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FIRST-ORDER PERTURBED EIGENVALUE PROBLEMS IN A UNIFORM EXPANSION FOR FREDHOLM INTEGRAL EQUATIONS

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Abstract

In this paper, a simple and efficient symbolic computing algorithm will be established for the first-order uniform expansion of the perturbed eigenvalue problem for Fredholm integral equations. Applications of the algorithm are also given.

1. Introduction

Integral equations are encountered in various fields of science and numerous applications, in Elasticity [1], Plasticity [2], Oscillation theory [3], Electrostatics [4], Thermal radiation [5], Artificial satellite neural network

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[6], Stellar statistics [7], Helioseismology [8] and in most branches of science and engineering.

Most of the integral equations of these and other problems involve difficulties that preclude solving them exactly. Consequently, solutions are approximated using numerical techniques, analytic techniques and combination of both. Foremost among the analytical techniques are the systematic methods of perturbations [9].

The numerical methods provide very accurate solutions. But certainly, if full analytical formulae are utilized via symbol manipulating digital computer programs, then they definitely become invaluable for obtaining solutions for any desired accuracy. Moreover, symbolic computing algorithms for scientific problems in general [10] represent a new branch of numerical methods that we may call “*algorithmization*” of problems.

Coping with this line of recent researches, and also due to the importance of the integral equations, the present paper is devoted to establish symbolic computing algorithm for the first-order uniform expansion of the perturbed eigenvalue problem for Fredholm integral equations of the form

$$\Phi(s) = \lambda \int_a^b [K(s, t) + \varepsilon K_1(s, t)]\Phi(t)\,dt; \quad \varepsilon \ll 1, \tag{1}$$

where K and K_1 are continuous for $a \le s,\ t \le b.$ The developments are done by using the software Mathematica.

2. Basic Formulation

Recalling equation (1),

$$\Phi(s) = \lambda \int_a^b [K(s, t) + \varepsilon K_1(s, t)]\Phi(t)\,dt; \quad \varepsilon \ll 1.$$

Let

$$\Phi(s) = \Phi_0(s) + \varepsilon\Phi_1(s) + \cdots,$$

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \cdots.$$

Using these two equations into the first equation and equating coefficients of like powers of ε, we get

$$\Phi_0(s) = \lambda_0 \int_a^b K(s, t)\Phi_0(t)dt, \tag{2}$$

$$\Phi_1(s) = \lambda_0 \int_a^b K_1(s, t)\Phi_0(t)dt + \lambda_0 \int_a^b K(s, t)\Phi_1(t)dt$$

$$+ \lambda_1 \int_a^b K(s, t)\Phi_0(t)dt. \tag{3}$$

Let the kernel be expressible in the form

$$K(s, t) = \sum_{i=1}^{n} P_i(t)Q_i(s). \tag{4}$$

Even when the kernel is not of this form, one may approximate it by such a kernel and achieve an approximate solution to the integral equation.

Substituting equation (4) into equation (2), we get

$$\Phi_0(s) = \lambda_0 \sum_{j=1}^{n} Q_j(s)X_j, \tag{5}$$

where

$$X_j = \int_a^b P_j(t)\Phi_0(t)dt; \quad j = 1, 2, ..., n. \tag{6}$$

Multiplying equation (5) by $P_i(s)$ and integrating the resulting equation from $s = a$ to $s = b$, we get a typical matrix eigenvalue problem of the form

$$(\gamma - \mu I)X = 0, \tag{7}$$

where γ is a square matrix of order n whose elements are:

$$\gamma_{ij} = \int_a^b P_i(y)Q_j(y)dy; \quad i = 1, 2, ..., n; \; j = 1, 2, ..., n, \tag{8}$$

I is the unit matrix of order n, X is $(n, 1)$ column vector whose components are X_j of equation (6) and $\mu = 1/\lambda_0$.

By solving equation (7), we obtain the eigenvalues μ_i; $i = 1, 2, ..., n$ and the corresponding eigenvectors $X_{j,i}$; $j = 1, 2, ..., n$. Consequently, we can get from equation (5), the zero order solutions

$$\Phi_0^{(i)}(s) = \frac{1}{\mu}\sum_{j=1}^{n} Q_j(s) X_{j,i}; \quad i = 1, 2, ..., n. \tag{9}$$

Now, using equation (4) into equation (3) and using equation (5) into the resulting equation, we get

$$\Phi_1(s) = \frac{1}{\mu}\sum_{j=1}^{n} Q_j(s) Y_j + \lambda_1 \mu \Phi_0(s) + \frac{1}{\mu}\int_a^b K_1(s, t)\Phi_0(t)dt, \tag{10}$$

where

$$Y_j = \int_a^b P_j(t)\Phi_1(t)dt; \quad j = 1, 2, ..., n. \tag{11}$$

Multiply equation (10) by $P_i(s)$ and integrate the resulting equation from $s = a$ to $s = b$, to obtain the linear system

$$(\gamma - \mu I)Y = q, \tag{12}$$

where Y is $(n, 1)$ column vector whose components are Y_j of equation (11) and q is $(n, 1)$ column vector whose components

$$q_j = -\lambda_1 \mu^2 X_j - \alpha_j; \quad j = 1, 2, ..., n, \tag{13}$$

where

$$\alpha_k = \int_a^b\int_a^b K_1(s, t)\Phi_0(t)P_k(s)dtds; \quad k = 1, 2, ..., n. \tag{14}$$

Since the homogeneous equation of the system (12) are the same as equation (7) and since the latter have a nontrivial solution, the inhomogeneous equation (equation (12)) having a solution if and only if a solvability condition [9] is satisfied. This condition can be expressed as

$$\mathrm{Det}(C) = 0, \tag{15}$$

where the c_{ij} elements are given as

$$c_{ij} = \begin{cases} \gamma_{ij} - \mu, & i = j < n, \\ \gamma_{ij}, & i = 1, 2, ..., n;\ j = 1, 2, ..., n-1;\ j \neq i, \\ q_i, & j = n. \end{cases} \tag{16}$$

Finally, the solution of equation (15) yields the value of λ_1.

3. Computational Developments

3.1. Symbolic computing algorithm

- **Input:** $a, b, k, K_1(s, t), P_k(t)$, and $Q_k(s)$, $k = 1, ..., n$.
- **Output:** $\Phi_{0,i}(s) \equiv \Phi_i(s)$ and λ_i; $i = 1, ..., n$.
- **Computational steps:**

1. Compute $\gamma_{i,j}$, $\forall i = 1, ..., n;\ \ j = 1, ..., n$ from equation (8).

2. Solve the matrix eigenvalue problem of equation (7) for μ_i and $X_{j,i}$, $\forall i = 1, ..., n;\ \ j = 1, ..., n$.

3. Compute $\Phi_i(s)$, $\forall i = 1, ..., n$ from equation (9).

4. Construct $q_{i,k}$, $\forall i = 1, ..., n;\ \ k = 1, ..., n$ from

$$q_{i,k} = -\lambda_{1,k} * \mu_k^2 * X_{i,k} - \int_a^b \int_a^b K_1(s, t)\Phi_i(t)P_k(s)dtds.$$

5. Construct $\forall k = 1, ..., n$ the elements $c_{i,j}^{(k)}$, $\forall i = 1, ..., n;\ \ j = 1, ..., n$ of the matrix C^k from

$$c_{ij} = \begin{cases} \gamma_{ij} - \mu, & i = j < n, \\ \gamma_{ij}, & i = 1, 2, ..., n;\ j = 1, 2, ..., n-1;\ j \neq i, \\ q_i, & j = n. \end{cases}$$

6. Solve $\forall k = 1, \ldots, n$ the equation $\mathrm{Det}(C^k) = 0$ for $\lambda_{i,k}$.

7. Compute λ_i, $\forall i = 1, \ldots, n$ from $\lambda_i = \dfrac{1}{\mu_i} + \varepsilon\lambda_{1,i}$.

8. End.

4. Examples

We will illustrate two cases of examples. In the first case, a solution of simple examples solved analytically, in the second case, solutions of more complicated examples solved with an algorithm which derived in this paper.

The accuracy of computed eigenvalues and corresponding eigenvectors for each example is verified by the condition given by equation (7).

Case I.

Example I.1.

$$\phi(s) = \lambda \int_0^{\pi} [\cos(s + t) + \varepsilon K_1(s, t)]\phi(t)dt; \quad \varepsilon \prec\prec 1.$$

In this equation, we can rewrite $K(s, t) = P_1(t)Q_1(s) + P_2(t)Q_2(s)$, where

$$P_1(t) = \cos(t),\ P_2(t) = \sin(t),\ Q_1(t) = \cos(s),\ Q_2(t) = -\sin(s).$$

Then the zero order solution will be

$$\phi^{(1)} = \cos s + \cdots,$$

$$\phi^{(2)} = -\sin s + \cdots,$$

$$\lambda^{(1)} = \frac{2}{\pi} + \varepsilon\lambda_1^{(1)} + \cdots,$$

$$\lambda^{(2)} = -\frac{2}{\pi} + \varepsilon\lambda_1^{(2)} + \cdots.$$

Example I.2.

$$\phi(s) = \lambda \int_{-1}^{1} [st + s^2t^2 + \varepsilon K_1(s, t)]\phi(t)dt; \quad \varepsilon \prec\prec 1.$$

In this equation, we can rewrite $K(s,\ t) = P_1(t)Q_1(s) + P_2(t)Q_2(s),$ where

$$P_1(t) = t,\ P_2(t) = t^2,\ Q_1(t) = s,\ Q_2(t) = s^2.$$

Then the zero order solution will be

$$\phi^{(1)} = s + \cdots,$$

$$\phi^{(2)} = s^2 + \cdots,$$

$$\lambda^{(1)} = \frac{2}{3} + \varepsilon\lambda_1^{(1)} + \cdots,$$

$$\lambda^{(2)} = \frac{2}{5} + \varepsilon\lambda_1^{(2)} + \cdots.$$

Example I.3.

$$\phi(s) = \lambda\int_{-1}^{1}[1 - st + \varepsilon K_1(s,\ t)]\phi(t)dt;\quad \varepsilon \prec\prec 1.$$

In this equation, we can rewrite $K(s,\ t) = P_1(t)Q_1(s) + P_2(t)Q_2(s),$ where

$$P_1(t) = 1,\ P_2(t) = t,\ Q_1(t) = 1,\ Q_2(t) = -s.$$

Then the zero order solution will be

$$\phi^{(1)} = 1 + \cdots,$$

$$\phi^{(2)} = -s + \cdots,$$

$$\lambda^{(1)} = 2 + \varepsilon\lambda_1^{(1)} + \cdots,$$

$$\lambda^{(2)} = -\frac{2}{3} + \varepsilon\lambda_1^{(2)} + \cdots.$$

Case II.

Example II.1.

$$a = 0 \qquad b = 1 \qquad n = 7$$

$$K_1(s,\ t) = s + t \qquad P_k(t) = t^k \qquad Q_k(s) = s^k.$$

The first-order uniform expansion will be

$$\phi_1(y) = 1.05031(4096 \cdot y - 65536 \cdot y^2 + 524288 \cdot y^5 + 1 \cdot y^7),$$

$$\phi_2(y) = 15.0189(-1.66748y - 0.28125y^2 - 0.28125y^3 + 0.6875y^4$$
$$+ 0.4375y^5 + 1.35938y^6 + 1 \cdot y^7),$$

$$\phi_3(y) = 331.121(0.883536y - 1.00119y^2 - 0.916723y^3 - 0.401687y^4$$
$$+ 0.143828y^5 + 0.616711y^6 + 1 \cdot y^7),$$

$$\phi_4(y) = 10257.4(-0.371244y + 1.24373y^2 - 0.286352y^3 - 0.917939y^4$$
$$- 0.671157y^5 + 0.0624553y^6 + 1 \cdot y^7),$$

$$\phi_5(y) = 473024(0.129831y - 0.864511y^2 + 1.42161y^3 + 0.136241y^4$$
$$- 1.04792y^5 - 0.764683y^6 + 1 \cdot y^7),$$

$$\phi_6(y) = 3.63862 \times 10^7(-0.0391449y + 0.441708y^2 - 1.57324y^3$$
$$+ 1.98309y^4 + 0.110357y^5 - 1.92098y^6 + 1 \cdot y^7),$$

$$\phi_7(y) = 6.19068 \times 10^9(0.010434y - 0.182268y^2 + 1.11268y^3$$
$$- 3.20246y^4 + 4.72905y^5 - 3.46715y^6 + 1 \cdot y^7),$$

$$\lambda_1 = 1.05031 + \varepsilon 0.719722,$$

$$\lambda_2 = 15.0189 - \varepsilon 71.9125,$$

$$\lambda_3 = 331.121 - \varepsilon 1647.82,$$

$$\lambda_4 = 10257.4 - \varepsilon 79914.9,$$

$$\lambda_5 = 473024 - \varepsilon 5.3166 \times 10^6,$$

$$\lambda_6 = 3.63862 \times 10^7 - \varepsilon 3.9391 \times 10^{10},$$

$$\lambda_7 = 6.19068 \times 10^9 - \varepsilon 2.44379 \times 10^{15}.$$

Example II.2.

$$a = 0 \qquad b = 1 \qquad n = 7$$

$$K_1(s, t) = st + s^2t^2 \qquad P_k(t) = T_k(t) \qquad Q_k(s) = T_k(s),$$

where $P_k(t) = T_k(t)$ is Chebyshev polynomials. The first-order uniform expansion will be

$$\begin{aligned}\phi_1(y) = {} & 0.804082(0.161411y + 1.19053(-1 + 2y^2) + 0.443308(-3y + 4y^3) \\ & - 1.17274(1 - 8y^2 + 8y^4) - 1.30963(5y - 20y^3 + 16y^5) + 0.116939 \\ & \cdot (-1 + 18y^2 - 48y^4 + 32y^5) + 1 \cdot (-7y + 56y^3 - 112y^5 + 64y^7)),\end{aligned}$$

$$\begin{aligned}\phi_2(y) = {} & 0.91158(0.591731y - 0.797612(-1 + 2y^2) - 1.66118(-3y + 4y^3) \\ & - 1.0249(1 - 8y^2 + 8y^4) + 0.607919(5y - 20y^3 + 16y^5) + 1.57945 \\ & \cdot (-1 + 18y^2 - 48y^4 + 32y^5) + 1 \cdot (-7y + 56y^3 - 112y^5 + 64y^7)),\end{aligned}$$

$$\begin{aligned}\phi_3(y) = {} & 1.72132(-1.49283y + 0.352848(-1 + 2y^2) + 1.004(-3y + 4y^3) \\ & + 0.661632(1 - 8y^2 + 8y^4) + 0.763428(5y - 20y^3 + 16y^5) + 1.29579 \\ & \cdot (-1 + 18y^2 - 48y^4 + 32y^5) + 1 \cdot (-7y + 56y^3 - 112y^5 + 64y^7)),\end{aligned}$$

$$\begin{aligned}\phi_4(y) = {} & 4.20399(-1.46565y - 1.47238(-1 + 2y^2) - 0.516111(-3y + 4y^3) \\ & - 0.0816287(1 - 8y^2 + 8y^4) - 0.951547(5y - 20y^3 + 16y^5) - 1.05713 \\ & \cdot (-1 + 18y^2 - 48y^4 + 32y^5) + 1 \cdot (-7y + 56y^3 - 112y^5 + 64y^7)),\end{aligned}$$

$$\phi_5(y) = 9.40644(0.898233y - 0.160067(-1 + 2y^2) + 0.236792(-3y + 4y^3)$$

$$+ 0.783652(1 - 8y^2 + 8y^4) + 0.783374(5y - 20y^3 + 16y^5) - 0.323078$$

$$\cdot (-1 + 18y^2 - 48y^4 + 32y^5) + 1 \cdot (-7y + 56y^3 - 112y^5 + 64y^7)),$$

$$\phi_6(y) = 1127.78(-0.755087y + 1.91797(-1 + 2y^2) - 1.30097(-3y + 4y^3)$$

$$-0.787086(1 - 8y^2 + 8y^4) + 2.48067(5y - 20y^3 + 16y^5) - 2.21551$$

$$\cdot (-1 + 18y^2 - 48y^4 + 32y^5) + 1 \cdot (-7y + 56y^3 - 112y^5 + 64y^7)),$$

$$\phi_7(y) = 1.32686 \times 10^6(4.09567y - 12.6455(-1 + 2y^2)$$

$$+ 18.3254(-3y + 4y^3) - 17.0238(1 - 8y^2 + 8y^4)$$

$$+ 10.8192(5y - 20y^3 + 16y^5) - 4.49065$$

$$\cdot (-1 + 18y^2 - 48y^4 + 32y^5) + 1 \cdot (-7y + 56y^3 - 112y^5 + 64y^7)),$$

$$\lambda_1 = 0.804082 - \varepsilon 0.0127054,$$

$$\lambda_2 = 0.91158 - \varepsilon 0.0417932,$$

$$\lambda_3 = 1.72132 - \varepsilon 0.839906,$$

$$\lambda_4 = 4.20399 - \varepsilon 2.13598,$$

$$\lambda_5 = 9.40644 - \varepsilon 5.2173,$$

$$\lambda_6 = 1127.78 - \varepsilon 121.995,$$

$$\lambda_7 = 1.32686 \times 10^6 - \varepsilon 524073.$$

Example II.3.

$a = 0$ $\qquad$ $b = 1$ $\qquad$ $n = 7$

$K_1(s, t) = st + s^2t^2$ $\qquad$ $P_k(t) = T_k(t)$ $\qquad$ $Q_k(s) = s^k.$

The first-order uniform expansion will be

$$\begin{aligned}
\phi_1(y) = {} & (1.37563 - 4.40872i)((1.89591 + 8.52867i)\,y \\
& - (5.3052 - 1.28648i)\,y^2 - (7.1608 + 4.26732i)\,y^3 \\
& - (2.42742 + 5.16934i)\,y^4 + (2.63596 - 2.92744i)\,y^5 \\
& + (3.22233 - 0.714727i)\,y^6 + 1\cdot y^7),
\end{aligned}$$

$$\begin{aligned}
\phi_2(y) = {} & (1.37563 + 4.40872i)((1.89591 - 8.52867i)\,y \\
& - (5.3052 + 1.28648i)\,y^2 - (7.1608 - 4.26732i)\,y^3 \\
& - (2.42742 - 5.16934i)\,y^4 + (2.63596 + 2.92744i)\,y^5 \\
& + (3.22233 + 0.714727i)\,y^6 + 1\cdot y^7),
\end{aligned}$$

$$\begin{aligned}
\phi_3(y) = {} & (30.7199 - 13.803i)((-0.800901 - 0.480759i)\,y \\
& + (1.22471 + 0.381368i)\,y^2 + (1.12124 + 0.629951i)\,y^3 \\
& - (0.736412 - 0.120558i)\,y^4 - (1.56301 + 0.403366i)\,y^5 \\
& - (0.355097 + 0.370529i)\,y^6 + 1\cdot y^7),
\end{aligned}$$

$$\begin{aligned}
\phi_4(y) = {} & (30.7199 + 13.803i)((-0.800901 + 0.480759i)\,y \\
& + (1.22471 - 0.381368i)\,y^2 + (1.12124 - 0.629951i)\,y^3 \\
& - (0.736412 + 0.120558i)\,y^4 - (1.56301 - 0.403366i)\,y^5 \\
& - (0.355097 - 0.370529i)\,y^6 + 1\cdot y^7),
\end{aligned}$$

$$\begin{aligned}
\phi_5(y) = {} & 2742.93(0.0666607y - 0.392401y^2 + 0.365783y^3 \\
& + 0.731568y^4 - 0.549014y^5 - 1.21489y^6 + 1\cdot y^7),
\end{aligned}$$

$$\phi_6(y) = 257928(-0.0164y + 0.203932y^2 - 0.691756y^3$$
$$+ 0.481645y^4 + 1.31086y^5 - 2.28652y^6 + 1 \cdot y^7),$$

$$\phi_7(y) = 6.63704 \times 10^7(0.00554114y - 0.121659y^2 + 0.854538y^3$$
$$- 2.70782y^4 + 4.29096y^5 - 3.32112y^6 + 1 \cdot y^7),$$

$$\lambda_1 = 1.36563 - 4.40872i - \varepsilon(5.05872 + 1.64721i),$$

$$\lambda_2 = 1.36563 + 4.40872i - \varepsilon(5.05872 - 1.64721i),$$

$$\lambda_3 = 30.7199 - 13.803i - \varepsilon(-2.3689 + 39.8684i),$$

$$\lambda_4 = 30.7199 + 13.803i - \varepsilon(-2.3689 - 39.8684i),$$

$$\lambda_5 = 2742.93 + \varepsilon 130.435,$$

$$\lambda_6 = 257928 + \varepsilon 31737.5,$$

$$\lambda_7 = 6.63704 \times 10^7 + \varepsilon 9.02964 \times 10^6.$$

5. Conclusion

A symbolic computing algorithm was established for the first-order perturbed eigenvalue problems in a uniform expansion for Fredholm integral equations. The efficiency due to its ability in dealing with any functions $P's$ and $Q's$ can be considered as possible approximation functions for the kernel $K(s, t)$.

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