Far East Journal of Applied Mathematics
Volume 79, Number 2, 2013, Pages 73-110
Published Online: September 2013
Available online at http://pphmj.com/journals/fjam.htm
Published by Pushpa Publishing House, Allahabad, INDIA

STRONG STABILITY OF SYMPLECTIC MATRICES USING A SPECTRAL DICHOTOMY METHOD

Mouhamadou Dosso[1], Namory Coulibaly[2] and Lassana Samassi[1]

[1]Laboratoire de Mathématiques Appliquées
UFR de Mathématiques et Informatique
Université Félix Houphouet-Boigny de Cocody-Abidjan
22 BP 582 Abidjan 22, Côte d'Ivoire
e-mail: mouhamadoudoss@yahoo.fr
samassilassana@yahoo.fr

[2]Laboratoire de Mécanique
UFR de Mathématique
Université Félix Houphouet-Boigny de Cocody-Abidjan
22 BP 582 Abidjan 22, Côte d'Ivoire
e-mail: coulnamson@hotmail.com

Abstract

A method of the analysis of strong stability of symplectic matrix is presented. This method is based on spectral dichotomy methods of a matrix with respect to a circle whose new variant has been proposed. Applying the original idea proposed by S. K. Godunov, an algorithm of the analysis of strong stability is proposed. Numerical examples are presented to confirm the theory.

2010 Mathematics Subject Classification: 15A21, 15A57, 65F10, 65F15.

Keywords and phrases: symplectic matrix, spectral dichotomy, spectral projector, spectral portrait, strong stability.

Received May 18, 2013

1. Introduction

Let $J \in \mathbb{R}^{2N \times 2N}$ be a skew-symmetric and nonsingular matrix. A matrix $W \in \mathbb{R}^{2N \times 2N}$ is J-symplectic if $W^T JW = J;$ we say also that W is J-orthogonal. The symplectic matrices belong to the group of structured matrices. They are important in optimal control theory [13, 15, 2] and in theory of parametric resonance (see, e.g., [21]). The eigenvalues of W can be classified in three groups with respect to the unit circle: a group of N_0 eigenvalues inside the unit circle, another of $N_\infty = N_0$ eigenvalues outside the unit circle and the third group of $2N_1 = 2(N - N_0)$ eigenvalues on the unit circle. For any eigenvalue λ of W, $\bar{\lambda}$, $1/\lambda$ and $1/\bar{\lambda}$ are eigenvalues of W.

A symplectic matrix is strongly stable if and only if it verifies the KGL criterion [21, 9] or, equivalently, its spectrum is on the unit circle, and is uniquely composed of red and/or green eigenvalues [4, 9]. The KGL criterion is due to Krein, Gelfand and Lidskii [21]. This criterion is based on the following definition which classifies the eigenvalues of W lying on the unit circle in three groups.

Definition 1.1. Let λ be a semisimple eigenvalue of W lying on the unit circle. Then λ is called an *eigenvalue of the first* (*second*) *kind* if the quadratic form (iJx, x) is positive (negative) on the eigenspace associated with λ. When $(Jx, x) = 0,$ then λ is of mixed kind.

In this definition, the notation (iJx, x) is the Euclidean scalar product and $i = \sqrt{-1}$. Recall that ± 1 are mixed eigenvalues, since for any eigenvector x associated with 1 or -1, we have $(Jx, x) = 0.$

Thus, the KGL criterion says that a symplectic matrix is strongly stable if and only if all its eigenvalues are either of first kind or second kind (see, e.g., [21, Chap. III], [12]). Another classification of eigenvalues of modulus 1 of symplectic matrix is given by the following definition.

Definition 1.2. Let λ be a semisimple eigenvalue of W. Then λ is a red (green) color or in short r-eigenvalues (g-eigenvalue) if $(S_0 x, x)$ is positive (negative) on the eigenspace associated with λ, where $S_0 = (1/2) J(W - W^{-1}) \equiv (JW + (JW)^T)$.

The latter classification given by Definition 1.2 is more appropriate for numerical calculations than Definition 1.1 because it uses symmetric matrices. The link between Definitions 1.1 and 1.2 is that if (λ, x) is an eigenvalue of W with $\lambda = e^{i\theta}$, then we have $(S_0 x, x) = \sin\theta (iJx, x)$. On the other hand, if $\lambda = e^{i\theta}$, with $0 < \theta < \pi$, is an eigenvalue of first kind, then $\overline{\lambda} = e^{-i\theta}$ is an eigenvalue of second kind associated with an eigenvector $\overline{x}$ and we have, for any linear combination $z = \alpha x + \beta \overline{x}$,

$$\begin{aligned}(S_0 z, z) &= |\alpha|^2 (S_0 x, x) + |\beta|^2 (S_0 \overline{x}, \overline{x}) \\ &= \sin\theta [|\alpha|^2 (iJx, x) - |\beta|^2 (iJ\overline{x}, \overline{x})] > 0.\end{aligned}$$

Thus, the second definition gives another formulation of the strong stability of a symplectic matrix suitable for numerical calculations. Indeed the KGL criterion is equivalent to the fact that all the eigenvalues of W must be either of red or green color (see [11, 4]).

The objective of this paper is to construct an algorithm using the spectral dichotomy method to analyze the strong stability of a symplectic matrix W. The method of spectral dichotomy was introduced by Godunov [6]. It allows to calculate the spectral projectors associated with eigenvalues on, inside and outside a contour γ of the complex plane. The computation of projectors is accompanied by that of the norms of Hermitian and positive definite matrices, called *criterion dichotomies* (or *numerical quality*). These norms allow to know if projectors are well computed. If these criteria are small, then the numerical quality of the projectors is the best. In this study devoted to the analysis of the strong stability of symplectic matrices, we consider the case where γ is a circle.

In Section 2, we give the more important steps of the method of spectral dichotomy of a matrix with respect to the unit circle proposed by Godunov and Sadkane [10] and estimate the cost of the elementary operations neglecting the lower strict terms to 3. In Section 3, we present another variant of spectral dichotomy method with respect to the unit circle which is four times less expensive. These two methods are stable and allow a good computation of projectors. Their numerical qualities are good. Section 4 is devoted to the application of the spectral dichotomy in the analysis of the strong stability of symplectic matrix. Using a Cayley transformation (see [5, 20]),

$$A = \mathcal{C}(W) = (W - I_{2N})(W + I_{2N})^{-1},$$

where I_{2N} is an identity matrix of order $2N$ and the symmetric matrix S_0, we determine the projectors $\mathbb{P}_r$ and $\mathbb{P}_g$ associated with r-eigenvalues and g-eigenvalues, respectively. In Section 5, we present numerical examples in which our methods are checked.

2. Spectral Dichotomy of a Matrix with Respect to a Unit Circle

Consider a matrix $A \in \mathbb{R}^{N \times N}$ such that A has not eigenvalues on the unit circle.

2.1. Notation and preliminaries

Suppose that $r = 1.$ The spectral projector on the invariant subspace associated to interior eigenvalues to $C(0, 1)$ of the matrix A is given by (see [16, p. 39]),

$$\mathbb{P} = \frac{1}{2i\pi}\int_{C(0,1)} (zI_N - A)^{-1} dz = \frac{1}{2\pi}\int_0^{2\pi} (I_N - e^{-i\theta}A)^{-1} d\theta, \qquad (1)$$

where I_N is the identity matrix of order N or just I when the order is clear from context.

The numerical computation of this projector is accompanied with a

Hermitian positive definite matrix $\mathbb{H}$ which is defined by

$$\mathbb{H} = \frac{1}{2\pi}\int_0^{2\pi}(I_N - e^{-i\theta}A)^{-*}(I_N - e^{-i\theta}A)^{-1}d\theta. \tag{2}$$

The norm of $\mathbb{H}$ is called *criterion of dichotomy*. It is a reliable indicator. However, we note that the pair $(\mathbb{P}, \mathbb{H})$ is the unique solution of the generalized Lyapunov system [8],

$$\begin{cases} \mathbb{H} - A^*\mathbb{H}A = \mathbb{P}^*\mathbb{P} - (I - \mathbb{P})^*(I - \mathbb{P}) \\ \mathbb{H} = \mathbb{H}^* > 0 \\ \mathbb{P}A = A\mathbb{P} \\ \mathbb{P}^2 = \mathbb{P} \\ \mathbb{H}\mathbb{P} = (\mathbb{H}\mathbb{P})^*. \end{cases} \tag{3}$$

Consider the following decomposition of the matrix A:

$$A = X\begin{pmatrix} \Gamma_\infty & \\ & \Gamma_0 \end{pmatrix}X^{-1}, \tag{4}$$

where $\Gamma_0 \in \mathbb{C}^{p\times p}$, $\Gamma_\infty \in \mathbb{C}^{(N-p)\times(N-p)}$ and the spectrum $\Lambda(A)$ of A is partitioned as $\Lambda(A) = \Lambda(\Gamma_0) \cup \Lambda(\Gamma_\infty)$ with

$$\Lambda(\Gamma_\infty) = \{\lambda \in \text{ such that } |\lambda| > 1\},$$

$$\Lambda(\Gamma_0) = \{\lambda \in \text{ such that } |\lambda| < 1\}.$$

The spectral projectors $\mathbb{P}$ and $I - \mathbb{P}$ onto the invariant subspaces of A associated to the eigenvalues of matrices Γ_0 and Γ_∞ are, respectively, given by

$$\mathbb{P} = X\begin{pmatrix} 0_{(N-p)} & 0 \\ 0 & I_p \end{pmatrix}X^{-1} \text{ and } I - \mathbb{P} = X\begin{pmatrix} I_{(N-p)} & 0 \\ 0 & 0_p \end{pmatrix}X^{-1}, \tag{5}$$

where $0_{(N-p)}$ is a null matrix of order $(N - p)$ or just 0 when the order is clear from context.

Throughout this paper, the symbol $\|\ \|$ denotes the Euclidean norm or its induced norm and $\kappa_2(X) = \| X \| \cdot \| X^{-1} \| = \sigma_N(X)/\sigma_1(X)$ denotes the condition number of a nonsingular matrix *X*, where $\sigma_N(X)$ and $\sigma_1(X)$ are, respectively, the largest and smallest singular values.

2.2. Presentation of the method

A method of computation of approximations of $\mathbb{P}$ and $\mathbb{H}$ has been shown [10, Sec. 2] a method. We recall the most important steps. Considering the decomposition in Fourier series

$$(I - e^{-i\theta}A)^{-1} = \sum_{k=-\infty}^{+\infty} Z_k e^{ik\theta}, \quad \text{where} \quad Z_k = \frac{1}{2\pi}\int_0^{2\pi} (I - e^{i\theta}A)^{-1} e^{ik\theta} d\theta$$

of the 2π-periodic function $\theta \to (I - e^{i\theta}A)^{-1}$, we get the projector $\mathbb{P} = Z_0$, the matrix $\mathbb{H} = \sum_{k=-\infty}^{+\infty} Z_k^* Z_k$ and the infinity system

$$\begin{cases} Z_k - AZ_{k+1} = 0 \text{ if } k \neq 0, \\ Z_0 - AZ_1 = I_N, \end{cases} \tag{6}$$

where $(Z_k)_{k\in\mathbb{Z}}$ are unknown. In [18], it is shown that the Fourier coefficients Z_k converge to zero when $|k|$ converges to infinity. Putting

$$Z_k^{(2^{j+1})} = \sum_{l=-\infty}^{+\infty} Z_{k+l2^{j+1}}, \quad k \in \mathbb{Z}, \tag{7}$$

where *j* is a given integer, the sequence $(Z_k^{(2^{j+1})})_{k\in\mathbb{Z}}$, is 2^{j+1}-periodic and satisfies $\lim_{j\to+\infty} Z_k^{(2^{j+1})} = Z_k$, $\forall k \in \mathbb{Z}$ (see [10]). From (6) and (7), we obtained the finished linear (and cyclic) system

$$\begin{cases} Z_0^{(2^{j+1})} - AZ_1^{(2^{j+1})} = I_N, \\ Z_k^{(2^{j+1})} - AZ_{k+1}^{(2^{j+1})} = 0, \quad 1 \le k \le 2^{j+1} - 1 \end{cases} \tag{8}$$

of unknowns $Z_0^{(2^{j+1})}, Z_1^{(2^{j+1})}, \ldots, Z_{2^{j+1}}^{(2^{j+1})} = Z_0^{(2^{j+1})}$ when j is great. This system has a unique solution if and only if the matrix A has not eigenvalues on the circle $\mathcal{C}(0, 1)$. Then $\mathbb{P}$ and $\mathbb{H}$ can be approximated iteratively with the following theorem (see [10] for the proof).

Theorem 2.1. *Put* $B_0 = I$, $A_0 = -A$ *and*

$$\Delta_j = B_0 Z_{2^j}^{(2^{j+1})} + A_0 Z_1^{(2^{j+1})},$$

$$\nabla_j = B_0 Z_{2^{j+1}}^{(2^{j+1})} + A_0 Z_{2^j+1}^{(2^{j+1})},$$

$$H_j = \sum_{k=1}^{2^{j+1}} (Z_k^{(2^{j+1})})^* Z_k^{(2^{j+1})}.$$

Then

$$H_j = \Delta_j^* H_{j-1} \Delta_j + \nabla_j^* H_{j-1} \nabla_j.$$

Moreover,

$$\lim_{j \to \infty} Z_0^{(2^{j+1})} = \lim_{j \to \infty} Z_{2^{j+1}}^{(2^{j+1})} = \mathbb{P},$$

$$\lim_{j \to \infty} Z_1^{(2^{j+1})} = \mathbb{P} - I,$$

$$\lim_{j \to \infty} H_j = \mathbb{H}.$$

This theorem shows, in particular, that

- the approximation of $\mathbb{P}$ requires only the computation of $Z_0^{(2^{j+1})} = Z_{2^{j+1}}^{(2^{j+1})}$ for j great,

- the approximation of $\mathbb{H}$ requires only the computation of $Z_1^{(2^{j+1})}$, $Z_{2^j}^{(2^{j+1})}$, $Z_{2^j+1}^{(2^{j+1})}$ and $Z_{2^{j+1}}^{(2^{j+1})}$ for j great.

The computation of the matrices $Z_1^{(2^{j+1})}$, $Z_{2^j}^{(2^{j+1})}$, $Z_{2^j+1}^{(2^{j+1})}$ and $Z_{2^{j+1}}^{(2^{j+1})}$ in [10] is done with the following system:

$$\begin{pmatrix} B_0 & A_0 & 0 & 0 \\ A_j & 0 & B_j & 0 \\ 0 & B_j & 0 & A_j \\ 0 & 0 & A_0 & B_0 \end{pmatrix} \begin{pmatrix} Z_{2^j}^{(2^{j+1})} \\ Z_{2^j+1}^{(2^{j+1})} \\ Z_1^{(2^{j+1})} \\ Z_{2^{j+1}}^{(2^{j+1})} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ I \end{pmatrix}, \tag{9}$$

where the matrices A_j and B_j are constructed iteratively from the B_0 and A_0 using a QR-factorization obtained by transformations of Householder in the form

$$\begin{pmatrix} B_0 & A_0 & 0 & 0 \\ A_{j-1} & 0 & B_{j-1} & 0 \\ 0 & B_{j-1} & 0 & A_{j-1} \end{pmatrix}$$

$$= Q^{(j-1)} \begin{pmatrix} R_{11}^{(j-1)} & R_{12}^{(j-1)} & R_{13}^{(j-1)} & R_{14}^{(j-1)} \\ 0 & R_{22}^{(j-1)} & R_{23}^{(j-1)} & R_{24}^{(j-1)} \\ 0 & 0 & B_j & A_j \end{pmatrix}, \tag{10}$$

where $Q^{(j-1)}$ is a unitary matrix. The important steps of the method proposed by Godunov and Sadkane are summarized below (see [10, Algorithm 1]).

Algorithm 2.1. (Spectral dichotomy of A by the unit circle)

1. **Initialization:** $B_0 = I$, $A_0 = -A$:

- Compute $Z_1^{(2)}$ and $Z_2^{(2)}$ solutions of the system

$$\begin{pmatrix} B_0 & A_0 \\ A_0 & B_0 \end{pmatrix} \begin{pmatrix} Z_1^{(2)} \\ Z_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

- Compute $H_0 = (Z_1^{(2)})^* Z_1^{(2)} + (Z_2^{(2)})^* Z_2^{(2)}$.

2. **Iteration:** Computation of $Z_1^{(2^{j+1})}$, $Z_{2^j}^{(2^{j+1})}$, $Z_{2^j+1}^{(2^{j+1})}$, $Z_{2^{j+1}}^{(2^{j+1})}$ and H_j for $j = 1, 2, \ldots$.

- Compute the QR-factorization

$$\begin{pmatrix} B_0 & A_0 & 0 & 0 \\ A_{j-1} & 0 & B_{j-1} & 0 \\ 0 & B_{j-1} & 0 & A_{j-1} \end{pmatrix}$$

$$= Q^{(j-1)} \begin{pmatrix} R_{11}^{(j-1)} & R_{12}^{(j-1)} & R_{13}^{(j-1)} & R_{14}^{(j-1)} \\ 0 & R_{22}^{(j-1)} & R_{23}^{(j-1)} & R_{24}^{(j-1)} \\ 0 & 0 & B_j & A_j \end{pmatrix}.$$

- Compute the solutions $Z_1^{(2^{j+1})}$ and $Z_{2^{j+1}}^{(2^{j+1})}$ of the system

$$\begin{pmatrix} B_j & A_j \\ A_0 & B_0 \end{pmatrix} \begin{pmatrix} Z_1^{(2^{j+1})} \\ Z_{2^{j+1}}^{(2^{j+1})} \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

- Compute the solutions $Z_{2^j}^{(2^{j+1})}$ and $Z_{2^j+1}^{(2^{j+1})}$ of the system

$$\begin{pmatrix} R_{11}^{(j-1)} & R_{12}^{(j-1)} \\ 0 & R_{22}^{(j-1)} \end{pmatrix} \begin{pmatrix} Z_{2^j}^{(2^{j+1})} \\ Z_{2^j+1}^{(2^{j+1})} \end{pmatrix} = -\begin{pmatrix} R_{13}^{(j-1)} & R_{14}^{(j-1)} \\ R_{23}^{(j-1)} & R_{24}^{(j-1)} \end{pmatrix} \begin{pmatrix} Z_1^{(2^{j+1})} \\ Z_{2^j+1}^{(2^{j+1})} \end{pmatrix}.$$

- Compute

$$\Delta_j = B_0 Z_{2^j}^{(2^{j+1})} + A_0 Z_1^{(2^{j+1})},$$

$$\nabla_j = B_0 Z_{2^{j+1}}^{(2^{j+1})} + A_0 Z_{2^j+1}^{(2^{j+1})},$$

$$H_j = \Delta_j^* H_{j-1} \Delta_j + \nabla_j^* H_{j-1} \nabla_j.$$

2.3. Operations cost of Algorithm 2.1

We now estimate the number of elementary operations in Algorithm 2.1. In these estimates, we consider only the most important terms, i.e., terms in N^3.

- **Initialization:** To compute $Z_1^{(2)}$ and $Z_2^{(2)}$, we solve the linear systems block $(I - A_0^2) Z_2^{(2)} = I$ and $Z_1^{(2)} = -AZ_2^{(2)}$. The first requires $2N^3$ operations (computation of A_0^2), and $(2/3)N^3$ operations (Gauss's factorization of $(I - A_0^2)$), and $2N^3$ operations (resolution of two triangular systems block). The computations of $Z_1^{(2)}$ and H_0 require, respectively, $2N^3 + 2N^3 = 4N^3$ operations.
- **Iteration** ***j*****:**
  - The *QR*-factorization by Householder's transformation requires $54N^3$ operations [14].
  - The computation of $Z_1^{(2^{j+1})}$ and $Z_{2^{j+1}}^{(2^{j+1})}$ requires $(31/3)N^3$ operations: $(2/3)(2N)^3$ for the *LU* factorization of $\begin{pmatrix} B_j & A_j \\ A_0 & B_0 \end{pmatrix}$, followed by resolution of two block-triangular systems, and considering the specificity of the second member $\begin{pmatrix} 0 \\ I \end{pmatrix}$, these systems require N^3 for *L* and $4N^3$ for *U*.

- The computation of $Z_{2^j}^{(2^{j+1})}$ and $Z_{2^{j+1}}^{(2^{j+1})}$ requires $12N^3 : 8N^3$ to compute $\begin{pmatrix} R_{13}^{(j-1)} & R_{14}^{(j-1)} \\ R_{23}^{(j-1)} & R_{24}^{(j-1)} \end{pmatrix}\begin{pmatrix} Z_1^{(2^{j+1})} \\ Z_{2^{j+1}}^{(2^{j+1})} \end{pmatrix}$ and $4N^3$ to solve the block-triangular system of matrix $\begin{pmatrix} R_{11}^{(j-1)} & R_{12}^{(j-1)} \\ 0 & R_{22}^{(j-1)} \end{pmatrix}$.

- Finally, the computations of Δ_j, ∇_j and H_j require, respectively, $2N^3$, $2N^3$ and $4N^3$ operations.

In total, one iteration requires of the order of $253/3N^3$ operations.

The cost of the operations is summarized in Table 1.

Table 1. Cost of Algorithm 2.1

Initialization	
Compute of $Z_1^{(2)}$, $Z_2^{(2)}$, H_0	$26/3N^3$
One iteration j	
QR-factorization	$54N^3$
Computation of $Z_1^{2^{j+1}}$, $Z_{2^{j+1}}^{(2^{j+1})}$	$31/3N^3$
Computation of $Z_{2^j}^{(2^{j+1})}$, $Z_{2^j+1}^{(2^{j+1})}$	$12N^3$
Computation of Δ_j	$2N^3$
Computation of ∇_j	$2N^3$
Computation of H_j	$4N^3$
Total	$\frac{26}{3}N^3 + \frac{253}{3}jN^3$

2.4. Spectral portrait

The spectral portrait of the matrix A is the graph of the function

$$r \mapsto f(r) = \| H(r) \|, \tag{11}$$

where

$$\mathbb{H}(r) = \frac{1}{2\pi}\int_0^{2\pi}\left(I - e^{i\theta}\frac{A^*}{r}\right)^{-1}\left(I - e^{-i\theta}\frac{A}{r}\right)^{-1}d\theta. \tag{12}$$

The spectral projector on the invariant subspace of A associated to eigenvalues inside the circle $\mathcal{C}(0, r)$ is

$$\mathbb{P}(r) = \frac{1}{2i\pi}\int_{\mathcal{C}(0,r)}(zI - A)^{-1}dz = \frac{1}{2\pi}\int_0^{2\pi}\left(I - e^{-i\theta}\frac{1}{r}\right)^{-1}d\theta.$$

When A has an eigenvalue λ on the circle $\mathcal{C}(0, r)$, then $f(r) \to \infty$, i.e., the graph of f has an asymptote of equation $|\lambda| = r$. It has been shown in [8, Sec. 13] that the function f is convex in each interval where it is defined. These intervals correspond to regions of absence of eigenvalues (the trace of projector remains constant).

The spectral portrait is another perspective on the pseudo-spectrum [19] of a matrix. This allows the determination of neighborhoods defined by eigenvalues of all the perturbations of the matrix in a given region. Several techniques are used to calculate, see, e.g., [19, 17, 1]. The spectral portrait allows a spectral stratification, i.e., the construction of regions (bands in our case) that share the spectrum.

3. Another Method of Spectral Dichotomy of a Matrix with Respect to a Unit Circle

The variant of Algorithm 2.1 proposed in this section essentially removes the QR-factorization and the computation of $Z_{2^j}^{(2^{j+1})}$ and $Z_{2^{j+1}}^{(2^{j+1})}$. Thus, the new variant is justified as follows.

3.1. Analysis of the algorithm

The system (8) gives 2^{j+1} solutions of expressions

$$Z_k^{(2^{j+1})} = A^{2^{j+1}-k}[I_N - A^{2^{j+1}}]^{-1}, \quad k = 0, 1, \ldots, 2^{j+1} - 1. \tag{13}$$

Note that the analytic computation of this expression is not stable. It is preferable to compute them of iterative manner. For that, we put, for $j = 0, 1, \ldots,$

$$K_{j+1} = A^{2^j}[I_N - A^{2^j}][I_N - A^{2^{j+1}}]^{-1}, \tag{14}$$

$$L_{j+1} = [I_N - A^{2^j}][I_N - A^{2^{j+1}}]^{-1}. \tag{15}$$

The following proposition gives the link between $Z_k^{(2^{j+1})}$ and $Z_k^{(2^j)}$, and H_{j+1} and H_j.

Proposition 3.1. *For* $j = 0, 1, \ldots$ *and* $k = 0, 1, \ldots, 2^j$, *we have*

$$Z_k^{(2^{j+1})} = Z_k^{(2^j)} K_{j+1}, \tag{16}$$

$$Z_{2^j+k}^{(2^{j+1})} = Z_k^{(2^j)} L_{j+1}, \tag{17}$$

$$H_{j+1} = K_{j+1}^* H_j K_{j+1} + L_{j+1}^* H_j L_{j+1}. \tag{18}$$

Proof. We have

$$\begin{aligned} Z_k^{(2^{j+1})} &= A^{2^{j+1}-k}[I_N - A^{2^{j+1}}]^{-1} \\ &= A^{2^j-k}[I_N - A^{2^j}]^{-1} A^{2^j}[I_N - A^{2^j}][I_N - A^{2^{j+1}}]^{-1} \\ &= Z_k^{(2^j)} K_{j+1} \end{aligned} \tag{19}$$

and

$$\begin{aligned} Z_{2^j+k}^{(2^{j+1})} &= A^{2^{j+1}-(2^j+k)}[I_N - A^{2^j}]^{-1}[I_N - A^{2^j}][I_N - A^{2^{j+1}}]^{-1} \\ &= A^{2^j-k}[I_N - A^{2^j}]^{-1}[I_N - A^{2^j}][I_N - A^{2^{j+1}}]^{-1} \\ &= Z_k^{(2^j)} L_{j+1}. \end{aligned} \tag{20}$$

Moreover,

$$
\begin{aligned}
H_{j+1} &= \sum_{k=1}^{2^{j+1}} (Z_k^{(2^{j+1})})^* Z_k^{(2^{j+1})} \\
&= \sum_{k=1}^{2^j} (Z_k^{(2^{j+1})})^* Z_k^{(2^{j+1})} + \sum_{k=2^j+1}^{2^{j+1}} (Z_k^{(2^{j+1})})^* Z_k^{(2^{j+1})} \\
&= \sum_{k=1}^{2^j} K_{j+1}^* (Z_k^{(2^j)})^* Z_k^{(2^j)} K_{j+1} + \sum_{k=1}^{2^j} L_{j+1} (Z_k^{(2^j)})^* Z_k^{(2^j)} L_{j+1} \\
&= K_{j+1}^* \left(\sum_{k=1}^{2^j} (Z_k^{(2^j)})^* Z_k^{(2^j)} \right) K_{j+1} + L_{j+1} \left(\sum_{k=1}^{2^j} L_{j+1} (Z_k^{(2^j)})^* Z_k^{(2^j)} \right) L_{j+1} \\
&= K_{j+1}^* H_j K_{j+1} + L_{j+1}^* H_j L_{j+1}. \qquad \square
\end{aligned}
$$

The following proposition allows to compute the matrices K_{j+1} and L_{j+1} iteratively.

Proposition 3.2. *For* $j = 0, 1, \ldots,$ *we have*

$$
\begin{pmatrix} B_j & A_j \\ A_j & B_j \end{pmatrix} \cdot \begin{pmatrix} K_{j+1} \\ L_{j+1} \end{pmatrix} = \begin{pmatrix} 0 \\ I_N \end{pmatrix} \tag{21}
$$

with

$$
A_j = -AZ_1^{(2^j)}, \quad B_j = Z_{2^j}^{(2^j)}.
$$

Proof. The second line of the system (8) written for $k = 2^j$ gives

$$
Z_{2^j}^{(2^{j+1})} - AZ_{2^j+1}^{(2^{j+1})} = 0
$$

or, according to Proposition 1

$$
Z_{2^j}^{(2^j)} K_{j+1} - AZ_1^{(2^j)} L_{j+1} = 0
$$

or again

$$B_j K_{j+1} + A_j L_{j+1} = 0. \tag{22}$$

Similarly, the first line can be written, knowing that $Z_0^{(2^j)} = Z_{2^j}^{(2^j)}$,

$$Z_{2^j}^{(2^j)} L_{j+1} - A Z_1^{(2^j)} K_{j+1} = I_N$$

or again

$$B_j L_{j+1} + A_j K_{j+1} = I_N. \tag{23}$$

□

For $j = 0, 1, \ldots,$ knowing K_{j+1} and L_{j+1}, we determine the matrices $Z_1^{(2^{j+1})}$ and $Z_{2^{j+1}}^{(2^{j+1})}$ from $Z_1^{(2^j)}$ and $Z_{2^j}^{(2^j)}$ by the application of Proposition 3.1, and K_{j+2} and L_{j+2} as solutions of the matrix system (21).

3.2. Analysis of $Z_{2^{j+1}}^{(2^{j+1})}$

The following lemma is a preliminary convergence result for the matrices Γ_0^n and Γ_∞^{-n}, for all $n \geq 0$ (see [3]). It was used to show the convergence of the sequence $(Z_{2^{j+1}}^{(2^{j+1})})_{j \geq j_0}$ to the projector $\mathbb{P}$ where j_0 is a given integer.

Lemma 3.1. *For all* $n \geq 0$, *we have*

$$\max(\| \Gamma_0^n \|, \| \Gamma_\infty^{-n} \|) \leq \omega\gamma^n$$

with $\omega \geq 1$ *and* $0 < \gamma < 1$.

Proof. The spectra of Γ_0 and Γ_∞^{-1} lie inside the unit disk. Then there exists symmetric positive definite matrices H_0 and H_∞ such that (see, e.g., [8, p. 148]):

$$H_0 - \Gamma_0^T H_0 \Gamma_0 = I_p \quad \text{and} \quad H_\infty - \Gamma_\infty^{-T} H_\infty \Gamma_\infty^{-1} = I_{N-p}$$

which give us

$$\begin{pmatrix} H_0 & \\ & H_\infty \end{pmatrix} - \begin{pmatrix} \Gamma_0 & \\ & \Gamma_\infty^{-1} \end{pmatrix}^T \begin{pmatrix} H_0 & \\ & H_\infty \end{pmatrix} \begin{pmatrix} \Gamma_0 & \\ & \Gamma_\infty^{-1} \end{pmatrix} = \begin{pmatrix} I_p & \\ & I_{N-p} \end{pmatrix}.$$

We obtain the following estimate $\forall n \geq 0$:

$$\left\| \begin{pmatrix} \Gamma_0 & \\ & \Gamma_\infty^{-1} \end{pmatrix}^n \right\| \leq \omega \gamma^n,$$

where $\omega = \max(\| H_0 \|, \| H_\infty \|) \geq 1$ and $\gamma = \sqrt{1 - 1/\max(\| H_0 \|, \| H_\infty \|)} < 1$ from a standard result on Lyapunov equations (see, e.g., [8, p. 149]). □

From Lemma 3.1, we deduct the following lemma:

Lemma 3.2. *There exists* $j_0 \in \mathbb{N}$ *such that for all* $j \geq j_0$, *we have*

$$\max(\| [I_p - \Gamma_0^{2^{j+1}}]^{-1} - I_p \|, \| [I_{(N-p)} - \Gamma_\infty^{2^{j+1}}]^{-1} \|) \leq \frac{\omega \gamma^{2^{j+1}}}{1 - \omega \gamma^{2^{j+1}}}.$$

Proof. Remark that

$$[I_p - \Gamma_0^{2^{j+1}}]^{-1} - I_p = [I_p - \Gamma_0^{2^{j+1}}]^{-1}[I_p - (I_p - \Gamma_0^{2^{j+1}})]$$

$$= [I_p - \Gamma_0^{2^{j+1}}]^{-1} \Gamma_0^{2^{j+1}}.$$

Then we have

$$\| [I_p - \Gamma_0^{2^{j+1}}]^{-1} - I_p \| \leq \| [I_p - \Gamma_0^{2^{j+1}}]^{-1} \| \| \Gamma_0^{2^{j+1}} \|.$$

All eigenvalues of the matrix Γ_0 are in the unit circle; so there exists $j_1 \in \mathbb{N}$ such that $\| \Gamma_0^{2^{j+1}} \| < 1.$ Then $[I_p - \Gamma_0^{2^{j+1}}]^{-1} = \sum_{k=0}^{+\infty} (\Gamma_0^{2^{j+1}})^k$; this implies that $\forall j \geq j_1,$ we have

$$\| [I_p - \Gamma_0^{2^{j+1}}]^{-1} - I_p \| \leq \left(\sum_{k=0}^{\infty} \| (\Gamma_0^{2^{j+1}})^k \| \right) \| \Gamma_0^{2^{j+1}} \|$$

$$\leq \left(\sum_{k=0}^{\infty} \| \Gamma_0^{2^{j+1}} \|^k \right) \| \Gamma_0^{2^{j+1}} \|$$

$$\leq \frac{\| \Gamma_0^{2^{j+1}} \|}{1 - \| \Gamma_0^{2^{j+1}} \|}.$$

From Lemma 3.1, we deduct

$$\| [I_p - \Gamma_0^{2^{j+1}}]^{-1} - I_p \| \leq \frac{\omega \gamma^{2^{j+1}}}{1 - \omega \gamma^{2^{j+1}}}, \quad \forall j \geq j_1.$$

We get again the estimate

$$\| [I_{(N-p)} - \Gamma_\infty^{2^{j+1}}]^{-1} \| \leq \frac{\omega \gamma^{2^{j+1}}}{1 - \omega \gamma^{2^{j+1}}}, \quad \forall j \geq j_2$$

remarking that $[I_{(N-p)} - \Gamma_\infty^{2^{j+1}}]^{-1} = [\Gamma_\infty^{-2^{j+1}} - I_{(N-p)}]^{-1} \Gamma_\infty^{-2^{j+1}}$ with the spectrum of Γ_∞^{-1} lies inside the unit disk and $\| \Gamma_\infty^{-2^{j+1}} \| < 1$ for a value $j_2 \in \mathbb{N}$. We consider $j_0 = \max(j_1, j_2)$. □

Lemma 3.2 shows that

$$[I_p - \Gamma_0^{2^{j+1}}]^{-1} \to I_p \text{ and } [I_{(N-p)} - \Gamma_\infty^{2^{j+1}}]^{-1} \to 0. \tag{24}$$

However, the matrix $Z_{2^{j+1}}^{(2^{j+1})}$ can be decomposed as

$$Z_{2^{j+1}}^{(2^{j+1})} = X \begin{pmatrix} [I_{(N-p)} - \Gamma_\infty^{2^{j+1}}]^{-1} & \\ & [I_p - \Gamma_0^{2^{j+1}}]^{-1} \end{pmatrix} X^{-1}. \tag{25}$$

From the decomposition (25), we have the following theorem which presents an estimation of the error $Z_{2^{j+1}}^{(2^{j+1})} - \mathbb{P}$ when j take the great values.

Theorem 3.1. *There exists* $j_0 \in \mathbb{N}$ *such that for all* $j \geq j_0$, *we have*

$$\| Z_{2^{j+1}}^{(2^{j+1})} - \mathbb{P} \| \leq \kappa_2(X) \frac{\omega\gamma^{2^{j+1}}}{1 - \omega\gamma^{2^{j+1}}}.$$

Proof. According to Lemma 3.2, there exists $j_0 \in \mathbb{N}$ such that $\forall j \geq j_0$, we have

$$\begin{aligned}
&\| Z_{2^{j+1}}^{(2^{j+1})} - \mathbb{P} \| \\
&= \left\| X \begin{bmatrix} [I_{(N-p)} - \Gamma_\infty^{2^{j+1}}]^{-1} & 0 \\ 0 & [I_p - \Gamma_0^{2^{j+1}}]^{-1} - I_p \end{bmatrix} X^{-1} \right\| \\
&\leq \kappa_2(X) \max(\| [I_p - \Gamma_0^{2^{j+1}}]^{-1} - I_p \|, \| [I_{(N-p)} - \Gamma_\infty^{2^{j+1}}]^{-1} \|) \\
&\leq \kappa_2(X) \frac{\omega\gamma^{2^{j+1}}}{1 - \omega\gamma^{2^{j+1}}}. \qquad \square
\end{aligned}$$

The speed of convergence of $\| Z_{2^{j+1}}^{(2^{j+1})} - \mathbb{P} \|$ depends mainly on $\gamma^{2^{j+1}}$. The quantity $\kappa_2(X)$ comes from the block-diagonalization (4). This theorem shows that $Z_{2^{j+1}}^{(2^{j+1})}$ converges to the projector $\mathbb{P}$ very rapidly.

3.3. Algorithm

All the results obtained apply to the case where A has not eigenvalues on the circle $C(0, r)$ of center 0 and of radius r. It is enough to replace A by $\frac{A}{r}$. The new variant of the algorithm of spectral dichotomy with respect to the circle $\mathcal{C}(0, r)$ is given below.

Algorithm 3.1. (Spectral dichotomy of A with respect to the circle $C(0, r)$)

(1) **Initialize**

(a) $A_0 = -\dfrac{A}{r}.$

(b) Resolve

$$\begin{pmatrix} A_0 & I_N \\ I_N & A_0 \end{pmatrix}\begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = \begin{pmatrix} 0 \\ I_N \end{pmatrix}.$$

(c) Put $Z_1^{(2)} = K_1$, $Z_2^{(2)} = L_1$ and compute $H_1 = (Z_1^{(2)})^* Z_1^{(2)} + (Z_2^{(2)})^* Z_2^{(2)}$.

(2) **Iterate:** for $j = 1, 2, \ldots$

(a) Put

$$A_j = A_0 Z_1^{(2^j)}, \quad B_j = Z_{2^j}^{(2^j)}.$$

(b) Resolve

$$\begin{pmatrix} B_j & A_j \\ A_j & B_j \end{pmatrix}\begin{pmatrix} K_{j+1} \\ L_{j+1} \end{pmatrix} = \begin{pmatrix} 0 \\ I_N \end{pmatrix}.$$

(c) Compute

$$Z_1^{(2^{j+1})} = Z_1^{(2^j)} K_{j+1},$$

$$Z_{2^{j+1}}^{(2^{j+1})} = Z_{2^j}^{(2^j)} L_{j+1},$$

$$H_{j+1} = K_{j+1}^* H_j K_{j+1} + L_{j+1}^* H_j L_{j+1}.$$

The cost of Algorithm 3.1 by neglecting the terms of power inferior to 3, is summarized in Table 2.

Table 2. Cost of Algorithm 3.1

Initialization	
Computation of K_1, L_1, H_1	$(26/3)N^3$
an iteration j	
Computation of A_j	$2N^3$
Computation of K_{j+1}, L_{j+1}	$(31/3)N^3$
Computation of $Z_1^{(2^{j+1})}$	$2N^3$
Computation of $Z_{2^{j+1}}^{(2^{j+1})}$	$2N^3$
Computation of H_{j+1}	$4N^3$
Total	$\frac{26}{3}N^3 + \frac{61}{3}jN^3$

If we compare the cost of elementary operations presented in Tables 1 and 2, we note that an iteration of Algorithm 2.1 is four times more expensive than an iteration of Algorithm 3.1. So Algorithm 3.1 represents a non-negligible gain in comparison to the algorithm of [10]. In general, about ten iterations are enough for Algorithm 3.1 (and to the algorithm proposed by Godunov and Sadkane) to construct very good approximations of projector $\mathbb{P}$ and of the matrix $\mathbb{H}$.

4. Application to the Strong Stability of Symplectic Matrices

We now give a matrix $W \in \mathbb{R}^{2N\times 2N}$ and a matrix $J \in \mathbb{R}^{2N\times 2N}$ invertible and anti-symmetric such that W is J-symplectic.

The study of strong stability of W leads us, in a first step, to verify if all the eigenvalues are on the unit circle. Then we determine the projectors $\mathbb{P}_0$, $\mathbb{P}_1$ and $\mathbb{P}_\infty$ associated, respectively, to eigenvalues of modulus less than 1, equal to 1 and greater than 1 using Algorithm 3.1. For this, we use two circles $\mathcal{C}(0, r_0)$ and $\mathcal{C}(0, r_\infty)$ with the same center 0 and radii r_0 and r_∞ such that the first contains all eigenvalues of W inside the unit circle (and excluding others) and the second contains all eigenvalues of modulus less than or equal to 1 (and excluding others). Then the spectral projector obtained by Algorithm 3.1 applied to W and to circle $\mathcal{C}(0, r_0)$ will be equal to $\mathbb{P}_0$ and this one obtained with $\mathcal{C}(0, r_\infty)$ will be equal to $I_{2N} - \mathbb{P}_\infty$. Thus, we obtain $\mathbb{P}_0$, $\mathbb{P}_\infty$ and so $\mathbb{P}_1 = I_{2N} - \mathbb{P}_0 - \mathbb{P}_\infty$.

Considering the fact that the eigenvalues of symplectic matrices W are symmetric with respect to the unit circle (within the meaning of inversion), it suffices to find r_0 and to take $r_\infty = 1/r_0$. To have r_0, it is enough, for example, to approach the first minimum of this function located just left of the asymptote $r = 1$. Since the function is convex outside asymptotes, this minimum exists and can be approached by different techniques.

If the projectors $\mathbb{P}_0 = \mathbb{P}_\infty = 0$ and $\mathbb{P}_1 = I_{2N}$, then we verify if all the eigenvalues are well defined, i.e., they are r-eigenvalues or g-eigenvalues. Denote by $\mathbb{P}_r$ and $\mathbb{P}_g$ the projectors onto the subspaces invariant associated, respectively, with the r-eigenvalues and g-eigenvalues of W. Denoting again by $S_r = \mathbb{P}_r^T S_0 \mathbb{P}_r = S_r^T$ and $S_g = \mathbb{P}_g^T S_0 \mathbb{P}_g = S_g^T$, then we have $\mathbb{P}_r^T S_0 \mathbb{P}_g = 0$ and $S_r - S_g > 0$ and the following conditions are equivalent (see [4, 11]):

- W is strongly stable, (26)
- The set of eigenvalues of W is formed only r-eigenvalues and g-eigenvalues, (27)
- $S_r \geq 0,\ S_g \leq 0,\ S_r - S_g > 0,$ (28)
- $\mathbb{P}_0 = \mathbb{P}_\infty = 0$ and $\mathbb{P}_r + \mathbb{P}_g = \mathbb{P}_1 = I_{2N}$. (29)

Note that two eigenvalues $\lambda, \mu \neq \pm 1$ of W such that $\bar{\lambda}\mu = 1$, are on the same circle, because $\bar{\lambda}\mu = 1$ implies that $\left|\frac{\lambda - 1}{\lambda + 1}\right| = \left|\frac{\mu - 1}{\mu + 1}\right|$ which is the equation of the circle. We apply the idea originally proposed in [7], which is to bring together all the eigenvalues λ of W at the intersection of the unit circle and the circle equation $\left|\frac{\lambda - 1}{\lambda + 1}\right| = C$ (where C is a positive constant). We know that the eigenvectors associated with eigenvalues λ and μ located on distinct circles $\left(\left|\frac{\lambda - 1}{\lambda + 1}\right| \neq \left|\frac{\mu - 1}{\mu + 1}\right| \text{ since } \bar{\lambda}\mu \neq 1\right)$ are J-orthogonal (see [9]). The result of this simple remark allows us to analyze strong stability (determination projectors $\mathbb{P}_r$ and $\mathbb{P}_g$). We summarize the main ideas:

Suppose that the application of Algorithm 3.1 to W, with a suitable choice of r_0, gives us $\mathbb{P}_0 = \mathbb{P}_\infty = 0$ and thus $\mathbb{P}_1 = I_{2N}$.

Suppose further that the matrix S_0 is invertible and consider the matrix A given by the classical Cayley transformation:

$$A = \mathcal{C}(W) = (W - I_{2N})(W + I_{2N})^{-1}$$

whose eigenvalues are of the form

$$l_k = \frac{\lambda_k - 1}{\lambda_k + 1} = i.\tan(\theta_k/2), \quad k = 1, 2, \ldots, m,$$

where $\lambda_k = e^{i\theta_k}$ with $\theta_k \in \left]-\pi, \pi\right]$, designates an eigenvalue of W of modulus 1. $\mathcal{C}(W)$ is only defined if $\lambda \in \{+1, -1\}$ is not eigenvalue of W and its eigenvalues are on the imaginary axis. Note that the eigenvectors (and invariant subspaces) of A and W are the same.

Let $a_1, a_2, \ldots$ be positive numbers that interweave strictly eigenvalues of modulus $|l_1|, |l_2|, \ldots$. In other words, $0 < a_k < |l_k| < a_{k+1}$ for $k = 1, \ldots, m$. Since W is a real matrix, its eigenvalues λ_k and therefore l_k are complex conjugate and so $m < N + 1$.

In practice, it is possible to choose, for example, a_1 in a neighborhood of 0 and a_m in the interval $\left]\|A\|, \infty\right[$, where $\|A\|$ denotes any matrix norm of

A. The others a_k can be obtained by examining the spectral portrait of A. Furthermore, since W must not have eigenvalues ± 1 to be strongly stable, there must exist two positive reals r_ε and M such that

$$r_\varepsilon < a_1 < |l_1| < \cdots < a_k < |l_k| < a_{k+1} < \cdots < |l_m| < a_{m+1} < M. \quad (30)$$

If $|l_1| = 0$ (or $|l_m| = +\infty$), there is not strong stability because this implies that $+1$ (or -1) is an eigenvalue of A. Hence the following proposition:

Proposition 4.1. *A symplectic matrix W having all the eigenvalues on the unit circle, has not eigenvalues ± 1 if and only if there exist two positive reals r_ε and M such that all the eigenvalues l_k of $A = (W - I)(W + I)^{-1}$ verifies*

$$r_\varepsilon < |l_k| < M, \quad k = 1, 2, \ldots, m.$$

Proof. The eigenvalue $\lambda = e^{i\theta}$ of W, where $\theta \in \,]-\pi, \pi]$, is different from ± 1 if and only if $\theta \notin \{0, \pi\}$. This implies that $\tan(\theta/2) \notin \{0, \infty\}$ or again there exist two positive reals r_ε and M such that $r_\varepsilon < |\tan(\theta/2)| < M$. □

We know from the above discussion that the eigenvectors (and invariant subspaces) associated with the eigenvalues and l_k, and l_j in different fields are J-orthogonal. These regions can be obtained by applying Algorithm 3.1 to the matrix A.

Denote by P_k, $k = 1, \ldots, m+1$ the projectors on the subspaces associated with eigenvalues of A in the circles $\mathcal{C}(0, a_k)$ with $P_1 = 0$ and $P_m = I$. Furthermore, each matrix

$$Q_k = P_{k+1} - P_k, \quad k = 1, 2, \ldots, m \quad (31)$$

is a projector on the invariant subspace of A (also of W) associated to eigenvalues $l = \frac{\lambda - 1}{\lambda + 1}$ of A located in the crown

$$a_k < \frac{|\lambda - 1|}{|\lambda + 1|} < a_{k+1}, \quad k = 1, 2, \ldots, m. \tag{32}$$

Consider now the matrix S_k defined by

$$S_k = Q_k^T S_0 Q_k \equiv S_k^T, \quad k = 1, 2, \ldots, m. \tag{33}$$

The idea is to gather all the eigenvalues λ of W for which S_k is positive (or negative) semi-definite. To have W strongly stable, all the matrices S_k should be well definite, i.e., they should be positive (negative) semi-definite. This discussion leads to Theorem 4.1 below.

Theorem 4.1. *The matrix W is strongly stable if and only if*

$$\mathbb{P}_0 = \mathbb{P}_\infty = 0 \quad \textit{and} \quad \mathbb{P}_r + \mathbb{P}_g = \mathbb{P}_1 = I_{2N}$$

with

$$\mathbb{P}_r = \sum_{S_k \geq 0} Q_k \quad \textit{and} \quad \mathbb{P}_g = \sum_{S_k \leq 0} Q_k.$$

The algorithm which allows to determine the strongly stable of a J-symplectic matrix W, is the following:

Algorithm 4.1.

1. Using Algorithm 3.1 and an appropriate choice of the parameter r_0, to determine the projectors $\mathbb{P}_0$, $\mathbb{P}_1$ and $\mathbb{P}_\infty$.

2. If $\mathbb{P}_0 \neq 0$ $(\text{or } \mathbb{P}_\infty \neq 0)$, then there is no strong stability. Else compute

$$S_0 = (1/2)((JW) + (JW)^T).$$

If S_0 is singular (poorly conditioned), then there is no stability. Else compute

$$A = (W - I)(W + I)^{-1}.$$

From the spectral portrait of A, determine the scalars $(a_k)_{1 \le k \le m+1}$ satisfying inequalities (30) and (32).

3. Determine the projectors $P_1 = 0,\ P_2,\ \ldots,\ P_m,\ P_{m+1} = I$ by the spectral dichotomy applied to A and circle $\mathcal{C}(0,\ a_k),\ k = 1,\ 2,\ \ldots,\ m+1$.

4. For $k = 1,\ 2,\ \ldots,\ m,$ compute

$$Q_k = P_{k+1} - P_k \quad \text{and} \quad S_k = Q_k^T S_0 Q_k.$$

If S_k is not definite, then there is not strong stability. If all the matrices S_k are semi-definite, then there is strong stability. In this case

$$\mathbb{P}_r = \sum_{S_k \ge 0} Q_k \quad \text{and} \quad \mathbb{P}_g = \sum_{S_k \le 0} Q_k.$$

Taking $a_1 > 0$ (if it exists) such that $a_1 < |\, l_k \,|,\ \ \forall k = 1,\ \ldots,\ m$ and $a_{m+1} > \|A\|,$ we obtain $P_1 = 0$ and $P_{m+1} = I.$ In Algorithm 4.1, the computation of each projector is accompanied of its dichotomy criterion. The algorithm stops if this criterion is too big. When there is strong stability, all the projectors must be computed with a good criterion dichotomy. To define the red and green projectors, all eigenvalues must be on the unit circle because we are primarily interested in the study of stability. But beyond the questions linked to the stability, it is entirely possible to define these projectors in a larger context, i.e., some eigenvalues cannot be necessarily on the unit circle. Indeed, if $\mathbb{P}_0,\ \mathbb{P}_\infty \ne 0,$ then $\mathbb{P}_1 \ne I,$ it is enough to apply Algorithm 4.1 to the matrix $A = (\mathbb{P}_1 W - I)(\mathbb{P}_1 W + I)^{-1}.$ A simple adaptation of Algorithm 4.1 shows that the projector $\mathbb{P}_1$ can be decomposed as $\mathbb{P}_1 = \mathbb{P}_r + \mathbb{P}_g$ with $\mathbb{P}_r = \sum_{S_k \ge 0} Q_k$ and $\mathbb{P}_g = \sum_{S_k \le 0} Q_k.$

5. Numerical Example

Example 5.1. Consider the *J*-symplectic matrices

$$W = \begin{pmatrix} \frac{4}{5} & 0 & 0 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{5} & 0 & \frac{4}{5} \\ -\frac{3}{5} & 0 & 0 & 0 & \frac{4}{5} & 0 \\ 0 & 0 & 0 & -\frac{4}{5} & 0 & -\frac{3}{5} \end{pmatrix}, \quad \text{where} \quad J = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

We will study the strong stability of this matrix W by the spectral dichotomy method. By applying the spectral dichotomy method (Algorithm 3.1) to the matrix W and the circles $\mathcal{C}(0, r_0)$ and $\mathcal{C}(0, 1/r_0)$, respectively, where $r_0 = 0.9992$, we deduce the projectors $\mathbb{P}_0 = 0$, $\mathbb{P}_1 = I_6$ and $\mathbb{P}_\infty = 0$ with $Tr(\mathbb{P}_i) = 0$, $i = 0, \infty$, $Tr(\mathbb{P}_1) = 6$, $\| \mathbb{P}_i^2 - \mathbb{P}_i \| = 0$, $i = 0, 1, \infty$ and $\| \mathbb{H}_i \|_2 = 665.9168$, $i = 0, 1, \infty$, where $Tr(\mathbb{P}_i)$ is the trace of the matrix $\mathbb{P}_i$. This implies that all the eigenvalues of W are on the unit circle. We can verify if all the eigenvalues are either of red color or either of green color. Thus, we consider the spectral portrait of the matrix $A = (W - I)(W + I)^{-1}$ to determine the constants $(a_k)_{k=1,2,\ldots}$ which allow to compute the projectors $\mathbb{P}_r$ and $\mathbb{P}_g$. This computation gives us $a_1 = 0.1667$, $a_2 = 0.6667$, $a_3 = 1.5$ and $a_4 = 4$ examining spectral portrait of the matrix A on Figure 1. We can remark that $a_1 \in]0, 1/3[$, $a_2 \in]1/3, 1[$, $a_3 \in]1, 2[$ and $a_4 \in]2, \infty[$. Algorithm 3.1 gives us the projectors

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } Q_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the symmetric matrices

$$S_1 = \begin{pmatrix} -3/5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \le 0,$$

$$S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \ge 0$$

and

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4/5 \end{pmatrix} \leq 0.$$

Then the symplectic matrix W is strongly stable, since $\mathbb{P}_0 = \mathbb{P}_\infty = 0$ and $\mathbb{P}_r + \mathbb{P}_g = I_6$ with $\mathbb{P}_r = Q_2$ and $\mathbb{P}_g = Q_1 + Q_3$. We verify that the conditions (27) and (28) are satisfied as follows:

$$S_r = (\mathbb{P}_r)^T S_0 \mathbb{P}_r = S_2 \geq 0, \quad S_g = (\mathbb{P}_g)^T S_0 \mathbb{P}_g = S_1 + S_3 \leq 0$$

and

$$S_r - S_g = \begin{pmatrix} 0.6000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8000 \end{pmatrix} > 0.$$

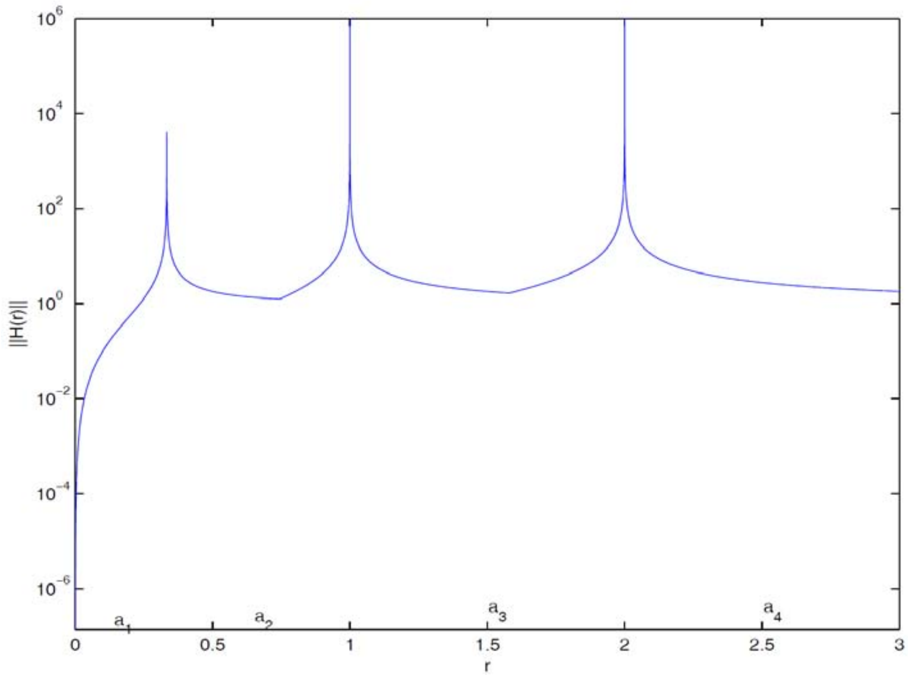

Figure 1. Spectral portrait of the matrix A.

Example 5.2. Let the skew-symmetric matrix $J = \begin{pmatrix} O_2 & -I_2 \\ I_2 & O_2 \end{pmatrix}$. Consider Example 2 of [4] given by the following of J-symplectic matrices for all $t \in [0,\ 2\pi]$:

$$W(t) = \begin{pmatrix} C(s(t))\cos\omega(t) & -(C(s(t)))^{-T}\sin\omega(t) \\ C(s(t))\sin\omega(t) & (C(s(t)))^{-T}\cos\omega(t) \end{pmatrix}$$

with $C(s) = \begin{pmatrix} 1-s^2 & -1 \\ s^2 & 1-s^2 \end{pmatrix}$ and $s(t) = 4\sin(t),\ \omega(t) = \pi((1/2) - (1/3)\sin 3t)$.

For all $t \in [0,\ 2\pi]$, $W(t)$ has eigenvalues inside, on, and outside the unit circle. We illustrate the strong stability with some value of t using the spectral dichotomy method (i.e., with Algorithm 2.1 or 3.1).

- At $t = 2.93$, we show numerically that the matrix $W(2.93)$ is strongly stable. Using Algorithm 3.1 to the matrix $W(2.93)$ and the circles $\mathcal{C}(0, r_0)$ and $\mathcal{C}(0, 1/r_0)$, respectively, where $r_0 = 9.9925 \times 10^{-1}$, we deduce the projectors $\mathbb{P}_0 = 0_4$, $\mathbb{P}_1 = I_4$ and $\mathbb{P}_\infty = 0_4$. Then all the eigenvalues of $W(2.93)$ are on the unit circle. We can verify if all eigenvalues are well defined. Thus, Algorithm 4.1 computes the matrix

$$A = [W(2.93) - I_4][W(2.93) + I_4]^{-1}$$

$$= \begin{pmatrix} 5.1800 \times 10^{-2} & -1.2089 \times 10^{0} & -1.1259 \times 10^{0} & -2.1528 \times 10^{-1} \\ 9.2818 \times 10^{-1} & -2.0319 \times 10^{-1} & -2.1528 \times 10^{-1} & -9.4351 \times 10^{-1} \\ 7.3189 \times 10^{-1} & -1.7711 \times 10^{-1} & -5.1800 \times 10^{-2} & -9.2818 \times 10^{-1} \\ -1.7711 \times 10^{-1} & 9.1431 \times 10^{-1} & 1.20890 \times 10^{0} & 2.0319 \times 10^{-1} \end{pmatrix}$$

and determines the constant $a_1 = 6.1404 \times 10^{-2}$, $a_2 = 1.0355$ and $a_3 = 2.5$ from the spectral portrait of Figure 2. Applying again Algorithm 3.1 three times to the matrix A and the circles $\mathcal{C}(0, a_k)$, $k = 1, 2, 3$, respectively, we get the projectors $P_1 = 0_4$, P_2 and $P_3 = I$, where

$$P_2 = \begin{pmatrix} 5 \times 10^{-1} & 4.9094 \times 10^{-2} & 0 & 5.6362 \times 10^{-1} \\ -3.4646 \times 10^{-2} & 5 \times 10^{-1} & -5.6362 \times 10^{-1} & -0 \\ -0 & -4.4657 \times 10^{-1} & 5 \times 10^{-1} & -3.4646 \times 10^{-2} \\ 4.4657 \times 10^{-1} & 0 & 4.9094 \times 10^{-2} & 5 \times 10^{-1} \end{pmatrix}.$$

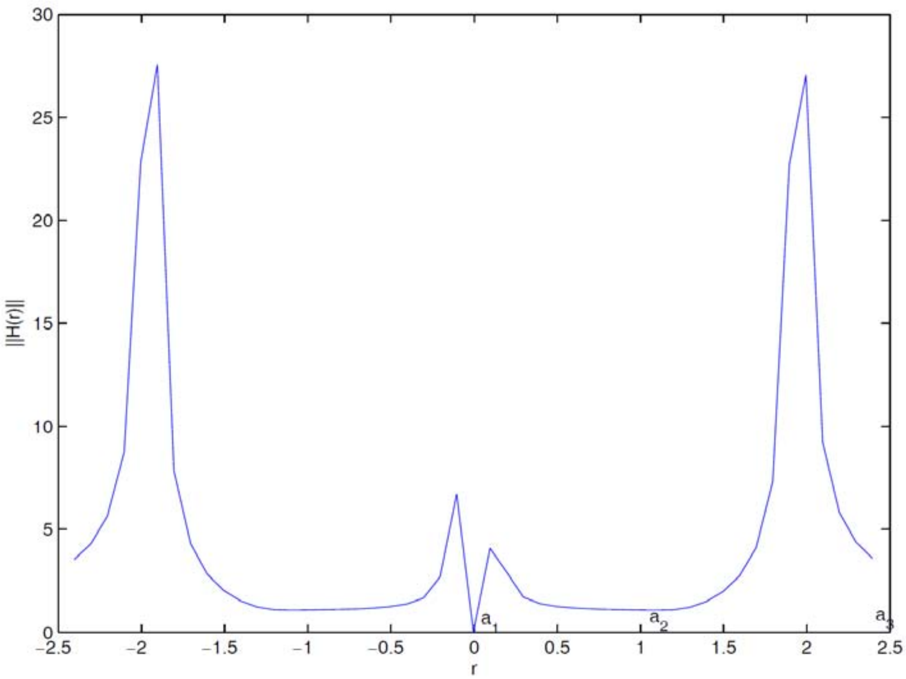

Figure 2. Spectral portrait of the matrix *A*.

This gives

$$Q_1 = P_2 - P_1$$

$$= \begin{pmatrix} 5\times10^{-1} & 4.9094\times10^{-2} & 0 & 5.6362\times10^{-1} \\ -3.4646\times10^{-2} & 5\times10^{-1} & -5.6362\times10^{-1} & 0 \\ 0 & -4.4657\times10^{-1} & 5\times10^{-1} & -3.4646\times10^{-2} \\ 4.4657\times10^{-1} & 0\times10^{-1} & 4.9094\times10^{-2} & 5\times10^{-1} \end{pmatrix},$$

$$Q_2 = P_3 - P_2$$

$$= \begin{pmatrix} 5\times10^{-1} & -4.9094\times10^{-2} & 0 & -5.6362\times10^{-1} \\ 3.4646\times10^{-1} & 5\times10^{-1} & 5.6362\times10^{-1} & 0 \\ 0 & 4.4657\times10^{-1} & 5\times10^{-1} & 3.4646\times10^{-2} \\ -4.4657\times10^{-1} & 0 & -4.9094\times10^{-2} & 5\times10^{-1} \end{pmatrix}$$

and

$$S_1 = Q_1^T S_0 Q_1$$

$$= \begin{pmatrix} 8.3583\times10^{-2} & 5.8106\times10^{-2} & -5.5869\times10^{-2} & 9.8090\times10^{-2} \\ 5.8106\times10^{-2} & 1.8009\times10^{-1} & -1.9524\times10^{-1} & 7.9030\times10^{-2} \\ -5.5869\times10^{-2} & -1.9524\times10^{-1} & 2.1246\times10^{-1} & -7.7701\times10^{-2} \\ 9.8090\times10^{-2} & 7.9030\times10^{-2} & -7.7701\times10^{-2} & 1.1595\times10^{-1} \end{pmatrix} \geq 0,$$

$$S_2 = Q_2^T S_0 Q_2$$

$$= \begin{pmatrix} -3.2292\times10^{-1} & 6.1564\times10^{-2} & 3.3428\times10^{-2} & 3.6633\times10^{-1} \\ 6.1564\times10^{-2} & -4.1943\times10^{-1} & -4.6284\times10^{-1} & -1.0147\times10^{-1} \\ 3.3428\times10^{-2} & -4.6284\times10^{-1} & -5.1454\times10^{-1} & -7.3337\times10^{-2} \\ 3.6633\times10^{-1} & -1.0147\times10^{-1} & -7.3337\times10^{-2} & -4.1803\times10^{-1} \end{pmatrix} \leq 0.$$

Then the symplectic matrix W(2.93) is strongly stable and the projectors $\mathbb{P}_r$ and $\mathbb{P}_g$ are given by

$$\mathbb{P}_r = Q_1$$

$$= \begin{pmatrix} 5\times10^{-1} & 4.9094\times10^{-2} & 0 & 5.6362\times10^{-1} \\ -3.4646\times10^{-2} & 5.0000\times10^{-1} & -5.6362\times10^{-1} & 0 \\ -0 & -4.4657\times10^{-1} & 5\times10^{-1} & -3.4646\times10^{-2} \\ 4.4657\times10^{-1} & 0 & 4.9094\times10^{-2} & 5\times10^{-1} \end{pmatrix},$$

$$\mathbb{P}_g = Q_2$$

$$= \begin{pmatrix} 5\times10^{-1} & -4.9094\times10^{-2} & 0 & -5.6362\times10^{-1} \\ 3.4646\times10^{-2} & 5\times10^{-1} & 5.6362\times10^{-1} & 0 \\ 0 & 4.4657\times10^{-1} & 5\times10^{-1} & 3.4646\times10^{-2} \\ -4.4657\times10^{-1} & 0 & -4.9094\times10^{-2} & 5\times10^{-1} \end{pmatrix}$$

of qualities $\| \mathbb{P}_r^2 - \mathbb{P}_r \| \approx \| \mathbb{P}_g^2 - \mathbb{P}_g \| \approx 6.6801 \times 10^{-17}$ and of traces 2. We can verify that the conditions (27) and (28) are all satisfied as follows:

$$S_r = (\mathbb{P}_r)^T S_0 \mathbb{P}_r = S_1 \geq 0, \qquad S_g = (\mathbb{P}_g)^T S_0 \mathbb{P}_g = S_2 \leq 0$$

and

$$S_r - S_g$$

$$= \begin{pmatrix} 4.0650 \times 10^{-1} & -3.4579 \times 10^{-3} & -8.9298 \times 10^{-2} & -2.6824 \times 10^{-1} \\ -3.4579 \times 10^{-3} & 5.9952 \times 10^{-1} & 2.6759 \times 10^{-1} & 1.8050 \times 10^{-1} \\ -8.9298 \times 10^{-2} & 2.6759 \times 10^{-1} & 7.2700 \times 10^{-1} & -4.3643 \times 10^{-3} \\ -2.6824 \times 10^{-1} & 1.8050 \times 10^{-1} & 4.3643 \times 10^{-3} & 5.3399 \times 10^{-1} \end{pmatrix} > 0.$$

- At $t = 0.20260,$ we briefly show that $W(0.20260)$ is strongly stable. Using Algorithm 3.1 to the matrix $W(0.2026)$ and the circles $\mathcal{C}(0, r_0)$ and $\mathcal{C}(0, 1/r_0),$ respectively, where $r_0 = 9.992 \times 10^{-1},$ we deduce the projectors $\mathbb{P}_0 = 0_4,$ $\mathbb{P}_1 = I_4$ and $\mathbb{P}_\infty = 0_4.$ Then all the eigenvalues of $W(0.20260)$ are on the unit circle. We can verify if all eigenvalues are well defined. Indeed, from Algorithm 4.1, we get

$$Q_1 = \begin{pmatrix} 5 \times 10^{-1} & 5.38815 \times 10^{-2} & 0 & 5.7125 \times 10^{-1} \\ -3.4905 \times 10^{-2} & 5 \times 10^{-1} & -5.7125 \times 10^{-1} & 0 \\ 0 & -4.4092 \times 10^{-1} & 5 \times 10^{-1} & -3.4905 \times 10^{-2} \\ 4.4092 \times 10^{-1} & 0 & 5.3881 \times 10^{-2} & 5 \times 10^{-1} \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 5 \times 10^{-1} & -5.3881 \times 10^{-2} & 0 & -5.7125 \times 10^{-1} \\ 3.4905 \times 10^{-2} & 5 \times 10^{-1} & 5.7125 \times 10^{-1} & 0 \\ 0 & 4.4092 \times 10^{-1} & 5 \times 10^{-1} & 3.4905 \times 10^{-2} \\ -4.4092 \times 10^{-1} & 0 & -5.3881 \times 10^{-1} & 5 \times 10^{-1} \end{pmatrix}$$

and

$$S_1 = Q_1^T S_0 Q_1$$

$$= \begin{pmatrix} 3.3967\times10^{-2} & 7.0005\times10^{-2} & -7.5233\times10^{-2} & 4.4060\times10^{-2} \\ 7.0005\times10^{-2} & 1.4723\times10^{-1} & -1.5841\times10^{-1} & 9.1040\times10^{-2} \\ -7.5233\times10^{-2} & -1.5841\times10^{-1} & 1.7044\times10^{-1} & -9.7854\times10^{-2} \\ 4.4060\times10^{-2} & 9.1040\times10^{-2} & -9.7854\times10^{-2} & 5.7170\times10^{-2} \end{pmatrix} \geq 0,$$

$$S_2 = Q_2^T S_0 Q_2$$

$$= \begin{pmatrix} -3.2503\times10^{-1} & 7.5528\times10^{-2} & 4.5928\times10^{-2} & 3.7456\times10^{-1} \\ 7.5528\times10^{-2} & -4.3830\times10^{-1} & -4.8780\times10^{-1} & -1.2034\times10^{-1} \\ 4.5928\times10^{-2} & -4.8780\times10^{-1} & -5.4754\times10^{-1} & -9.0698\times10^{-2} \\ 3.7456\times10^{-1} & -1.2034\times10^{-1} & -9.0698\times10^{-2} & -4.3427\times10^{-1} \end{pmatrix} \leq 0.$$

Then the symplectic matrix W(2.93) is strongly stable and the projectors $\mathbb{P}_r$ and $\mathbb{P}_g$ are given by

$$\mathbb{P}_r = Q_1$$

$$= \begin{pmatrix} 5\times10^{-1} & 5.3881\times10^{-2} & 0 & 5.7125\times10^{-1} \\ -3.4905\times10^{-1} & 5\times\times10^{-1} & -5.7125\times10^{-1} & 0 \\ 0 & -4.4092\times10^{-1} & 5\times10^{-1} & -3.4905\times10^{-2} \\ 4.4092\times10^{-1} & 0 & 5.3881\times10^{-2} & 5\times10^{-1} \end{pmatrix}$$

$$\mathbb{P}_g = Q_2$$

$$= \begin{pmatrix} 5\times10^{-1} & -5.3881\times10^{-2} & 0 & -5.7125\times10^{-1} \\ 3.4905\times10^{-1} & 5\times10^{-1} & 5.7125\times10^{-1} & 0 \\ 0 & 4.4092\times10^{-1} & 5\times10^{-1} & 3.4905\times10^{-2} \\ -4.4092\times10^{-1} & 0 & -5.3881\times10^{-2} & 5\times10^{-1} \end{pmatrix}$$

of qualities $\| \mathbb{P}_r^2 - \mathbb{P}_r \| \approx 1.1208 \times 10^{-16}$, $\| \mathbb{P}_g^2 - \mathbb{P}_g \| \approx 5.8268 \times 10^{-17}$ and of traces 2. We can verify that the conditions (27) and (28) are all satisfied as follows:

$$S_r = (\mathbb{P}_r)^T S_0 \mathbb{P}_r = S_1 \geq 0, \qquad S_g = (\mathbb{P}_g)^T S_0 \mathbb{P}_g = S_2 \leq 0$$

and

$$S_r - S_g$$

$$= \begin{pmatrix} 3.5900 \times 10^{-1} & -5.5234 \times 10^{-3} & -1.2116 \times 10^{-1} & -3.3050 \times 10^{-1} \\ -5.5234 \times 10^{-3} & 5.8554 \times 10^{-1} & 3.2939 \times 10^{-1} & 2.1138 \times 10^{-1} \\ -1.2116 \times 10^{-1} & 3.2939 \times 10^{-1} & 7.1798 \times 10^{-1} & -7.1561 \times 10^{-3} \\ -3.3050 \times 10^{-1} & 2.1138 \times 10^{-1} & -7.1561 \times 10^{-3} & 4.9144 \times 10^{-1} \end{pmatrix} > 0.$$

- At $t = 0.1413505$, we show numerically that $W(0.1413505)$ is not stable. Using Algorithm 3.1 to the matrix $W(0.1413505)$ and the circles $\mathcal{C}(0, r_0)$ and $\mathcal{C}(0, 1/r_0)$, respectively, where $r_0 = 9.9991 \times 10^{-1}$, we get the projectors:

$$\mathbb{P}_0 = \begin{pmatrix} 7.1304 \times 10^{-1} & -1.7127 \times 10^{-1} & 2.7154 \times 10^{-1} & 8.0452 \times 10^{-1} \\ 6.1430 \times 10^{-1} & -1.4756 \times 10^{-1} & 2.3394 \times 10^{-1} & 6.9311 \times 10^{-1} \\ -5.5943 \times 10^{-1} & 1.3438 \times 10^{-1} & -2.1304 \times 10^{-1} & -6.3120 \times 10^{-1} \\ 5.7393 \times 10^{-1} & -1.3786 \times 10^{-1} & 2.1856 \times 10^{-1} & 6.4756 \times 10^{-1} \end{pmatrix},$$

$$\mathbb{P}_1 = \begin{pmatrix} 5 \times 10^{-1} & -4.7287 \times 10^{-2} & 0 & -5.7057 \times 10^{-1} \\ 1.6896 \times 10^{-2} & 5 \times 10^{-1} & 5.7057 \times 10^{-1} & 0 \\ 0 & 4.3955 \times 10^{-1} & 0.5 \times 10^{-1} & 1.6896 \times 10^{-2} \\ -4.3955 \times 10^{-1} & 0 & -4.7287 \times 10^{-2} & 5 \times 10^{-1} \end{pmatrix},$$

$$\mathbb{P}_\infty = \begin{pmatrix} -2.1304\times10^{-1} & 2.1856\times10^{-1} & -2.7154\times10^{-1} & -2.3394\times10^{-1} \\ -6.3120\times10^{-1} & 6.4756\times10^{-1} & -8.0452\times10^{-1} & -6.9311\times10^{-1} \\ 5.5943\times10^{-1} & -5.7393\times10^{-1} & 7.1304\times10^{-1} & 6.1430\times10^{-1} \\ -1.3438\times10^{-1} & 1.3786\times10^{-1} & -1.7127\times10^{-1} & -1.4756\times10^{-1} \end{pmatrix},$$

where

$$\| \mathbb{P}_0^2 - \mathbb{P}_0 \| \approx 7.652518846055654\times10^{-16},$$

$$\| \mathbb{P}_1^2 - \mathbb{P}_1 \| \approx 2.728044157407159\times10^{-15}$$

and $\| \mathbb{P}_\infty^2 - \mathbb{P}_\infty \| \approx 5.856243848391748\times10^{-16}$, and the trace $Tr(\mathbb{P}_0) = 1$, $Tr(\mathbb{P}_1) = 2$ and $Tr(\mathbb{P}_\infty) = 1$. This implies that all the eigenvalues are not on the unit circle; so W is not stable.

- At $t = 0.141350433896871535$, we show numerically that $W(0.141350433896871535)$ is not strongly stable. Using Algorithm 3.1 to the matrix $W(0.1413504339...)$ and the circles $\mathcal{C}(0, r_0)$ and $\mathcal{C}(0, 1/r_0)$, respectively, where $r_0 = 9.9880\times10^{-1}$, we get the projectors $\mathbb{P}_0 = 0_4$, $\mathbb{P}_1 = I_4$ and $\mathbb{P}_\infty = 0_4$. Then all the eigenvalues of $W(0.1413504339...)$ are on the unit circle. However, the determination of a_1 gives $a_1 < | l_1 | = 1.139866233037296\times10^{-8}$. That is approximately equal to zero. Then W has an eigenvalue $+1$ or -1. So W is not strongly stable.

6. Conclusion

In this paper, we proposed a method to analyze the strong stability of symplectic matrices using methods of spectral dichotomy. This method determines, using the spectral portrait, a parameter r_0 such that the circle $\mathcal{C}(0, r_0)$ contains all the eigenvalues of modulus strictly less than 1 of a symplectic matrix *W*. Then we apply Algorithm 2.1 or 3.1 two times: one with matrix *W* and $\mathcal{C}(0, r_0)$ and another with the matrix *W* and $\mathcal{C}(0, 1/r_0)$ to finally obtain the three projectors $\mathbb{P}_0$, $\mathbb{P}_\infty$ and $\mathbb{P}_1$ on the invariant subspaces associated, respectively, with eigenvalues of modulus of less than 1, more than 1 and equal to 1. The computation of these projectors are accompanied with the dichotomy criterions which give an information on the qualities of the projectors.

In the case where $\mathbb{P}_0 = \mathbb{P}_\infty = 0$, Algorithm 4.1 is verified if we can decompose the projector $\mathbb{P}_1 = I$ in two projectors $\mathbb{P}_r$ and $\mathbb{P}_g$ on the invariant subspaces associated, respectively, with *r*- and *g*-eigenvalues. This decomposition is possible if and only if the matrix *W* is strongly stable. Two examples show how to analyze the strong stability of a symplectic matrix.

References

[1] C. Bekas and E. Gallopoulos, Cobra: parallel path following for computing the matrix pseudospectrum, Parallel Comput. 27 (2001), 1879-1896.

[2] B. N. Datta, Numerical Methods in Linear Control Systems and Design Analysis, Elsevier Academic Press, 2004.

[3] M. Dosso and M. Sadkane, A spectral trichotomy method for symplectic matrices, Numer. Algor. 52 (2009), 187-212.

[4] M. Dosso and M. Sadkane, On the strongly stable of symplectic matrices, Numer. Linear Algebra Appl. 20(2) (2011), 234-249.

[5] S. M. Fallet and M. J. Tsatsomeros, On the Cayley transform of positivity class of matrices, Electr. J. Linear Algebra 9 (2002), 190-196.

[6] S. K. Godunov, Problem of dichotomy of the spectrum of a matrix, Siberian Math. J. 27 (1986), 649-660.

[7] S. K. Godunov, Stability of iterations of symplectic transformations, Siberian Math. J. 30 (1989), 54-63.

[8] S. K. Godunov, Modern Aspects of Linear Algebra, Amer. Math. Soc., Vol. 175, 1998.

[9] S. K. Godunov and M. Sadkane, Numerical determination of a canonical form of a symplectic matrix, Siberian Math. J. 42 (2001), 629-647.

[10] S. K. Godunov and M. Sadkane, Some new algorithms for the spectral dichotomy methods, Linear Algebra Appl. 358 (2003), 173-194.

[11] S. K. Godunov and M. Sadkane, Spectral analysis of symplectic matrices with application to the theory of parametric resonance, SIAM J. Matrix Anal. Appl. 28 (2006), 1083-1096.

[12] I. Gohberg, P. Lancaster and L. Rodman, Matrices and Indefinite Scalar Products, Birkhäuser Verlag, Basel, 1983.

[13] I. Gohberg, P. Lancaster and I. Rodman, Indefinite Linear Algebra and Applications, Birkhäuser Verlag, Basel, 2005.

[14] G. H. Golub and C. F. Van Loan, Matrix Computations, 2nd ed., The Johns Hopkins University Press, Baltimore, MD, 1989.

[15] B. Hassibi, A. H. Sayed and T. Kailath, Indefinite-quadratic Estimation and Control, SIAM, Philadelphia, PA, 1999.

[16] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1976.

[17] P.-F. Lavalle and M. Sadkane, Computation of pseudospectra by spectral dichotomy methods in a parallel environment, Numer. Algor. 33 (2003), 343-355.

[18] M. Sadkane, Estimates from the discrete-time Lyapunov equation, Appl. Math. Lett. 16 (2003), 313-316.

[19] L. N. Trefethen and M. Embree, Spectra and Pseudospectra. The Behavior of Non-normal Matrices and Operators, Princeton University Press, Princeton, 2005.

[20] Panagiotis Tsiotras, John L. Junkins and Hanspeter Schaub, Higher order Cayley transforms with applications to attitude representations, J. Guidance, Control and Dynamics 20(3) (1997), 528-536.

[21] V. A. Yakubovich and V. M. Starzhinskii, Linear Differential Equations with Periodic Coefficients, Vols. 1 and 2, Wiley, New York, 1975.