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WHITNEY MAP FOR HYPERSPACES OF CONTINUA WITH THE PROPERTY OF KELLEY

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Abstract

The main purpose of this paper is to study the properties of continua with the property of Kelley which admit a Whitney map either for hyperspace $C(X)$ or for hyperspace $C^2(X)$. In particular, it is proved that an arcwise connected continuum X with the property of Kelley admits a Whitney map for $C(X)$ if and only if it is metrizable.

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $w(X)$. We shall use the notion of an inverse system as in [4, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points (end points) x, y. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

We say that a space X is *arcwise connected* if for every pair x, y of points of X there exists a generalized arc L with end points x, y.

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Let X be a space. We define its hyperspaces as the following sets:

$$2^X = \{F \subseteq X : F \text{ is closed and nonempty}\},$$

$$C(X) = \{F \in 2^X : F \text{ is connected}\},$$

$$C^2(X) = C(C(X)),$$

$$X(n) = \{F \in 2^X : F \text{ has at most } n \text{ points}\}, \ n \in \mathbb{N}.$$

The topology on 2^X is the Vietoris topology and $C(X)$, $X(n)$ are subspaces of 2^X. Moreover, $X(1)$ is homeomorphic to X.

Let X and Y be the spaces and let $f : X \to Y$ be a mapping. Define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [10, Theorem 5.10, p. 170] 2^f is continuous and $2^f(C(X)) \subset C(Y)$, $2^f(X(n)) \subset (Y)$. The restriction $2^f \mid C(X)$ is denoted by $C(f)$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is σ-*directed* if for each sequence $a_1, a_2, \ldots, a_k, \ldots$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Theorem 1.1. *Let X be a compact Hausdorff space such that $w(X) \geq \aleph_1$. Then there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a and surjective bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$. Moreover, if X is a Hausdorff continuum, then each coordinate space X_a can be chosen as a metric continuum.*

Proof. In [8, Theorem 1.8, p. 397] it is proved that there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a such that X is homeomorphic to $\lim \mathbf{X}$. From the proof of [8, Theorem 1.8, p. 397] it follows that the bonding mappings are surjective. It is clear that if X is a Hausdorff continuum, then each $X_a = p_a(X)$ is a metric continuum.

2. Hereditarily Irreducible Mappings

The notion of an irreducible mapping was introduced by Whyburn [14, p. 162]. If X is a continuum, then a surjection $f : X \to Y$ is *irreducible* provided no proper subcontinuum of X maps onto all of Y under f.

A mapping $f : X \to Y$ is said to be *hereditarily irreducible* [11, (1.212.3), p. 204] provided that for any given subcontinuum Z of X, no proper subcontinuum of Z maps onto $f(Z)$.

Lemma 2.1. *If a hereditarily irreducible mapping $f : X \to Y$ is monotone, then it is one-to-one. Moreover, if f is a surjection, then f is a homeomorphism.*

A mapping $f : X \to Y$ is *light* (*zero-dimensional*) if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [4, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subsets of cardinality larger than one $(\dim f^{-1}(y) \leq 0)$. Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

An easy proof of the following lemma is left to the reader.

Lemma 2.2. *Every hereditarily irreducible mapping is light.*

A continuum X is called a *D-continuum* if for every pair C, D of its disjoint non-degenerate subcontinua there exists a subcontinuum $E \subset X$ such that $C \cap E \neq \varnothing \neq D \cap E$ and $(C \cup D) \setminus E \neq \varnothing$.

Lemma 2.3. *If X is an arcwise connected continuum, then X is a D-continuum.*

Proof. Let C, D be a pair of disjoint non-degenerate subcontinua of X. Take the points $c \in C$ and $d \in D$. There exists an arc L with the end points c and d. We have two cases. First, D is not a proper subset of L. Now, $E = C \cup L$ is a subcontinuum which contains C and $D \cap E$ is a non-empty proper subset of D. Secondly, let D be a proper subset of L. Then D is an arc with end points d and e. It is clear that e is not in C. Let $E = C \cup [c, e]$, where $[c, e]$ is a subarc of L with end points c and e. The continuum E contains C and $E \cap D = \{e\} \subset D$. Finally, we infer that X is a D-continuum.

Lemma 2.4. *If X is a locally connected continuum, then X is a D-continuum.*

Proof. Let C, D be a pair of disjoint non-degenerate subcontinua of X. Let d be a point of D. Let U_d be a connected neighborhood of d such that $Cl(U_d) \cap C = \emptyset$ and $D \setminus Cl(U_d) \neq \emptyset$. Set $U = X \setminus (Cl(U_d) \cup D)$. Because of the Boundary Bumping Theorem [12, p. 73, Theorem 5.4] there exists a component K of $Cl(U)$ such that $C \subset K$ and $K \cap Bd(U) \neq \emptyset$. If $K \cap C = \emptyset$, then $K \cap Cl(U_d) \neq \emptyset$. Set $E = C \cup K$. From $d \in U_d$, $Cl(U_d) \cap C = \emptyset$ and $D \setminus Cl(U_d) \neq \emptyset$ it follows that $C \subset E$, $E \cap D \neq \emptyset$ and $D \setminus E \neq \emptyset$. The required continuum E is constructed. If $K \cap Cl(U_d) = \emptyset$, then $K \cap (D \setminus Cl(U_d)) \neq \emptyset$. Set $E = K$. It follows that $C \subset E$, $E \cap D \neq \emptyset$ and $D \setminus E \neq \emptyset$ since $Cl(U_d) \subset D \setminus E$. Hence, X is a D-continuum.

3. Whitney Map and Hereditarily Irreducible Mappings

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [11, p. 24, (0.50)] we will mean any mapping $g : \Lambda \to [0, +\infty)$ satisfying

(a) if $A, B \in \Lambda$ such that $A \subset B$ and $A \neq B$, then $g(A) < g(B)$ and

(b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([11, pp. 24-26], [5, p. 106]). On the other hand, if X is non-metrizable, then it admits no Whitney map for 2^X [2]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [2].

The following theorem is an external characterization of non-metric continua which admit a Whitney map.

Theorem 3.1. *Let X be a non-metric continuum. Then X admits a Whitney map for $C(X)$ if and only if for each* σ*-directed inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *of continua which admit Whitney maps for $C(X_a)$ and* $X = \lim \mathbf{X}$ *there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b : \lim \mathbf{X} \to X_b$ is hereditarily irreducible and the projection* $C(p_b) : C(\lim \mathbf{X}) \to C(X_b)$ *is light.*

Proof. See [8, Theorem 2.3, p. 398]. The lightness of $C(p_b) : C(\lim X)$ $\to C(X_b)$ follows from the fact that, for a mapping $f : X \to Y$ of a continuum X into a continuum Y, $C(f) : C(X) \to C(Y)$ is light if and only if *f* is hereditarily irreducible [11, p. 204, (1.212.3)].

As a consequence of Theorem 3.1 we have the following result.

Corollary 3.2. *If a continuum X admits a Whitney map for $C^2(X)$, then it admits a Whitney map for $C(X)$.*

Proof. From Theorem 1.1 it follows that there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua and surjective bonding mappings such that X is homeomorphic to lim $\mathbf{X}$. Consider the inverse systems $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ whose limit is $C(X)$ and $C^2(\mathbf{X}) = \{C^2(X_a), C^2(p_{ab}), A\}$ whose limit is $C^2(X)$. If X admits a Whitney for $C^2(X)$, then by Theorem 3.1 there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $C(p_b) : C(\lim X) \to$ $C(X_b)$ is hereditarily irreducible. This means that $C(p_b)$ is light (Lemma 2.2). Using [11, (1.212.3), p. 204] we infer that p_b is hereditarily irreducible. Now, Theorem 3.1 completes the proof.

Theorem 3.3. *If a D-continuum X admits a Whitney map for $C(X)$, then $C(X) \setminus X(1)$ is metrizable and $w(C(X) \setminus X(1)) \leq \aleph_0$.*

Proof. It is clear that the theorem is true if X is a metric continuum. Let X be a non-metric continuum which admits a Whitney map for $C(X)$. From Theorem 1.1 it follows that there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua and surjective bonding mappings such that X is homeomorphic to $\lim \mathbf{X}$. Consider inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ whose limit is $C(X)$. From Theorem 3.1 it follows that the projections p_a are hereditarily irreducible and $C(p_a)$ are light. If $C(p_a)$ are one-to-one, then we have a homeomorphism $C(p_a)$ of $C(X)$ onto $C(p_a)(X)$. Since $C(p_a)(X)$ is metric, $C(X)$ is metrizable. It follows that X is metrizable since X is homeomorphic to $X(1)$. Suppose that $C(p_a)$ is not one-to-one. Then there exists a continuum C_a in X_a and two continua C, D in X such that $p_a(C) = p_a(D) = C_a$. It is impossible that $C \subset D$ or $D \subset C$ since p_a is hereditarily irreducible. Otherwise, if $C \cap D \neq \emptyset$, then for a continuum $Y = C \cup D$ we have that C and D are subcontinua of Y and $p_a(Y) = p_a(C) = p_a(D) = C_a$ which is impossible since p_a is hereditarily irreducible. We infer that $C \cap D = \emptyset$. There exists a subcontinuum E such that $C \subset E$, $D \neq D \cap E \neq \emptyset$ since X is a D-continuum. Now $p_a(E \cup D) = p_a(E)$ which is impossible since p_a is hereditarily irreducible. Furthermore, $C(p_a)^{-1}(X_a(1)) = X(1)$ since from the hereditarily irreducibility of p_a it follows that no non-degenerate subcontinuum of X maps under p_a onto a point. Let $Y_a = C(p_a)(C(X))$. We infer that $C(p_a)^{-1}[Y_a \setminus X_a(1)] = C(X) \setminus X(1)$. It follows that the restriction $P_a = C(p_a)|(C(X) \setminus X(1))$ is one-to-one and closed [4, Proposition 2.1.4, p. 95]. From $C(p_a)^{-1}[Y_a \setminus X_a(1)] = C(X) \setminus X(1)$ it follows that P_a is surjective. Hence, P_a is a homeomorphism and $C(X) \setminus X(1)$ is metrizable. Moreover, $w(C(X) \setminus X(1)) \leq \aleph_0$ since Y_a as a compact metrizable space is separable and, consequently, second-countable [4, p. 320].

It is known that if X is a continuum, then $C(X)$ is arcwise connected [9, p. 1209, Theorem]. Hence, using Lemma 2.3 and Theorem 3.3, we have the following corollary.

Corollary 3.4. *If X is a continuum which admits a Whitney map for the hyperspace $C^2(X)$, then $C^2(X) \setminus C(X)(1)$ is metrizable and $w(C^2(X) \setminus C(X)(1)) \leq \aleph_0$.*

Similarly, for D-continua, we have the following theorem.

Theorem 3.5. *If a D-continuum X admits a Whitney map for $C(X)$, then $C(C(X) \setminus X(1))$ is metrizable and $w(C(C(X) \setminus X(1))) \leq \aleph_0$.*

Proof. The theorem follows from Theorem 3.3 and the fact that if (X, d) is a metric space, then the hyperspace of all compact subsets of X in Vietoris topology is metrizable [5, Theorem 3.1, p. 16].

4. The Property of Kelley

We say that a continuum X has the *property of Kelley* at a point $p \in X$ if for every subcontinuum $K \subset X$ containing p and for every open neighborhood $\mathcal{U}$ of K in the hyperspace $C(X)$ there exists a neighborhood U of p in X such that if $q \in U$, then there exists a continuum $L \in C(X)$ with $q \in L \in \mathcal{U}$. A continuum X has the *property of Kelley* if it has the property of Kelley at each of its points.

For a given continuum X we define the function $\alpha_X : X \to C^2(X)$ by

$$\alpha_X(x) = \{A \in C(X) : x \in A\}$$

for each point $x \in X$ [3, p. 91].

It is clear that

$$\alpha_X(X) \subset C(C(X) \setminus X(1))$$

since $\alpha_X(x)$ is a continuum in $C(X)$ which contains $\{x\}$ and $\{X\}$, i.e., $\alpha_X(x)$ is a nondegenerate subcontinuum of $C(X)$. Moreover, if $x \neq y$, then $\alpha_X(x) \neq \alpha_X(y)$ since $\{x\} \notin \alpha_X(y)$.

Lemma 4.1. *The function α_X is upper semi-continuous.*

Proof. See [13, (2.1) Theorem, p. 292] or [1, (2.1) Proposition, p. 210].

Theorem 4.2 [3, Theorem 3.1, p. 92]**.** *The function* α_X *is continuous if and only if X has the property of Kelley*.

Hence, we have the following lemma.

Lemma 4.3. *The function* $\alpha_X : X \to C^2(X)$ *is an embedding if X has the property of Kelley*.

Now we are ready to prove the following theorem.

Theorem 4.4. *If a continuum X with the property of Kelley admits a Whitney map for* $C^2(X)$, *then it is metrizable.*

Proof. The hyperspace $C(X)$ is arcwise connected [9, Theorem, p. 1209,]. By Corollary 3.4 the space $C^2(X)\setminus C(X)(1)$ is metrizable and $w(C^2(X)\setminus C(X)(1)) \leq \aleph_0$. Using Lemma 4.3 we see that $\alpha_X(X) \subset C^2(X)\setminus C(X)(1)$ is metrizable. Moreover, X is homeomorphic to $\alpha_X(X)$. Hence, X is metrizable. Let us observe that X admits also a Whitney map for $C(X)$ (Corollary 3.2).

If a continuum X with the property of Kelley is a D-continuum, then we have the following theorem.

Theorem 4.5. *If a D-continuum X with the property of Kelley admits a Whitney map for* $C(X)$, *then it is metrizable.*

Proof. By virtue of Theorem 3.5 the space $C(C(X)\setminus X(1))$ is metrizable. Lemma 4.3 completes the proof.

Problem 1. Is it true that a continuum X with the property of Kelley is metrizable if it admits a Whitney map for $C(X)$?

A locally connected continuum is a D-continuum (Lemma 2.4) and has the property of Kelley [6, Theorem 9, p. 46]. Thus, we have the following theorem.

Theorem 4.6. *If a locally connected continuum X admits a Whitney map for* $C(X)$, *then X is metrizable.*

Remark. For another proof of this theorem see [7, Theorem 8, p. 4].

We say that a continuum X is *hereditarily indecomposable* if no subcontinuum of X can be written as the union of two proper subcontinua.

Lemma 4.7 [1, Proposition 2.7, p. 211]**.** *Hereditarily indecomposable continua have the property of Kelley.*

From Theorem 4.4 and Lemma 4.7 we obtain the following result.

Theorem 4.8. *If a hereditarily indecomposable continuum X admits a Whitney map for $C^2(X)$, then X is metrizable.*

An *arboroid* is a hereditarily unicoherent arcwise connected continuum. A metrizable arboroid is a *dendroid*.

We close this section with the following theorem.

Theorem 4.9. *Let X be an arboroid with the property of Kelley. Then X admits a Whitney map for $C(X)$ if and only if it is metrizable.*

Proof. Apply Theorem 4.5.

5. Concluding Remarks

It is known [4, Corollary 3.1.20, p. 171] that if a compact space X is the countable union of its subspaces X_n, $n \in \mathbb{N}$, such that $w(X_n) \leq \aleph_0$, then $w(X) \leq \aleph_0$. Using this fact and theorems proved in the previous section we obtain the following theorems.

Theorem 5.1. *Let a continuum X be the countable union of its locally connected subcontinua. Then X admits a Whitney map for $C(X)$ if and only if it is metrizable.*

Theorem 5.2. *If a continuum X is the countable union of its arcwise connected subcontinua with the property of Kelley, then X admits a Whitney map for $C(X)$ if and only if it is metrizable.*

Finally, applying Theorem 4.4 we obtain the following theorem.

Theorem 5.3. *If a continuum X is the countable union of its subcontinua with the property of Kelley and if X admits a Whitney map for $C^2(X)$, then X is metrizable.*

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