative root spaces of $\mathfrak{g}$ and $L(\Lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight Λ. Then, as $\mathfrak{h}$-modules, we have

$$H_p(\mathfrak{n}^-, L(\Lambda)) = \bigoplus_{\lambda \text{ s.t.} (\lambda+\rho|\lambda+\rho)=(\Lambda+\rho|\Lambda+\rho)} \mathbb{C}(\lambda), \quad (1.2)$$

where $\mathbb{C}(\lambda)$ denotes the irreducible highest weight $\mathfrak{h}$-module with highest weight λ (see Theorem 3.1).

Now, we observe that formula (1.1) and the decomposition (1.2) share the same condition $(\lambda + \rho | \lambda + \rho) = (\Lambda + \rho | \Lambda + \rho)$ for the weights parameterizing their summands. This observation reflects us certain relationships between the non-vanishing terms of (1.1) and the irreducible components of $H_p(\mathfrak{n}^-, L(\Lambda))$.

To state our concern in more details, let us introduce the following definition:

Definition 1.1. Let $\lambda \in \mathfrak{h}^*$ and let Λ be a dominant integral weight arising from a BKM Lie algebra $\mathfrak{g}$. Then λ is called a *special primitive weight* associated $L(\Lambda)$ if we have both $c(\lambda) \neq 0$ and $\mathbb{C}(\lambda)$ is an irreducible component of $H_p(\mathfrak{n}^-, L(\Lambda))$ for some p. Here, $c(\lambda)$ is the coefficient appearing in (1.1).

Then the following question arises naturally:

Question 1.2. Let μ be a given special primitive weight associated with $L(\Lambda)$. Then determine the degree p such that Lie algebra homology space $H_p(\mathfrak{n}^-, L(\Lambda))$ contains an irreducible component $\mathbb{C}(\mu)$.

In this paper, we aim to answer for Question 1.2. For this, we first study the properties of the special primitive weights arising from the irreducible highest weight representations of BKM Lie algebras. Among those properties, we emphasize that our special primitive weights satisfy the interesting property Corollary 4.4. In general, the isotropy group of a dominant weight is generated by the fundamental reflections which it contains (see [4, Proposition 3.12]). Thus, we may regard Corollary 4.4 as a peculiar property of our special primitive weights. By decomposing the primitive weights into a sum of weights obtained from $\wedge(\mathfrak{n}^-)$ and $L(\Lambda)$, we can analyze the special primitive weight through the highest weight Λ and imaginary simple roots. Finally, by applying our results about special primitive weights, we calculate explicitly the weight space corresponding to a given special primitive weight. This yields our main results which answer to Question 1.2 (see Theorem 4.8).

2. Preliminaries

In this section, we fix notations and review basic facts about the representation theory of BKM Lie algebras. See [1, 4, 10, 11] for more details.

Let $A = (a_{ij})_{1\le i,\, j\le n}$ be an $n\times n$ real matrix satisfying the following conditions:

(1) $a_{ii} = 2$ or $a_{ii} \le 0$;

(2) $a_{ij} \le 0$ if $i \ne j$;

(3) $a_{ij} = 0$ implies $a_{ji} = 0$;

(4) *If* $a_{ii} = 2$, then $a_{ij} \in \mathbb{Z}$ for all j.

If a real $n \times n$ matrix A satisfies the above conditions (1), (2), (3) and (4), then the matrix A is called a *BKM matrix*.

Let $\mathfrak{h}$ be a complex vector space. Then a realization of the BKM matrix $A = (a_{ij})_{1\le i,\, j\le n}$ is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ satisfying the following conditions:

(1) both $\Pi = \{\alpha_1, ..., \alpha_n\}$ and $\Pi^\vee = \{\alpha_1^\vee, ..., \alpha_n^\vee\}$ are linearly independent subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively;

(2) $\alpha_j(\alpha_i^\vee) = a_{ij}$ for $i, j = 1, 2, ..., n$;

(3) $a_{ij} = 0$ implies $a_{ji} = 0$;

(4) $\dim_{\mathbb{C}} \mathfrak{h} = 2n - rank(A)$.

We denote by Π^{re} (resp. Π^{im}) the subset $\{\alpha_i \in \Pi \mid a_{ii} = 2\}$ (resp.

$\{\alpha_i \in \Pi \mid a_{ii} \leq 0\})$ of Π. We also write I^{re} (resp. I^{im}) for the subset $\{i \in I \mid a_{ii} = 2\}$ (resp. $\{i \in I \mid a_{ii} \leq 0\}$) of $I = \{1, 2, ..., n\}$.

Let $\tilde{\mathfrak{g}}(A)$ be a Lie algebra generated by $\mathfrak{h}$ and $2n$ letters e_i, f_i $(1 \leq i \leq n)$ under the following relations:

(1) $[h, h'] = 0$ $(h, h' \in \mathfrak{h})$.

(2) $[h, e_i] = \alpha_i(h)e_i$ $(h \in \mathfrak{h}, 1 \leq i \leq n)$.

(3) $[h, f_i] = -\alpha_i(h) f_i$ $(h \in \mathfrak{h}, 1 \leq i \leq n)$.

(4) $[e_i, f_i] = \delta_{ij}\alpha_i^\vee$.

To define a BKM Lie algebra, let τ be the largest proper ideal of $\tilde{\mathfrak{g}}(A)$ satisfying $\tau \cap \mathfrak{h} = \{0\}$. Then $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\tau$ is called a *BKM Lie algebra*.

If the BKM matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ is decomposed as $A = DB$ for a diagonal matrix D and symmetric matrix B, then the BKM matrix A is said to be *symmetrizable*. We also call the corresponding Lie algebra $\mathfrak{g}(A)$ the *symmetrizable BKM Lie algebra*.

As is well-known, there uniquely exists a non-degenerate symmetric bilinear form $(\,|\,)$ on $\mathfrak{g}(A)$ if $\mathfrak{g}(A)$ is a symmetrizable BKM Lie algebra. For $i \in I$ with $a_{ii} \neq 0$, it is easy to check that the bilinear form $(\,|\,)$ satisfies

$$\frac{2(\alpha_i \mid \lambda)}{(\alpha_i \mid \alpha_i)} = \frac{2\lambda(\alpha_i^\vee)}{a_{ii}} \tag{2.1}$$

for $\lambda \in \mathfrak{h}^*$.

Throughout this paper, we fix an element ρ of $\mathfrak{h}^*$ satisfying

$$(\rho \mid \alpha_i) = \frac{(\alpha_i \mid \alpha_i)}{2} \tag{2.2}$$

for all $\alpha_i \in \Pi$.

We have the root space decomposition $\mathfrak{g}(A) = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \mathfrak{h}^* - \{0\}} \mathfrak{g}_\alpha \right)$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) | [hx] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. An element $\alpha \in \mathfrak{h}^* - \{0\}$ is called a *root* if $\mathfrak{g}_\alpha \neq 0$. We write Δ for the set of all roots of $\mathfrak{g}(A)$. We call $\mathfrak{g}_\alpha$ the root space of a root α if $\alpha \in \Delta$, and denote by $mult(\alpha)$ the dimension of $\mathfrak{g}_\alpha$. A root α is called a *positive root* (resp. *negative root*) if $\alpha \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ $\left(\text{resp. } \alpha \in \sum_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i\right)$. Let Δ_+ (resp. Δ_-) be the set of all positive roots (resp. all negative roots). Define a partial order $\preceq$ on $\mathfrak{h}^*$ by $\mu \preceq \lambda$ if and only if $\lambda - \mu \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. Put $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha$ and $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$. Then we have the triangular decomposition $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

The Weyl group W of a BKM Lie algebra $\mathfrak{g}(A)$ is a subgroup of $GL(\mathfrak{h}^*)$ generated by reflections r_{α_i} $(i \in I^{re})$, where r_{α_i} is a reflection defined as $r_{\alpha_i}(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i$ $(\lambda \in \mathfrak{h}^*)$. For $w \in W$, write $l(w)$ for the length of w.

Henceforth, $\mathfrak{g}(A)$ will be a symmetrizable BKM Lie algebra unless otherwise specified. We also simply denote by $\mathfrak{g}$ the symmetrizable BKM Lie algebra $\mathfrak{g}(A)$ associated to the BKM matrix A if no confusion is likely to arise.

Let V be a representation of $\mathfrak{g}$. For $\lambda \in \mathfrak{h}^*$, we set

$$V_\lambda = \{v \in V \,|\, h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\lambda \neq 0$, then λ is called a *weight* of the representation V and we call V_λ the *weight space* of weight λ. We denote by $P(V)$ the set of all weights of the representation V. We also call the representation V the *weight* $\mathfrak{g}$*-module* if each weight space V_λ is finite dimensional and V admits a weight space decomposition $V = \bigoplus_{\lambda \in P(V)} V_\lambda$. When V is a weight $\mathfrak{g}$-module, the character of V is defined as $ch(V) = \sum_{\lambda \in P(V)} (\dim V_\lambda) e^\lambda$.

A weight $\mathfrak{g}$-module V is said to be *integrable* if the generators e_i and f_i act on V locally nilpotently for all $i \in I^{re}$. For a given integrable weight $\mathfrak{g}$-module V and $\lambda \in P(V)$, it is well-known that $\dim V_\lambda = \dim V_{w\lambda}$ for $w \in W$ and hence $ch(V)$ is invariant under the action of the Weyl group W.

For each $\lambda \in \mathfrak{h}^*$, recall that there exists a unique irreducible highest weight $\mathfrak{g}$-module $L(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$.

We now set $D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \preceq \lambda\}$ for $\lambda \in \mathfrak{h}^*$. Let O be the category of all weight $\mathfrak{g}$-modules V such that there exist a finite number of elements $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ satisfying $P(V) \subset \bigcup_{i=1}^{k} D(\lambda_i)$. If V is an object of O, then any submodule U of V and a quotient module V/U are both objects of O. Also, direct sums and tensor products of a finite number of modules in the category O are again in O.

For a given symmetrizable BKM Lie algebra $\mathfrak{g}$, we now take two pairs of basis $\{u_i\}_{1 \le i,\, j \le \dim \mathfrak{h}}$ and $\{u^i\}_{1 \le i,\, j \le \dim \mathfrak{h}}$ of $\mathfrak{h}$ such that $(u_i \mid u^i) = \delta_{ij}$. In addition, for each $\alpha \in \Delta_+$, we choose bases $\{e_\alpha^{(i)}\}_{1 \le i \le mult(\alpha)}$ of $\mathfrak{g}_\alpha$ and $\{e_{-\alpha}^{(i)}\}_{1 \le i \le mult(\alpha)}$ of $\mathfrak{g}_{-\alpha}$ satisfying $(e_\alpha^{(i)} \mid e_{-\alpha}^{(i)}) = \delta_{ij}$. We also define the Casmir element of $\mathfrak{g}$ as $\Omega = 2\rho + \sum_{i=1}^{\dim \mathfrak{h}} u^i u_i + 2 \sum_{\alpha \in \Delta_+} \sum_{i=1}^{mult(\alpha)} e_{-\alpha}^{(i)} e_\alpha^{(i)}$. Then it is well-known that $[e_i, \Omega] = [f_i, \Omega] = 0$ for $i = 1, 2, \ldots, n$. If V is a highest weight $\mathfrak{g}$-module with highest weight λ, then the Casmir element Ω acts on V as a scalar operator with eigenvalue $(\lambda \mid \lambda + 2\rho)$.

Let $(\mathfrak{h}, \Pi, \Pi^{\vee})$ be a realization of the BKM matrix $A = (a_{ij})_{1 \le i,\, j \le n}$. Then we can associate the quasi-Dynkin diagram to A as follows:

(1) The quasi-Dynkin diagram has n vertices $\alpha_1, \ldots, \alpha_n$, where $\Pi = \{\alpha_1, \ldots, \alpha_n\}$.

(2) If $i \neq j$ and $a_{ij} < 0$, then we connect two vertices α_i and α_j.

(3) If $i \neq j$ and $a_{ij} = 0$, then we do not connect two vertices α_i and α_j.

Let $\alpha = \sum_{i=1}^{n} m_i \alpha_i$ for the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Then we define the support of α to be a sub-diagram of the quasi-Dynkin diagram which consists of the vertices α_i such that $m_i \neq 0$ and of all edges joining these vertices.

Lemma 2.1. *If α is a root of a BKM Lie algebra $\mathfrak{g}$, then the support of α is connected.*

Proof. See [4, Section 5.3] or [11, Lemma 2.35]. □

The following proposition is well-known for Kac-Moody algebras.

Proposition 2.2. *Let $\mathfrak{g}$ be a symmetrizable BKM algebra, and let W be the Weyl group of $\mathfrak{g}$. For $w \in W$, let $\Phi_w = \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) \in \Delta_-\}$. If $w = r_{\alpha_{i_1}} \cdots r_{\alpha_{i_m}}$ is a reduced expression (i.e., $m = l(w)$) for $\{\alpha_{i_1}, \ldots, \alpha_{i_m}\} \subset \Pi^{re}$, then we have*

(1) $\Phi_w = \{\alpha_{i_1}, r_{\alpha_{i_1}}(\alpha_{i_2}), \ldots, r_{\alpha_{i_1}}, \ldots, r_{\alpha_{i_{m-1}}}(\alpha_{i_m})\}$.

(2) $\rho - w\rho = \sum_{\phi \in \Phi_w} \phi$.

Proof. Notice that the Weyl group W of the BKM Lie algebra $\mathfrak{g}$ is a Coxeter group (see [10, Section 2.3]). Thus, the statement (1) follows from the general theory of Coxeter groups (see [7, Lemma 1.4.14]). In addition, [7, Corollary 1.3.22] is applied to our case and yields that

$$\lambda - w\lambda = \sum_{k=1}^{m} \lambda(\alpha_{i_k}^{\vee})(r_{\alpha_{i_1}} \cdots r_{\alpha_{i_{k-1}}} \alpha_{i_k})$$

for $\lambda \in \mathfrak{h}^*$.

We also obtain from (2.2) that $\rho(\alpha_{i_k}^{\vee}) = \dfrac{2(\rho \mid \alpha_{i_k})}{(\alpha_{i_k} \mid \alpha_{i_k})} = 1$ for each $\alpha_{i_k} \in \Pi^{re}$. The second result now follows from (1). □

3. Lie Algebra Homology

In this section, we review the notion of Lie algebra homology. We refer to [6, 7] for more details.

We first take the irreducible highest weight $\mathfrak{g}$-module $L(\Lambda)$. Write $\wedge(\mathfrak{n}^-)$ for the exterior algebra of $\mathfrak{n}^-$. We denote by $\wedge^k(\mathfrak{n}^-)$ the space of kth homogeneous elements in $\wedge(\mathfrak{n}^-)$. Let us consider the chain complex

$$\to \wedge^p(\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda) \xrightarrow{\partial_p} \wedge^{p-1}(\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda) \to \cdots \to L(\Lambda) \xrightarrow{\partial_0} 0, \quad (3.1)$$

where

$$\begin{aligned} &\partial_p(x_1 \wedge \cdots \wedge x_p \otimes v) \\ &= \sum_{i<j} (-1)^{i+j} [x_i x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_p \otimes v \\ &\quad + \sum_i (-1)^i x_i \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_p \otimes (x_i \cdot v). \end{aligned}$$

Then the homology of the chain complex (3.1) is called the *Lie algebra homology* of $\mathfrak{n}^-$ with coefficients in $L(\Lambda)$, and we write $H_p(\mathfrak{n}^-, L(\Lambda))$ for this Lie algebra homology. Notice that $H_p(\mathfrak{n}^-, L(\Lambda))$ acquires an $\mathfrak{h}$-module structure in a natural way.

It is easy to see that $\wedge^p(\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda) \in O$ (see the proof of Theorem 3.1 below). This implies that $H_p(\mathfrak{n}^-, L(\Lambda)) \in O$. Thus, $H_p(\mathfrak{n}^-, L(\Lambda))$ is decomposed into irreducible components $\mathbb{C}(\mu)$ as $\mathfrak{h}$-modules because $H_p(\mathfrak{n}^-, L(\Lambda))$ is a weight $\mathfrak{g}$-module. Here $\mathbb{C}(\mu)$ means the one dimensional irreducible $\mathfrak{h}$-module with highest weight $\mu \in \mathfrak{h}^*$. By the definition of weight $\mathfrak{g}$-modules, each irreducible component $\mathbb{C}(\mu)$ occurs only finitely many times in the decomposition of $H_p(\mathfrak{n}^-, L(\Lambda))$.

By the arguments of [8, Proposition 18] and [7, Theorem 3.2.7] for symmetrizable Kac-Moody Lie algebras, we obtain the following theorem for symmetrizable BKM Lie algebras.

Theorem 3.1. *Let* $\mathfrak{g}$ *be a symmetrizable BKM algebra, and suppose that* $\mathfrak{g}$ *has the triangular decomposition* $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. *Let* $\mathbb{C}(\mu)$ *be an irreducible* $\mathfrak{h}$*-module component of* $H_p(\mathfrak{n}^-, L(\Lambda))$. *Then* $\mathbb{C}(\mu)$ *is an irreducible* $\mathfrak{h}$*-module component of* $\wedge^p(\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda)$ *and the weight* μ *satisfies that* $|\mu + \rho|^2 = |\Lambda + \rho|^2$.

Proof. Put $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}^+$ for the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Then we obtain that as $\mathfrak{h}$-modules

$$\mathbb{C} \otimes_{U(\mathfrak{n}^-)} ((U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \wedge^p(\mathfrak{g}/\mathfrak{p})) \otimes_{\mathbb{C}} L(\Lambda)) \tag{3.2}$$

$$\simeq \mathbb{C} \otimes_{U(\mathfrak{n}^-)} (U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \wedge^p(\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda)) \tag{3.3}$$

$$\simeq \wedge^p(\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda). \tag{3.4}$$

From the isomorphism (3.3), we see that $(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \wedge^p (\mathfrak{g}/\mathfrak{p})) \otimes_{\mathbb{C}} L(\Lambda) \in O$, and hence the action of Casmir element Ω on $(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \wedge^p (\mathfrak{g}/\mathfrak{p})) \otimes_{\mathbb{C}} L(\Lambda)$ is well-defined. Here, the action of Ω on $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \wedge^p (\mathfrak{g}/\mathfrak{p})$ is given by left multiplication only on the first factor $U(\mathfrak{g})$.

Let us denote by Ω_p the operator on $(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \wedge^p (\mathfrak{g}/\mathfrak{p})) \otimes_{\mathbb{C}} L(\Lambda)$, that is, induced from the action of the Casmir element Ω. On each $\mathbb{C} \otimes_{U(\mathfrak{n}^-)} ((U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \wedge^p (\mathfrak{g}/\mathfrak{p})) \otimes_{\mathbb{C}} L(\Lambda))$, the operator $1 \otimes \Omega_p$ is equal to the action of the element $2\rho + \sum_{i=1}^{\dim \mathfrak{h}} u^i u_i$ because $1 \otimes e_{-\alpha}^{(i)} e_{\alpha}^{(i)} x = 1 e_{-\alpha}^{(i)} \otimes e_{\alpha}^{(i)} x = 0$ for $x \in (U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \wedge^p (\mathfrak{g}/\mathfrak{p})) \otimes_{\mathbb{C}} L(\Lambda)$. If we combine the $\mathfrak{h}$-module isomorphism (3.4) with the fact $2\rho + \sum_{i=1}^{\dim \mathfrak{h}} u^i u_i \in \mathfrak{h}$, then we have an $\mathfrak{h}$-module map

$$1 \otimes \Omega_p : \wedge^p (\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda) \to \wedge^p (\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda).$$

On the other hand, by the statement (6) in the proof of [7, Theorem 3.2.7], we see that $1 \otimes \Omega_p : \wedge^p (\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda) \to \wedge^p (\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda)$ yields a scalar operator $(\Lambda | \Lambda + 2\rho)$ on the homology $H_p(\mathfrak{n}^-, L(\Lambda))$. Hence, we get that $(\mu | \mu + 2\rho) = (\Lambda | \Lambda + 2\rho)$ on the irreducible component $\mathbb{C}(\mu)$ of $H_p(\mathfrak{n}^-, L(\Lambda))$ because $2\rho + \sum_{i=1}^{\dim \mathfrak{h}} u^i u_i$ yields a scalar map $(\mu | \mu + 2\rho)$ on the highest weight module $\mathbb{C}(\mu)$. The theorem now follows. □

4. Main Results

In this section, we fix the BKM Lie algebra $\mathfrak{g}$ associated to a BKM matrix $A = (a_{ij})_{1 \leq i, j \leq n}$, and set

$$P_+ = \{\lambda \in \mathfrak{h}^* \,|\, \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I^{re}\}$$

and

$$P_{(+)} = \{\lambda \in P_+ \mid \lambda(\alpha_i^\vee) \geq 0 \text{ for all } i \in I\}.$$

We call an element of $P_{(+)}$ a *dominant integral weight*.

Henceforth, we fix $\Lambda \in P_{(+)}$.

By [11, Proposition 2.58] and [11, Proposition 2.61], every irreducible highest weight $\mathfrak{g}$-module with a dominant integral highest weight is integrable. Thus, the character $ch(L(\Lambda))$ is invariant under the action of the Weyl group W.

Let us write R for $\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{mult(\alpha)}$. Then formula (1.1) yields that

$$ch(L(\Lambda)) = \frac{1}{e^{\rho}R} \sum_{\lambda\leq\Lambda, |\lambda+\rho|^2 = |\Lambda+\rho|^2} c(\lambda)e^{\lambda+\rho}. \tag{4.1}$$

From equation (4.1), we obtain that $e^{\rho}Rch(L(\Lambda))$ is W-skew-invariant (i.e., $w(e^{\rho}RchL(\Lambda)) = (-1)^{l(w)}e^{\rho}RchL(\Lambda))$ because we have $w(e^{\rho}R) = (-1)^{l(w)}e^{\rho}R$ and $\Lambda \in P_{(+)}$.

Let us take a special primitive weight μ associated with $L(\Lambda)$. By the definition of a special primitive weight, we get that

$$|\mu+\rho|^2 = |\Lambda+\rho|^2 \text{ and } c(\mu) \neq 0. \tag{4.2}$$

Since $e^{\rho}Rch(L(\Lambda))$ is W-skew-invariant and $c(\mu) \neq 0$, we obtain that for any $w \in W$,

$$w(\mu+\rho) = \lambda+\rho \tag{4.3}$$

for some $\lambda \leq \Lambda$ satisfying $|\lambda+\rho|^2 = |\Lambda+\rho|^2$.

We now have the following proposition:

Proposition 4.1. *Let* $\mu \in P_{(+)}$ *and let* μ *be a special primitive weight associated with* $L(\Lambda)$. *Then there exists an element* $w' \in W$ *such that* $w'(\mu + \rho) \in P_+$.

Proof. Immediate from (4.3) and [11, Lemma 2.72]. □

Lemma 4.2. *Let* $\lambda \in P_+$. *If* $w(\lambda) \in P_+$ *for some* $w \in W$, *then* $w(\lambda) = \lambda$.

Proof. Assume that $w = r_{\alpha_{i_1}} \cdots r_{\alpha_{i_s}}$ is a reduced expression of $w \in W$, where $\{\alpha_{i_1}, \ldots, \alpha_{i_s}\} \subset \Pi^{re}$. Since $\lambda \in P_+$, we have $\lambda(\alpha_i^{\vee}) \geq 0$ for all $i \in I^{re}$. So we get that $(\alpha_i, \lambda) \geq 0$ for all $i \in I^{re}$. This implies that $(w(\alpha_i), w(\lambda)) \geq 0$ for all $i \in I^{re}$. In particular, we obtain that

$$(w(\alpha_{i_s}), w(\lambda)) \geq 0. \tag{4.4}$$

On the other hand, we notice from [7, Lemma 1.3.13] that $w(\alpha_{i_s}) \preceq 0$. Moreover, it is easy to see from a direct computation that $w(\alpha_{i_s})$ is a linear combination of $\{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ with integer coefficients. Thus, we should have $w(\alpha_{i_s}) = \sum_{j=1}^{s} k_j \alpha_{i_j}$ with $k_j \in \mathbb{Z}_{\leq 0}$. Hence, by the assumption $w(\lambda) \in P_+$, we have

$$(w(\alpha_{i_s}), w(\lambda)) \leq 0. \tag{4.5}$$

By combining (4.4) and (4.5), we obtain that $(w(\alpha_{i_s}), w(\lambda)) = (\alpha_{i_s}, \lambda) = 0$. This yields that $r_{\alpha_{i_s}}(\lambda) = \lambda$. The lemma now follows by induction on $l(w)$.

□

In the following theorem, we prove that there exists such a unique element $w \in W$ in Proposition 4.1.

Theorem 4.3. *Let* μ *be a special primitive weight associated with* $L(\Lambda)$. *Then there exists uniquely* $w \in W$ *such that* $w(\mu+\rho) \in P_+$.

Proof. Suppose that $w_1(\mu+\rho) \in P_+$ and $w_2(\mu+\rho) \in P_+$ for some $w_1, w_2 \in W$. Since $c(\mu) \neq 0$, we obtain from the W-skew-invariance of $e^{\rho}Rch(L(\Lambda))$ that $w_1(\mu+\rho) = \lambda_1 + \rho$ (resp. $w_2(\mu+\rho) = \lambda_2 + p$) for some $\lambda_1 \leq \Lambda$ (resp. $\lambda_2 \leq \Lambda$). Then we have $\mu + \rho = w_1^{-1}(\lambda_1+\rho) = w_2^{-1}(\lambda_2+\rho)$. This implies that $w_2w_1^{-1}(\lambda_1+\rho) = \lambda_2 + \rho \in P_+$. Thus, Lemma 4.2 yields that $w_2w_1^{-1}(\lambda_1+\rho) = \lambda_1 + \rho$ because of $\lambda_1 + \rho = w_1(\mu+\rho) \in P_+$. Hence, we have

$$w_2w_1^{-1}(\lambda_1+\rho) = \lambda_2 + \rho = \lambda_1 + \rho. \tag{4.6}$$

On the other hand, if $w(\mu+\rho)(\alpha_i^{\vee}) = 0$ for some $i \in I^{re}$ and $w \in W$, then we have

$$r_{\alpha_i}(w(\mu+\rho)) = w(\mu+\rho) - w(\mu+\rho)(\alpha_i^{\vee})\alpha_i = w(\mu+\rho). \tag{4.7}$$

By equation (4.3), we obtain that $w(\mu+\rho) = \lambda + \rho$ for some $\lambda \leq \Lambda$. So, the W-skew-invariance of $e^{\rho}RchL(\Lambda)$ and (4.7) yield that $c(\lambda) = -c(\lambda)$. This contradicts to the fact $c(\mu) \neq 0$ because $c(\mu) = (-1)^{l(w)}c(\lambda)$.

Hence, our assumption $c(\mu) \neq 0$ and equation (4.6) implies that

$$(\lambda_1+\rho)(\alpha_i^{\vee}) \geq 1 \text{ for all } i \in I^{re}. \tag{4.8}$$

In equation (4.6), if $w_2w_1^{-1} = r_{\alpha_{j_1}} \cdots r_{\alpha_{j_k}}$ for $\{\alpha_{j_1}, ..., \alpha_{j_k}\} \subset \Pi^{re}$, then by the proof of Lemma 4.2, we have $(\alpha_{j_k}, \lambda_1 + \rho) = 0$. This contradicts to (4.8).

Therefore, we should have $w_2w_1^{-1} = 1$. The theorem now follows. $\square$

The following corollary provides an interesting property of special primitive weights.

Corollary 4.4. *Let* μ *be a special primitive weight associated with* $L(\Lambda)$. *Let* w *be the element of the Weyl group* W *satisfying* $w(\mu + \rho) \in P_+$. *Then the isotropy group* $W_{w(\mu+\rho)}$ *of the dominant weight* $w(\mu + \rho)$ *is the trivial group*.

Proof. Immediate from the proof of Theorem 4.3. □

Now, we decompose a special primitive weight as a sum of weights of the spaces $\wedge^p(\mathfrak{n}^-)$ and $L(\Lambda)$. More explicitly, let us assume that μ is a special primitive weight associated with $L(\Lambda)$. Then, by the definition, $\mathbb{C}(\mu)$ is an irreducible $\mathfrak{h}$-module component of $H_p(\mathfrak{n}^-, L(\Lambda))$. So, we see from Theorem 3.1 that μ is a weight of $\wedge^p(\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda)$. Thus, we can decompose the weight μ as

$$\mu = \nu + \theta \tag{4.9}$$

for $\nu \in P(\wedge^p(\mathfrak{n}^-))$ and $\theta \in P(L(\Lambda))$.

The following lemma is an important ingredient in our main results.

Lemma 4.5. *Suppose that* Λ *is a dominant integral weight of the BKM Lie algebra* $\mathfrak{g}$. *Let* μ *be a special primitive weight associated with* $L(\Lambda)$, *and let* w *be the unique element of* W *satisfying* $w(\mu + \rho) \in P_+$. (*The existence and uniqueness of the element* w *are guaranteed by Proposition* 4.1 *and Theorem* 4.3.) *Then we have* $w(\mu + \rho) - \rho = \Lambda - (\alpha_{i_1} + \cdots + \alpha_{i_k})$, *where* $\{i_1, ..., i_k\} \subset I^{im}$, $(\alpha_{i_s}, \alpha_{i_t}) = 0$ *for* $i_s \neq i_t$ *and* $(\Lambda | \alpha_{i_s}) = 0$ *for all* $s = 1, ..., k$.

Proof. Set $\lambda + \rho = w(\mu + \rho)$. Then we get from (4.2) and (4.3) that $|\lambda + \rho|^2 = |\Lambda + \rho|^2$ and $\lambda + \rho \leq \Lambda + \rho$, respectively. Thus, by applying

[11, Lemma 2.7.4] to λ, we obtain that $\lambda = \Lambda - \sum_{s=1}^{k} m_{i_s}\alpha_{i_s}$, where $\{i_1, ..., i_k\} \subset I^{im}$, $(\alpha_{i_s}, \alpha_{i_t}) = 0$ for $i_s \neq i_t$, $(\Lambda | \alpha_{i_s}) = 0$ for all $s = 1, ..., k$ and $m_{i_s} = 1$ for i_s satisfying $(\alpha_{i_s} | \alpha_{i_s}) < 0$. However, we will show that $m_{i_s} = 1$ for all $s = 1, ..., k$.

Consider the irreducible $U(\mathfrak{g})$-module $L(\Lambda)$ with highest weight vector v_Λ of the highest weight Λ. Since we have $(\Lambda | \alpha_{i_s}) = 0$ for $s = 1, ..., k$, we see that $f_{i_s}^{m_s} v_\Lambda$ cannot appear in $L(\Lambda)$ for all s and $m_s \in \mathbb{Z}_{\geq 1}$. (Otherwise, $U(\mathfrak{g}) \cdot f_{i_s}^{m_s} v_\Lambda$ yields a non-trivial $U(\mathfrak{g})$-submodule of $L(\Lambda)$ because $(\Lambda | \alpha_{i_s})$ $= 0$.)

Hence, $\Lambda - m_s\alpha_{i_s}$ is not a weight of $L(\Lambda)$. Similarly, we obtain that

$$\Lambda - m_1\alpha_{j_1} - \cdots - m_l\alpha_{j_l} \text{ is not a weight of } L(\Lambda) \tag{4.10}$$

for $\{j_1, ..., j_l\} \subset \{i_1, ..., i_k\}$ and $(m_1, ..., m_l) \in \mathbb{Z}_{\geq 0}^{l} - \{(0, ..., 0)\}$. This is due to the conditions $(\alpha_{i_s}, \alpha_{i_t}) = 0$ for $i_s \neq i_t$ and $(\Lambda | \alpha_{i_s}) = 0$ for all s.

In addition, both Lemma 2.1 and the condition $(\alpha_{i_s}, \alpha_{i_t}) = 0$ for $i_s \neq i_t$ yield that

$$\text{any sum of roots in } \{\alpha_{i_1}, ..., \alpha_{i_k}\} \text{ cannot be a root.} \tag{4.11}$$

We also recall from (4.1) that

$$ch(L(\Lambda)) \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{mult(\alpha)} = \sum_{\tau \leq \Lambda, |\tau + \rho|^2 = |\Lambda + \rho|^2} c(\tau)e^{\tau}. \tag{4.12}$$

We note that $c(\mu) \neq 0$ because μ is a special primitive weight. Thus, it follows from $\lambda + \rho = w(\mu + \rho)$ that $c(\lambda) = (-1)^{l(w)}c(\mu) \neq 0$. So, the coefficient $c(\lambda)$ appears on the right hand side of (4.12). We already show that λ is of the form $\lambda = \Lambda - m_{i_1}\alpha_{i_1} - \cdots - m_{i_k}\alpha_{i_k}$. However, on the left hand side of (4.12), the term $e^{\Lambda - m_{i_1}\alpha_{i_1} - \cdots - m_{i_k}\alpha_{i_k}}$ only can be obtained from the product $e^{\Lambda}(1 - e^{-\alpha_{i_1}}) \cdots (1 - e^{-\alpha_{i_k}})$ because of the facts (4.10) and (4.11). (Here, we point out that $mult(\alpha_{i_s}) = 1$.) Hence, the weight $\Lambda - m_{i_1}\alpha_{i_1} - \cdots - m_{i_k}\alpha_{i_k}$ should be of the form $\Lambda - \alpha_{i_1} - \cdots - \alpha_{i_k}$. The lemma now follows. □

In Lemma 4.5, we should notice that the imaginary simple roots $\alpha_{i_1}, \ldots, \alpha_{i_k}$ are uniquely determined by the elements μ, ρ, w and Λ.

By using the following lemma, we can express the weights ν and θ appearing on (4.9) in terms of the highest weight Λ and imaginary simple roots.

Lemma 4.6. *Let* ν *and* θ *be the weights of* $P(\wedge(\mathfrak{n}^-))$ *and* $L(\Lambda)$, *respectively, that are defined in* (4.9), *and let* $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ *and* w *be as in Lemma* 4.5. *Then we have*

(1) $\theta = w^{-1}\Lambda$.

(2) $\rho - w(\rho + \nu) = \alpha_{i_1} + \cdots + \alpha_{i_k}$.

Proof. We first notice from (4.9) and Lemma 4.5 that $w(\nu + \theta + \rho) = \Lambda + \rho - (\alpha_{i_1} + \cdots + \alpha_{i_k})$. This yields that

$$\rho - w(\rho + \nu) = w\theta - \Lambda + (\alpha_{i_1} + \cdots + \alpha_{i_k}). \tag{4.13}$$

On the other hand, for $i \in I^{re}$ and $n(\alpha) \in \mathbb{Z}_{\geq 0}$, we have

$$r_{\alpha_i}\left(\rho - \sum_{\alpha \in \Delta_+} n(\alpha)\alpha\right) = \rho - \alpha_i - \sum_{\alpha \in \Delta_+ - \{\alpha_i\}} n(\alpha)\alpha + n(\alpha_i)\alpha_i \tag{4.14}$$

because $\rho(\alpha_i^\vee) = 1$ for $i \in I^{re}$ and $r_{\alpha_i}(\alpha_i) = -\alpha_i$.

If $-\sum_{\alpha\in\Delta_+} n(\alpha)\alpha \in P(\wedge(\mathfrak{n}^-))$, then we immediately see that $n(\alpha_i) \leq mult(\alpha_i) = 1$. Thus, equation (4.14) implies that $\rho + P(\wedge(\mathfrak{n}^-))$ is W-invariant. This gives that $\rho - w(\rho+\nu) \in -P(\wedge(\mathfrak{n}^-))$. Hence, we get that

$$\rho - w(\rho+\nu) \in \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i. \tag{4.15}$$

However, we obtain from $w\theta \leq \Lambda$ that

$$w\theta - \Lambda = \sum_{i\in I} \mathbb{Z}_{\leq 0}\alpha_i. \tag{4.16}$$

Since $\{\alpha_{i_1}, ..., \alpha_{i_k}\}$ is a linearly independent subset of $\mathfrak{h}^*$, the conditions (4.13), (4.15) and (4.16) yield that

$$w\theta - \Lambda = -(\alpha_{j_1} + \cdots + \alpha_{j_m}) \text{ for some } \{\alpha_{j_1}, ..., \alpha_{j_m}\} \subset \{\alpha_{i_1}, ..., \alpha_{i_k}\}$$

and

$$\rho - w(\rho+\nu) = \alpha_{l_1} + \cdots + \alpha_{l_n},$$

where $\{l_1, ..., l_n\} \sqcup \{j_1, ..., j_m\} = \{i_1, ..., i_k\}$.

Thus, we have $w\theta = \Lambda - (\alpha_{j_1} + \cdots + \alpha_{j_m})$, and this implies that $\Lambda - (\alpha_{j_1} + \cdots + \alpha_{j_m})$ is a weight of $L(\Lambda)$. But, we see from (4.10) that $\Lambda - \alpha_{j_1} - \cdots - \alpha_{j_m}$ cannot be a weight of $L(\Lambda)$ if $m \neq 0$. Hence, we should have $m = 0$. The lemma now follows. □

We recall from Proposition 2.2 that $w^{-1}\rho - \rho = -(\phi_1 + \cdots + \phi_{l(w)})$, and $-(\phi_1 + \cdots + \phi_{l(w)})$ is a weight of $e_{-\phi_1} \wedge \cdots \wedge e_{-\phi_{l(w)}}$. Here $\Phi_{w^{-1}} = \{\phi_1, ..., \phi_{l(w)}\}$ and $e_{-\phi_i}$ is a root vector corresponding to the negative root $-\phi_i$. (Each $e_{-\phi_i}$ is unique up to constant factor because $-\phi_i$ is a real root.) In addition, we obtain from Lemma 4.6 that $\nu = (w^{-1}\rho - \rho) + w^{-1}(-\alpha_{i_1} - \cdots - \alpha_{i_k})$.

Let us now take a nonzero vector $e_{w^{-1}(-\alpha_{i_s})}$ in each $\mathfrak{g}_{w^{-1}(-\alpha_{i_s})}$. (This is possible because we have $\dim \mathfrak{g}_{w^{-1}(-\alpha_{i_s})} = 1$ for each j.) Then it is clear that $e_{w^{-1}(-\alpha_{i_1})} \wedge \cdots \wedge e_{w^{-1}(-\alpha_{i_k})}$ is a nonzero weight vector with the weight $w^{-1}(-\alpha_{i_1} - \cdots - \alpha_{i_k})$.

The following theorem is the main result of this paper.

Theorem 4.7. *Assume that* Λ *is a dominant integral weight of the BKM Lie algebra* $\mathfrak{g}$. *Let* μ *be a special primitive weight associated with* $L(\Lambda)$ *which is decomposed as* (4.9). *We also let* w *and* $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ *be as in Lemma* 4.5. *Then the weight space of* $\wedge(\mathfrak{n}^-)$ *corresponding to the weight* $\nu = (w^{-1}\rho - \rho) + w^{-1}(-\alpha_{i_1} - \cdots - \alpha_{i_k})$ *is one dimensional space and is spanned by*

$$e_{-\phi_1} \wedge \cdots \wedge e_{-\phi_{l(w)}} \wedge e_{w^{-1}(-\alpha_{i_1})} \wedge \cdots \wedge e_{w^{-1}(-\alpha_{i_k})} \in \wedge^{l(w)+k}(\mathfrak{n}^-),$$

where $\Phi_{w^{-1}} = \{\phi_1, \ldots, \phi_{l(w)}\}$.

Proof. It is obvious that the weight of

$$e_{-\phi_1} \wedge \cdots \wedge e_{-\phi_{l(w)}} \wedge e_{w^{-1}(-\alpha_{i_1})} \wedge \cdots \wedge e_{w^{-1}(-\alpha_{i_k})}$$

is $(w^{-1}\rho - \rho) + w^{-1}(-\alpha_{i_1} - \cdots - \alpha_{i_k})$.

Suppose that $0 \neq e_{-\beta_1} \wedge \cdots \wedge e_{-\beta_l} \in \wedge(\mathfrak{n}^-)$ satisfies

$$-\beta_1 - \cdots - \beta_l = (w^{-1}\rho - \rho) + w^{-1}(-\alpha_{i_1} - \cdots - \alpha_{i_k}), \tag{4.17}$$

where $\{\beta_1, \ldots, \beta_l\} \subset \Delta_+$. Then equation (4.17) yields that

$$-\beta_1 - \cdots - \beta_l = -\sum_{\phi \in \Phi_{w^{-1}}} \phi + w^{-1}(-\alpha_{i_1} - \cdots - \alpha_{i_k}). \tag{4.18}$$

If $w\beta_i \preceq 0$ for some i, then by the definition of $\Phi_{w^{-1}}$, we have $\beta_i \in \Phi_{w^{-1}}$, and hence β_i is a real root. So, we get that $mult(\beta_i) = 1$. This implies that $\beta_i \neq \beta_j$ for all $j = 1, \ldots, i-1, i+1, \ldots, l$ because we assume that $e_{-\beta_1} \wedge \cdots \wedge e_{-\beta_l} \neq 0$.

We now cancel the term $-\beta_i$ from the both sides of (4.18). (On the right hand side of (4.18), we omit the $-\beta_i$ from the term $-\sum_{\phi \in \Phi_{w^{-1}}} \phi$.)

Continuing this way, equation (4.18) gives

$$-\beta_{m_1} - \cdots - \beta_{m_p} = -\sum_{\psi \in Q} \psi + w^{-1}(-\alpha_{i_1} - \cdots - \alpha_{i_k}) \tag{4.19}$$

for some $\{m_1, \ldots, m_p\} \subset \{1, 2, \ldots, l\}$ and $Q \subset \Phi_{w^{-1}}$.

In (4.19), we should notice that $w\beta_{m_s} \succeq 0$ for all s. We also obtain from (4.19) that

$$w\beta_{m_1} + \cdots + w\beta_{m_p} - \alpha_{i_1} - \cdots - \alpha_{i_k} = \sum_{\psi \in Q} w\psi \preceq 0 \tag{4.20}$$

because $w\psi \preceq 0$ for all $\psi \in Q$.

Recall from (4.11) that any sum of the roots $\alpha_{i_1}, \ldots, \alpha_{i_k}$ cannot be a root. However, each $w\beta_{m_s}$ is a positive root. Thus, equation (4.20) implies that each term $w\beta_{m_s}$ satisfies

$$w\beta_{m_s} = \alpha_{i_s} \text{ for some } s. \tag{4.21}$$

By summarizing our argument so far, each β_i in (4.17) satisfies

$$\text{either } \beta_i \in \Phi_{w^{-1}} \text{ or } \beta_i = w^{-1}(\alpha_{i_s}) \text{ for some } s. \tag{4.22}$$

On the other hand, we get from equation (4.18) that

$$-\beta_1 - \cdots - \beta_l + w^{-1}(\alpha_{i_s}) + \cdots + w^{-1}(\alpha_{i_k}) + \sum_{\phi \in \Phi_{w^{-1}}} \phi = 0. \qquad (4.23)$$

Recall that $w'(\Delta_+^{im}) = \Delta_+^{im}$ for any $w' \in W$, where Δ_+^{im} indicates the set of the positive imaginary roots (see [11, Section 2.2]). In addition, $\sum_{\phi \in \Phi_{w^{-1}}} \phi$ is a sum of positive roots.

Thus, by combining (4.22) and (4.23), we obtain that $l = k + |\Phi_{w^{-1}}|$. This yields that

$$\{\beta_1, \ldots, \beta_l\} = \{w^{-1}(\alpha_{i_1}), \ldots, w^{-1}(\alpha_{i_k})\} \sqcup \Phi_{w^{-1}}$$

$$= \{w^{-1}(\alpha_{i_1}), \ldots, w^{-1}(\alpha_{i_k})\} \sqcup \{\phi_1, \ldots, \phi_{l(w)}\}.$$

The theorem now follows because $\dim \mathfrak{g}_{-\phi_p} = \dim \mathfrak{g}_{w^{-1}(-\alpha_{i_s})} = 1$ for all $1 \leq p \leq l(w)$ and $1 \leq s \leq k$. □

As a by-product of our results, we now answer to Question 1.2.

Theorem 4.8. *Let* $\Lambda \in P_{(+)}$, *and let* $L(\Lambda)$ *be the irreducible highest weight* $\mathfrak{g}$*-module with highest weight* Λ. *Let* μ *be a special primitive weight associated with* $L(\Lambda)$, *and let us decompose* μ *as* $\mu = \nu + \theta$ *for* $\nu \in P(\wedge(\mathfrak{n}^-))$ *and* $\theta \in P(L(\Lambda))$. *Then we have*:

(1) *There exists uniquely an element* $w \in W$ *such that* $w(\mu + \rho) \in P_+$.

(2) $\theta = w^{-1}\Lambda$ *and* $\nu = (w^{-1}\rho - \rho) + w^{-1}(-\alpha_{i_1} - \cdots - \alpha_{i_k})$, *where* α_{i_1}, ..., α_{i_k} *are uniquely determined imaginary simple roots when we express* $\Lambda - (w(\mu + \rho) - \rho)$ *as a sum of simple roots.*

(3) *The irreducible component* $\mathbb{C}(\mu)$ *only appears in the Lie algebra homology* $H_p(\mathfrak{n}^-, L(\Lambda))$ *at the degree* $p = l(w) + k$.

Proof. The statements (1) and (2) are immediate from Theorem 4.3, Lemma 4.5 and Lemma 4.6.

For the statement (3), we obtain from Theorem 4.7 that μ is a weight of $\wedge(\mathfrak{n}^-) \otimes_{\mathbb{C}} L(\Lambda)$ with multiplicity 1. In addition, the corresponding weight vector of μ is

$$e_{-\phi_1} \wedge \cdots \wedge e_{-\phi_{l(w)}} \wedge e_{w^{-1}(-\alpha_{i_1})} \wedge \cdots \wedge e_{w^{-1}(-\alpha_{i_k})} \otimes x_{w^{-1}\Lambda},$$

where $x_{w^{-1}\Lambda}$ is a nonzero weight vector in $L(\Lambda)_{w^{-1}\Lambda}$.

(Notice that $\dim L(\Lambda)_{w^{-1}\Lambda} = \dim L(\Lambda)_\Lambda = 1$ because $L(\Lambda)$ is integrable.)

The theorem now follows. □

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