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GENERATING RELATIONS BETWEEN EXTON'S FUNCTIONS AND TRIPLE HYPERGEOMETRIC FUNCTION

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Abstract

In this paper, we aim at presenting the generating relations that involve between some Exton's functions and triple hypergeometric functions. We develop these relations via Laplace integral representations of Exton's functions $X_1, X_2, ..., X_{10}$. Also, we have recovered some of Exton's results through these generating relations as the particular cases of our results.

1. Introduction

The unification of generating functions has great importance in connection with ideas and principles of special functions. In this direction,

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some important steps have been made by researchers namely Singhal and Srivastava [1], Chatterjea [2, 3] and Chongdar [4]. Also, the special functions have great deal with applications in pure and applied mathematics. They appear in different frameworks and are used most frequently in combinatorial analysis [5], and even in statistics [6]. In their study, Desale and Qashash [7] have obtained a new general class of generating functions for the generalized modified Laguerre polynomials $L_n^{(\alpha)}(x)$ by group theoretic method. Also, they have introduced the bilateral generating function for the generalized modified Laguerre and Jacobi polynomials with the help of two linear partial differential operators. Further continuing their study [8, 9] they used the group theoretic method to obtain proper and improper partial bilateral as well as trilateral generating functions.

In connection with class of generating functions, we extend our ideas to obtain new generating relations that involve between Exton's functions and hypergeometric functions, in particularly Appell's functions, Lauricella function, Horn's functions, Saran's functions and Gaussian hypergeometric functions. We use integral form of Exton's functions (in the form of Laplace integral) to obtain the generating relations between Exton's functions and hypergeometric functions.

Exton in [10, 11] gave integral representations of some hypergeometric functions of three variables, which are denoted by $X_1, X_2, ..., X_{10}$ and defined as follows (cf. [10, 11]):

$$X_1(a, b; c, d; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+2n+p}(b)_p x^m y^n z^p}{(c)_m (d)_{n+p} m! n! p!}, \tag{1.1}$$

$$X_2(a, b; c_1, c_2, c_3; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+2n+p}(b)_p x^m y^n z^p}{(c_1)_m (c_2)_n (c_3)_p m! n! p!}, \tag{1.2}$$

$$X_3(a, b; c, d; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+2n+p}(b)_{n+p} x^m y^n z^p}{(c)_{m+n} (d)_p m! n! p!}, \tag{1.3}$$

$$X_4(a, b; c_1, c_2, c_3; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n+p}(b)_{n+p} x^m y^n z^p}{(c_1)_m (c_2)_n (c_3)_p m! n! p!}, \tag{1.4}$$

$$X_5(a, b_1, b_2; c; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n+p}(b_1)_n (b_2)_p x^m y^n z^p}{(c)_{m+n+p} m! n! p!}, \tag{1.5}$$

$$X_6(a, b_1, b_2; c, d; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n+p}(b_1)_n (b_2)_p x^m y^n z^p}{(c)_{m+n} (d)_p m! n! p!}, \tag{1.6}$$

$$X_7(a, b_1, b_2; c, d; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n+p}(b_1)_n (b_2)_p x^m y^n z^p}{(c)_m (d)_{n+p} m! n! p!}, \tag{1.7}$$

$$X_8(a, b_1, b_2; c_1, c_2, c_3; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n+p}(b_1)_n (b_2)_p x^m y^n z^p}{(c_1)_m (c_2)_n (c_3)_p m! n! p!}, \tag{1.8}$$

$$X_9(a, b; c; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+2p} x^m y^n z^p}{(c)_{m+n+p} m! n! p!}, \tag{1.9}$$

$$X_{10}(a, b; c, d; x, y, z) = \sum_{m, n, p=0}^{\infty} \frac{(a)_{2m+n}(b)_{n+2p} x^m y^n z^p}{(c)_{m+n} (d)_p m! n! p!}. \tag{1.10}$$

The integral representations of these Exton's functions in terms of Laplace integral are given in the following section.

2. Laplace Integral Representations

$$X_1(a, b; c, d; x, y, z)$$

$$= \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \exp(-s - t) s^{a-1} t^{b-1} {}_0F_1(-; c; xs^2)$$

$$\cdot\ {}_0F_1(-; d; ys^2 + zst) ds dt, \tag{2.1}$$

$$X_2(a, b; c_1, c_2, c_3; x, y, z) = \frac{1}{\Gamma(a)}\int_0^\infty \exp(-s)s^{a-1}\,{}_0F_1(-; c_1; xs^2)$$

$$\cdot\,{}_0F_1(-; c_2; ys^2)\,{}_1F_1(b; c_3; zs)ds, \tag{2.2}$$

$$X_3(a, b; c, d; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b)}\int_0^\infty\int_0^\infty \exp(-s-t)s^{a-1}t^{b-1}$$

$$\cdot\,{}_0F_1(-; c; xs^2 + yst)\,{}_0F_1(-; d; zst)dsdt, \tag{2.3}$$

$$X_4(a, b; c_1, c_2, c_3; x, y, z) = \frac{1}{\Gamma(a)}\int_0^\infty \exp(-s)s^{a-1}$$

$$\cdot\,{}_0F_1(-; c_1; xs^2)\Psi_2(b; c_2, c_3; ys, zs)ds, \tag{2.4}$$

$$X_5(a, b_1, b_2; c; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)}\int_0^\infty\int_0^\infty\int_0^\infty \exp(-s-t-u)$$

$$\cdot\, s^{a-1}t^{b_1-1}u^{b_2-1}\,{}_0F_1(-; c; xs^2 + yst + zsu)dsdtdu, \tag{2.5}$$

$$X_6(a, b_1, b_2; c, d; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b_1)}\int_0^\infty\int_0^\infty \exp(-s-t)s^{a-1}t^{b_1-1}$$

$$\cdot\,{}_0F_1(-; c; xs^2 + yst)\,{}_1F_1(b_2; d; zs)dsdt, \tag{2.6}$$

$$X_7(a, b_1, b_2; c, d; x, y, z) = \frac{1}{\Gamma(a)}\int_0^\infty \exp(-s)s^{a-1}\,{}_0F_1(-; c; xs^2)$$

$$\cdot\,\Phi_2(b_1, b_2; d; ys, zs)ds, \tag{2.7}$$

$$X_8(a, b_1, b_2; c_1, c_2, c_3; x, y, z) = \frac{1}{\Gamma(a)}\int_0^\infty \exp(-s)s^{a-1}\,{}_0F_1(-; c_1; xs^2)$$

$$\cdot\,{}_1F_1(b_1; c_2; ys)\,{}_1F_1(b_2; c_3; zs)ds, \tag{2.8}$$

$$X_9(a, b; c; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \exp(-s-t) s^{a-1} t^{b-1}$$

$$\cdot {}_0F_1(-; c; xs^2 + yst + zt^2) dsdt, \tag{2.9}$$

$$X_{10}(a, b; c, d; x, y, z) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \exp(-s-t) s^{a-1} t^{b-1}$$

$$\cdot {}_0F_1(-; c; xs^2 + yst) {}_0F_1(-; d; zt^2) dsdt. \tag{2.10}$$

We use these integral representations of Exton’s functions to obtain the new class of generating relations, which are summarized in the form of results (3.1) to (3.30) in the following section.

3. Results

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_1(\alpha+n, \beta+n; \gamma, \delta; x^2, y, z) = (1+2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1+2x}\right)^n$$

$$\cdot X_6\left(\alpha+n, \beta+n, \gamma-\frac{1}{2}; \delta, 2\gamma-1; \frac{y}{(1+2x)^2}, \frac{z}{1+2x}, \frac{4x}{1+2x}\right), \tag{3.1}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_1(\alpha+n, \beta+n; \gamma, \delta; x^2, y, z)$$

$$= \sum_{n, r=0}^{\infty} \frac{(\alpha+n)_{2r} w^n x^{2r}}{(\gamma)_r n! r!} H_3(\alpha+n+2r, \beta+n; \delta; y, z), \tag{3.2}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_2(\alpha+n, \beta; \gamma, \delta, \lambda; x^2, y^2, z)$$

$$= (1-z)^{-\beta} \sum_{n=0}^{\infty} \frac{w^n}{n!} \cdot F_4\left(\frac{\alpha+n}{2}, \frac{\alpha+n+1}{2}; \gamma, \delta; 4x^2, 4y^2\right), \tag{3.3}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_2(\alpha+n,\ \beta;\ \gamma,\ \delta,\ \lambda;\ x,\ y^2,\ z)=(1+2y)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2y}\right)^n$$

$$\cdot X_8\left(\alpha+n,\ \beta,\ \delta-\frac{1}{2};\ \gamma,\ \lambda,\ 2\delta-1;\ \frac{x}{(1+2y)^2},\ \frac{z}{1+2y},\ \frac{4y}{1+2y}\right), \quad (3.4)$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_2(\alpha+n,\ \beta;\ \gamma,\ \delta,\ \lambda;\ x^2,\ y^2,\ z)$$

$$=(1+2x+2y)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2x+2y}\right)^n$$

$$\cdot F_A^{(3)}\left(\alpha+n,\ \gamma-\frac{1}{2},\ \delta-\frac{1}{2},\ \beta;\ 2\gamma-1,\ 2\delta-1,\ \lambda;\right.$$

$$\left.\frac{4x}{1+2x+2y},\ \frac{4y}{1+2x+2y},\ \frac{z}{1+2x+2y}\right), \quad (3.5)$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_2(\alpha+n,\ \beta;\ \gamma,\ \delta,\ \lambda;\ x,\ y,\ z)=(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-z}\right)^n$$

$$\cdot X_2\left(\alpha+n,\ \lambda-\beta;\ \gamma,\ \delta,\ \lambda;\ \frac{x}{(1-z)^2},\ \frac{y}{(1-z)^2},\ \frac{z}{z-1}\right), \quad (3.6)$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_3(\alpha+n,\ \beta+n;\ \gamma,\ \delta;\ x,\ y,\ zk)$$

$$=(1+z)^{-\alpha}(1+k)^{-\beta}\sum_{n,r=0}^{\infty}\frac{(\beta+n)_r}{n!\,r!}\left(\frac{w}{(1+z)(1+k)}\right)^n\left(\frac{k}{1+k}\right)^r$$

$$\cdot X_6\left(\alpha+n,\ \beta+n+r,\ \delta+r;\ \gamma,\ \delta;\ \frac{x}{(1+z)^2},\ \frac{y}{(1+z)(1+k)},\ \frac{z}{1+z}\right), \quad (3.7)$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_4(\alpha+n,\ \beta;\ \gamma,\ \delta,\ \lambda;\ x^2,\ y,\ z)=(1+2x)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2x}\right)^n$$

$$\cdot F_E\left(\alpha+n,\ \alpha+n,\ \alpha+n,\ \gamma-\frac{1}{2},\ \beta,\ \beta;\ 2\gamma-1,\ \delta,\ \lambda;\right.$$

$$\left.\frac{4x}{1+2x},\ \frac{y}{1+2x},\ \frac{z}{1+2x}\right), \tag{3.8}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_4(\alpha+n,\ \beta;\ \gamma,\ \beta,\ \beta;\ x,\ y,\ z)=(1-y-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-y-z}\right)^n$$

$$\cdot F_4\left(\frac{\alpha+n}{2},\ \frac{\alpha+n+1}{2};\ \gamma,\ \beta;\ \frac{4x}{(1-y-z)^2},\ \frac{4yz}{(1-y-z)^2}\right), \tag{3.9}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_4(\alpha+n,\ \beta;\ \gamma,\ \beta,\ \beta;\ x^2,\ y,\ y)$$

$$=(1+2x-2y)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2x-2y}\right)^n$$

$$\cdot H_4\left(\alpha+n,\ \gamma-\frac{1}{2};\ \beta,\ 2\gamma-1;\ \left(\frac{y}{1+2x-2y}\right)^2,\ \frac{4x}{1+2x-2y}\right), \tag{3.10}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_4(\alpha+n,\ \beta;\ \gamma,\ \beta,\ \beta;\ x^2,\ y,\ y)=(1+2x)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2x}\right)^n$$

$$\cdot F_2\left(\alpha+n,\ \beta-\frac{1}{2},\ \gamma-\frac{1}{2};\ 2\beta-1,\ 2\gamma-1;\ \frac{4y}{1+2x},\ \frac{4x}{1+2x}\right), \tag{3.11}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_6(\alpha+n,\ \beta+n,\ \gamma;\ \varepsilon,\ \delta;\ x,\ y,\ z)=(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-z}\right)^n$$

$$\cdot X_6\left(\alpha+n,\ \beta+n,\ \delta-\gamma;\ \varepsilon,\ \delta;\ \frac{x}{(1-z)^2},\ \frac{y}{1-z},\ \frac{z}{z-1}\right), \tag{3.12}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_6(\alpha+n,\ \beta+n,\ \gamma;\ \varepsilon,\ \delta;\ x,\ y,\ z)=(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-z}\right)^n$$

$$\cdot H_3\left(\alpha+n, \beta+n; \varepsilon; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right){}_2F_1\left(\delta-\gamma, \alpha+n+2p+q; \delta; \frac{z}{z-1}\right), \tag{3.13}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_6(\alpha+n, \beta+n, \gamma; \delta, \gamma; x, y, z) = (1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-z}\right)^n$$

$$\cdot H_3\left(\alpha+n, \beta+n; \delta; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right), \tag{3.14}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_7(\alpha+n, \beta, \gamma; \delta, \lambda; x^2, y, z) = (1+2x)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2x}\right)^n$$

$$\cdot F_G\left(\alpha+n, \alpha+n, \alpha+n, \delta-\frac{1}{2}, \beta, \gamma; 2\delta-1, \lambda, \lambda;\right.$$

$$\left.\frac{4x}{1+2x}, \frac{y}{1+2x}, \frac{z}{1+2x}\right), \tag{3.15}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_7(\alpha+n, \beta, \delta-\beta; \gamma, \delta; x, y, z) = (1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-z}\right)^n$$

$$\cdot H_4\left(\alpha+n, \beta; \gamma, \delta; \frac{x}{(1-z)^2}, \frac{y-z}{1-z}\right), \tag{3.16}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_7(\alpha+n, \beta, \delta-\beta; \gamma, \delta; x^2, y, z) = (1-z+2x)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}$$

$$\cdot\left(\frac{w}{1-z+2x}\right)^n F_2\left(\alpha+n, \beta, \gamma-\frac{1}{2}; \delta, 2\gamma-1; \frac{y-z}{1-z+2x}, \frac{4x}{1-z+2x}\right), \tag{3.17}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n, \beta, \gamma; \delta, \lambda, \varepsilon; x^2, y, z) = (1+2x)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2x}\right)^n$$

$$\cdot F_A^{(3)}\left(\alpha+n, \delta-\frac{1}{2}, \beta, \gamma; 2\delta-1, \lambda, \varepsilon; \frac{4x}{1+2x}, \frac{y}{1+2x}, \frac{z}{1+2x}\right), \tag{3.18}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \varepsilon;\ x,\ y,\ z)=(1-y)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-y}\right)^n$$

$$\cdot X_8\left(\alpha+n,\ \lambda-\beta,\ \gamma;\ \delta,\ \lambda,\ \varepsilon;\ \frac{x}{(1-y)^2},\ \frac{y}{y-1},\ \frac{z}{1-y}\right), \tag{3.19}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \varepsilon;\ x,\ y,\ z)=(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-z}\right)^n$$

$$\cdot X_8\left(\alpha+n,\ \beta,\ \varepsilon-\gamma;\ \delta,\ \lambda,\ \varepsilon;\ \frac{x}{(1-z)^2},\ \frac{y}{1-z},\ \frac{z}{z-1}\right), \tag{3.20}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \varepsilon;\ x,\ y,\ z)$$

$$=(1-y-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-y-z}\right)^n$$

$$\cdot X_8\left(\alpha+n,\ \lambda-\beta,\ \varepsilon-\gamma;\ \delta,\ \lambda,\ \varepsilon;\ \frac{x}{(1-y-z)^2},\ \frac{y}{y+z-1},\ \frac{z}{y+z-1}\right), \tag{3.21}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n,\ \beta,\ \gamma;\ \delta,\ \beta,\ \lambda;\ x,\ y,\ z)=(1-y)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-y}\right)^n$$

$$\cdot H_4\left(\alpha+n,\ \gamma;\ \delta,\ \lambda;\ \frac{x}{(1-y)^2},\ \frac{z}{1-y}\right), \tag{3.22}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \gamma;\ x,\ y,\ z)=(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-z}\right)^n$$

$$\cdot H_4\left(\alpha+n,\ \beta;\ \delta,\ \lambda;\ \frac{x}{(1-z)^2},\ \frac{y}{1-z}\right), \tag{3.23}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n,\ \beta,\ \gamma;\ \delta,\ \beta,\ \gamma;\ x,\ y,\ z)=(1-y-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-y-z}\right)^n$$

$$\cdot\ {}_2F_1\left(\frac{\alpha+n}{2}, \frac{\alpha+n+1}{2}; \delta; \frac{4x}{(1-y-z)^2}\right), \tag{3.24}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n, \beta, \gamma; \delta, \beta, \gamma; x^2, y, z) = (1+2x-y-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}$$

$$\cdot\left(\frac{w}{1+2x-y-z}\right)^n {}_2F_1\left(\alpha+n, \delta-\frac{1}{2}; 2\delta-1; \frac{4x}{1+2x-y-z}\right), \tag{3.25}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n, \beta, \gamma; \delta, \beta, \gamma; x^2, y, y) = (1+2x-2y)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}$$

$$\cdot\left(\frac{w}{1+2x-2y}\right)^n {}_2F_1\left(\alpha+n, \delta-\frac{1}{2}; 2\delta-1; \frac{4x}{1+2x-2y}\right), \tag{3.26}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_9(\alpha+n, \beta+n; \gamma; x^2, 2xy, y^2) = (1+2x)^{-\alpha}(1+2y)^{-\beta}\sum_{n=0}^{\infty}\frac{1}{n!}$$

$$\cdot\left(\frac{w}{(1+2x)(1+2y)}\right)^n F_1\left(\gamma-\frac{1}{2}, \alpha+n, \beta+n; 2\gamma-1; \frac{4x}{1+2x}, \frac{4y}{1+2y}\right), \tag{3.27}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_9(\alpha+n, \beta+n; \gamma; x^2, 2x^2, x^2) = (1+2x)^{-(\alpha+\beta)}\sum_{n=0}^{\infty}\frac{1}{n!}$$

$$\cdot\left(\frac{w}{(1+2x)^2}\right)^n {}_2F_1\left(\gamma-\frac{1}{2}, \alpha+\beta+2n; 2\gamma-1; \frac{4x}{1+2x}\right), \tag{3.28}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_9(\alpha+n, 2\gamma-\alpha-n-1; \gamma; x^2, 2x^2, x^2)$$

$$= (1+2x)^{\frac{1}{2}-\gamma}(1-2x)^{\frac{1}{2}-\gamma}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{(1+2x)^2}\right)^n, \tag{3.29}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_{10}(\alpha + n, \beta + n; \gamma, \delta; x, y, z^2) = (1 + 2z)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{w}{(1 + 2z)}\right)^n$$

$$\cdot X_{14}\left(\alpha + n, \beta + n, \delta - \frac{1}{2}; \gamma, 2\delta - 1; x, \frac{y}{1 + 2z}, \frac{4z}{1 + 2z}\right). \tag{3.30}$$

In all the above relations, F_1, F_2, F_3, F_4 are the Appell's functions, $F_A^{(3)}$ is the three variables Lauricella function, ${}_2F_1$ is called *Gaussian hypergeometric function*, H_3, H_4 are Horn's functions and F_E, F_G are Saran's functions of three variables. For more details about these functions, one may refer books of Srivastava [12, 13].

4. Proofs of Results

To prove the above results, we need the following formulae (cf. [12, 13, 14] and [15]):

$$\Psi_2(c; c, c; x, y) = \exp(x + y)\,{}_0F_1(-; c; xy), \tag{4.1}$$

$$\Phi_2(a, c - a; c; x, y) = \exp(y)\,{}_1F_1(a; c; x - y), \tag{4.2}$$

$${}_1F_1(a; c; x) = \exp(x)\,{}_1F_1(c - a; c; -x), \tag{4.3}$$

$${}_0F_1(-; c; x^2) = \exp(-2x)\,{}_1F_1\left(c - \frac{1}{2}; 2c - 1; 4x\right), \tag{4.4}$$

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}, \quad \lambda \neq 0, -1, -2, ..., \tag{4.5}$$

$$(\lambda)_{m+n} = (\lambda)_m(\lambda + m)_n, \tag{4.6}$$

$$(\lambda)_{2n} = 2^{2n}\left(\frac{\lambda}{2}\right)_n\left(\frac{\lambda + 1}{2}\right)_n, \quad n = 0, 1, 2, ..., \tag{4.7}$$

$$(n - k)! = \frac{(-1)^k n!}{(-n)_k}, \quad 0 \leq k \leq n, \tag{4.8}$$

$$L\{t^{2\sigma-1}{}_mF_n(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_n; \lambda^2t^2)\} = \Gamma(2\sigma)\, p^{-2\sigma}$$

$$\cdot\ {}_{m+2}F_n\left(\alpha_1, \ldots, \alpha_n, \frac{\sigma}{2}, \frac{\sigma+1}{2}; \beta_1, \ldots, \beta_n; 4\lambda^2 p^{-2}\right), \tag{4.9}$$

$$L\{t^{\gamma-1}{}_1F_1(\alpha, \gamma; \lambda t)\} = \Gamma(\gamma)\, p^{\alpha-\gamma}(p-\lambda)^{-\alpha},$$

$$\mathrm{Re}(\gamma) > 0, \quad \mathrm{Re}(p) > 0, \quad \mathrm{Re}(\lambda) > 0, \tag{4.10}$$

$$L\{t^{v}\} = \Gamma(v+1)\, p^{-v-1}, \quad \mathrm{Re}(v) > -1, \quad \mathrm{Re}(p) > 0, \tag{4.11}$$

$$L\{x^{\mu}{}_1F_1(a_1; b_1; \sigma x)_1F_1(a; b; \omega x)\}$$

$$= \Gamma(\mu+1)\, p^{-\mu-1}F_2\left(\mu+1, a, a_1; b, b_1; \frac{\omega}{p}, \frac{\sigma}{p}\right), \tag{4.12}$$

$$\mathrm{Re}(\mu) > -1, \ \ \mathrm{Re}(p-\sigma), \ \ \mathrm{Re}(p-\omega), \ \ \mathrm{Re}(p-\sigma-\omega) > 0,$$

where L is the Laplace transform, Φ_2, Ψ_2 are Humbert's functions, ${}_1F_1$ is Kummer's function (or confluent hypergeometric function) and ${}_mF_n$ is generalized hypergeometric function.

Proof of result (3.1). Let us denote the left hand side of (3.1) by I and using (2.1), we have

$$I = \sum_{n=0}^{\infty} \frac{w^n}{n!\,\Gamma(\alpha+n)\Gamma(\beta+n)} \int_0^{\infty}\int_0^{\infty} \exp(-s-t)\, s^{\alpha+n-1}t^{\beta+n-1}$$

$$\cdot\ {}_0F_1(-; \gamma; x^2s^2)\,{}_0F_1(-; \delta; ys^2+zst)\, dsdt$$

and then using (4.4), we get

$$I = \sum_{n=0}^{\infty} \frac{w^n}{n!\,\Gamma(\alpha+n)\Gamma(\beta+n)} \int_0^{\infty}\int_0^{\infty} \exp(-s(1+2x))\exp(-t)$$

$$\cdot\ s^{\alpha+n-1}t^{\beta+n-1}{}_1F_1\left(\gamma-\frac{1}{2}; 2\gamma-1; 4xs\right){}_0F_1(-; \delta; ys^2+zst)\, dsdt.$$

The functions ${}_1F_1$ and ${}_0F_1$ in the integrand can be written in its series form, and then interchanging the order of summation as well as the order of integral (which is permissible here), we get

$$I = \sum_{n,\,p,\,q,\,r=0}^{\infty} \frac{\left(\gamma - \frac{1}{2}\right)_p (4x)^p w^n y^q z^r}{(2\gamma-1)_p (\delta)_{q+r}\, n!\, p!\, q!\, r!\, \Gamma(\alpha+n)\Gamma(\beta+n)}$$

$$\cdot \int_0^{\infty}\int_0^{\infty} \exp(-s(1+2x))\exp(-t)\, s^{\alpha+n+p+2q+r-1} t^{\beta+n+r-1} ds dt.$$

Now, use of (4.11) and (4.5) in above equation and then simplifying with series manipulation completes the proof of result (3.1).

Remark 1. The proofs of all results run in the same way, considering the appropriate integral representation and Laplace transform during the proof.

5. Special Cases

Some generating relations, which believed to be new, can be established in this section as the special cases of the results were obtained in previous section.

1. Put $y = 0$ and $z = 0$ independently in (3.1), we obtain new generating relations between Horn's function and Appell's function:

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha+n,\, \beta+n;\, \gamma,\, \delta;\, x^2,\, z) = (1+2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{w}{1+2x}\right)^n$$

$$\cdot F_2\left(\alpha+n,\, \gamma - \frac{1}{2},\, \beta+n;\, 2\gamma-1,\, \delta;\, \frac{4x}{1+2x},\, \frac{z}{1+2x}\right), \tag{5.1}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_4\left(\frac{\alpha+n}{2},\, \frac{\alpha+n+1}{2};\, \gamma,\, \delta;\, 4x^2,\, 4y\right) = (1+2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\cdot \left(\frac{w}{1+2x}\right)^n H_4\left(\alpha+n,\, \gamma - \frac{1}{2};\, \delta,\, 2\gamma-1;\, \frac{y}{(1+2x)^2},\, \frac{4x}{1+2x}\right). \tag{5.2}$$

2. If we put $y = 0$ in (3.3), then we have generating relation between

Horn's function H_4 and Gaussian hypergeometric function ${}_2F_1$:

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta; \gamma, \lambda; x^2, z)$$

$$= (1-z)^{-\beta} \sum_{n=0}^{\infty} \frac{w^n}{n!} {}_2F_1\left(\frac{\alpha + n}{2}, \frac{\alpha + n + 1}{2}; \gamma; 4x^2\right). \tag{5.3}$$

3. Put $z = 0$ in (3.5) and $x = 0$ in (3.7), we have the following generating relations between two Appell's functions:

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_4\left(\frac{\alpha + n}{2}, \frac{\alpha + n + 1}{2}; \gamma, \delta; 4x^2, 4y^2\right)$$

$$= (1 + 2x + 2y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2x + 2y}\right)^n$$

$$\cdot F_2\left(\alpha + n, \gamma - \frac{1}{2}, \delta - \frac{1}{2}; 2\gamma - 1, 2\delta - 1; \frac{4x}{1 + 2x + 2y}, \frac{4y}{1 + 2x + 2y}\right), \tag{5.4}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_4(\alpha + n, \beta + n; \gamma, \delta; y, zk)$$

$$= (1+z)^{-\alpha}(1+k)^{-\beta} \sum_{n, r=0}^{\infty} \frac{(\beta + n)_r}{n!\, r!} \left(\frac{w}{(1+z)(1+k)}\right)^n \left(\frac{k}{1+k}\right)^r$$

$$\cdot F_2\left(\alpha + n, \beta + n + r, \delta + r; \gamma, \delta; \frac{y}{(1+z)(1+k)}, \frac{z}{1+z}\right). \tag{5.5}$$

4. Put $x = 0$ in equations (3.4), (3.9), (3.10), (3.12), (3.13), (3.16), (3.21), (3.24) and (3.30), we obtain the following generating relations:

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta; \delta, \lambda; y^2, z) = (1+2y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1+2y}\right)^n$$

$$\cdot F_2\left(\alpha + n, \beta, \delta - \frac{1}{2}; \lambda, 2\delta - 1; \frac{z}{1+2y}, \frac{4y}{1+2y}\right), \tag{5.6}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_4(\alpha+n,\ \beta;\ \beta,\ \beta;\ y,\ z) = (1-y-z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1-y-z} \right)^n$$

$$\cdot\ {}_2F_1\left(\frac{\alpha+n}{2},\ \frac{\alpha+n+1}{2};\ \beta;\ \frac{4yz}{(1-y-z)^2} \right), \tag{5.7}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_4(\alpha+n,\ \beta;\ \beta,\ \beta;\ y,\ y) = (1-2y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1-2y} \right)^n$$

$$\cdot\ {}_2F_1\left(\frac{\alpha+n}{2},\ \frac{\alpha+n+1}{2};\ \beta;\ \left(\frac{2y}{1-2y} \right)^2 \right), \tag{5.8}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_2(\alpha+n,\ \beta+n,\ \gamma;\ \varepsilon,\ \delta;\ y,\ z) = (1-z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1-z} \right)^n$$

$$\cdot\ F_2\left(\alpha+n,\ \beta+n,\ \delta-\gamma;\ \varepsilon,\ \delta;\ \frac{y}{1-z},\ \frac{z}{z-1} \right), \tag{5.9}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_2(\alpha+n,\ \beta+n,\ \gamma;\ \varepsilon,\ \delta;\ y,\ z) = (1-z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1-z} \right)^n$$

$$\cdot\ {}_2F_1\left(\alpha+n,\ \beta+n;\ \varepsilon;\ \frac{y}{1-z} \right) {}_2F_1\left(\delta-\gamma,\ \alpha+n+q;\ \delta;\ \frac{z}{z-1} \right), \tag{5.10}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_1(\alpha+n,\ \beta,\ \delta-\beta;\ \delta;\ y,\ z) = (1-z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1-z} \right)^n$$

$$\cdot\ {}_2F_1\left(\alpha+n,\ \beta;\ \delta;\ \frac{y-z}{1-z} \right), \tag{5.11}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_2(\alpha+n,\ \beta,\ \gamma;\ \lambda,\ \varepsilon;\ y,\ z) = (1-y-z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1-y-z} \right)^n$$

$$\cdot\ F_2\left(\alpha+n,\ \lambda-\beta,\ \varepsilon-\gamma;\ \lambda,\ \varepsilon;\ \frac{y}{y+z-1},\ \frac{z}{y+z-1} \right), \tag{5.12}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} F_2(\alpha + n, \beta, \gamma; \beta, \gamma; y, z) = (1 - y - z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 - y - z} \right)^n, \tag{5.13}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta + n; \gamma, \delta; y, z^2) = (1 + 2z)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1 + 2z)} \right)^n$$

$$\cdot F_2\left(\beta + n, \alpha + n, \delta - \frac{1}{2}; \gamma, 2\delta - 1; \frac{y}{1 + 2z}, \frac{4z}{1 + 2z} \right). \tag{5.14}$$

5. If we put $y = 0$ into equations (3.8), (3.11), (3.12), (3.13), (3.17), (3.18), (3.20), (3.23) and (3.26), then we obtain the following relations:

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta; \gamma, \lambda; x^2, z) = (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\cdot \left(\frac{w}{1 + 2x} \right)^n F_2\left(\alpha + n, \gamma - \frac{1}{2}, \beta; 2\gamma - 1, \lambda; \frac{4x}{1 + 2x}, \frac{z}{1 + 2x} \right), \tag{5.15}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} {}_2F_1\left(\frac{\alpha + n}{2}, \frac{\alpha + n + 1}{2}; \gamma; 4x^2 \right) = (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2x} \right)^n$$

$$\cdot {}_2F_1\left(\alpha + n, \gamma - \frac{1}{2}; 2\gamma - 1; \frac{4x}{1 + 2x} \right), \tag{5.16}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \gamma; \varepsilon, \delta; x, z) = (1 - z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 - z} \right)^n$$

$$\cdot {}_2F_1\left(\frac{\alpha + n}{2}, \frac{\alpha + n + 1}{2}; \varepsilon; \frac{4x}{(1 - z)^2} \right) {}_2F_1\left(\delta - \gamma, \alpha + n + 2p; \delta; \frac{z}{z - 1} \right), \tag{5.17}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \gamma; \varepsilon, \delta; x, z) = (1 - z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 - z} \right)^n$$

$$\cdot H_4\left(\alpha + n, \delta - \gamma; \varepsilon, \delta; \frac{x}{(1 - z)^2}, \frac{z}{z - 1} \right), \tag{5.18}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n,\ \delta - \beta;\ \gamma,\ \delta;\ x^2,\ z) = (1 - z + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 - z + 2x} \right)^n$$

$$\cdot F_2\left(\alpha + n,\ \beta,\ \gamma - \frac{1}{2};\ \delta,\ 2\gamma - 1;\ \frac{z}{z - 2x - 1},\ \frac{4x}{1 - z + 2x} \right), \tag{5.19}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n,\ \gamma;\ \delta,\ \varepsilon;\ x^2,\ z) = (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2x} \right)^n$$

$$\cdot F_2\left(\alpha + n,\ \delta - \frac{1}{2},\ \gamma;\ 2\delta - 1,\ \varepsilon;\ \frac{4x}{1 + 2x},\ \frac{z}{1 + 2x} \right), \tag{5.20}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n,\ \gamma;\ \delta,\ \varepsilon;\ x,\ z) = (1 - z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 - z} \right)^n$$

$$\cdot H_4\left(\alpha + n,\ \varepsilon - \gamma;\ \delta,\ \varepsilon;\ \frac{x}{(1 - z)^2},\ \frac{z}{z - 1} \right), \tag{5.21}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n,\ \gamma;\ \delta,\ \gamma;\ x,\ z) = (1 - z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 - z} \right)^n$$

$$\cdot {}_2F_1\left(\frac{\alpha + n}{2},\ \frac{\alpha + n + 1}{2};\ \delta;\ \frac{4x^2}{(1 - z)^2} \right), \tag{5.22}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} {}_2F_1\left(\frac{\alpha + n}{2},\ \frac{\alpha + n + 1}{2};\ \delta;\ 4x^2 \right) = (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2x} \right)^n$$

$$\cdot {}_2F_1\left(\alpha + n,\ \delta - \frac{1}{2};\ 2\delta - 1;\ \frac{4x}{1 + 2x} \right). \tag{5.23}$$

6. Now put $z = 0$ in (3.15), (3.19), (3.22), (3.25) and (3.28), we obtain the following relations:

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n,\ \beta;\ \delta,\ \lambda;\ x^2,\ y) = (1 + 2x)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2x} \right)^n$$

$$\cdot F_2\left(\alpha + n,\ \delta - \frac{1}{2},\ \beta;\ 2\delta - 1,\ \lambda;\ \frac{4x}{1 + 2x},\ \frac{y}{1 + 2x} \right), \tag{5.24}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta; \delta, \lambda; x, y) = (1-y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1-y}\right)^n$$

$$\cdot H_4\left(\alpha + n, \lambda - \beta; \delta, \lambda; \frac{x}{(1-y)^2}, \frac{y}{y-1}\right), \tag{5.25}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta; \delta, \beta; x, y) = (1-y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1-y}\right)^n$$

$$\cdot {}_2F_1\left(\frac{\alpha + n}{2}, \frac{\alpha + n + 1}{2}; \delta; \frac{4x}{(1-y)^2}\right), \tag{5.26}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta; \delta, \beta; x^2, y) = (1 + 2x - y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1 + 2x - y}\right)^n$$

$$\cdot {}_2F_1\left(\alpha + n, \delta - \frac{1}{2}; 2\delta - 1; \frac{4x}{1 + 2x - y}\right). \tag{5.27}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_4(\alpha + n, \beta + n; \gamma, \delta; x, yk)$$

$$= (1+y)^{-\alpha}(1+k)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1+y)(1+k)}\right)^n$$

$$\cdot X_{17}\left(\alpha + n, \delta, \beta + n; \gamma, \delta, \delta; \frac{x}{(1+y)^2}, \frac{y}{1+y}, \frac{k}{1+k}\right). \tag{5.28}$$

7. Now assign the value to x and z to be zero in (3.29), we get the relation

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} {}_2F_1(\alpha + n, \beta + n; \delta; yk)$$

$$= (1+y)^{-\alpha}(1+k)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{(1+y)(1+k)}\right)^n$$

$$\cdot F_2\left(\delta, \alpha + n, \beta + n; \delta, \delta; \frac{y}{1+y}, \frac{k}{1+k}\right). \tag{5.29}$$

8. Here we determine some new relations, which are obtained by substituting $n = 0$ into equations (3.2), (3.3), (3.7), (3.10), (3.11), (3.13), (3.17), (3.22), (3.24), (3.25) and (3.26):

$$X_1(\alpha, \beta; \gamma, \delta; x^2, y, z) = \sum_{r=0}^{\infty} \frac{(\alpha)_{2r} x^{2r}}{(\gamma)_r r!} H_3(\alpha + 2r, \beta; \delta; y, z), \tag{5.30}$$

$$X_2(\alpha, \beta; \gamma, \delta, \lambda; x^2, y^2, z) = (1-z)^{-\beta} F_4\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \gamma, \delta; 4x^2, 4y^2\right), \tag{5.31}$$

$$X_3(\alpha, \beta; \gamma, \delta; x, y, zk) = (1+z)^{-\alpha}(1+k)^{-\beta} \sum_{r=0}^{\infty} \frac{(\beta)_r}{r!}\left(\frac{k}{1+k}\right)^r$$

$$\cdot X_6\left(\alpha, \beta + r, \delta + r; \gamma, \delta; \frac{x}{(1+z)^2}, \frac{y}{(1+z)(1+k)}, \frac{z}{1+z}\right), \tag{5.32}$$

$$X_4(\alpha, \beta; \gamma, \beta, \beta; x^2, y, y)$$

$$= (1+2x-2y)^{-\alpha} H_4\left(\alpha, \gamma - \frac{1}{2}; \beta, 2\gamma - 1; \left(\frac{y}{1+2x-2y}\right)^2, \frac{4x}{1+2x-2y}\right), \tag{5.33}$$

$$X_4(\alpha, \beta; \gamma, \beta, \beta; x^2, y, y)$$

$$= (1+2x)^{-\alpha} F_2\left(\alpha, \beta - \frac{1}{2}, \gamma - \frac{1}{2}; 2\beta - 1, 2\gamma - 1; \frac{4y}{1+2x}, \frac{4x}{1+2x}\right), \tag{5.34}$$

$$X_6(\alpha, \beta, \gamma; \varepsilon, \delta; x, y, z)$$

$$= (1-z)^{-\alpha} H_3\left(\alpha, \beta; \varepsilon; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right) {}_2F_1\left(\delta - \gamma, \alpha + 2p + q; \delta; \frac{z}{z-1}\right), \tag{5.35}$$

$$X_7(\alpha, \beta, \delta - \beta; \gamma, \delta; x^2, y, z)$$

$$= (1-z+2x)^{-\alpha} F_2\left(\alpha, \beta, \gamma - \frac{1}{2}; \delta, 2\gamma - 1; \frac{y-z}{1-z+2x}, \frac{4x}{1-z+2x}\right), \tag{5.36}$$

$$X_8(\alpha, \beta, \gamma; \delta, \beta, \lambda; x, y, z) = (1-y)^{-\alpha} H_4\left(\alpha, \gamma; \delta, \lambda; \frac{x}{(1-y)^2}, \frac{z}{1-y}\right), \tag{5.37}$$

$$X_8(\alpha, \beta, \gamma; \delta, \beta, \gamma; x, y, z)$$

$$= (1-y-z)^{-\alpha} {}_2F_1\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}; \delta; \frac{4x}{(1-y-z)^2}\right), \tag{5.38}$$

$$X_8(\alpha, \beta, \gamma; \delta, \beta, \gamma; x^2, y, z)$$

$$= (1+2x-y-z)^{-\alpha} {}_2F_1\left(\alpha, \delta - \frac{1}{2}; 2\delta - 1; \frac{4x}{1+2x-y-z}\right), \tag{5.39}$$

$$X_8(\alpha, \beta, \gamma; \delta, \beta, \gamma; x^2, y, y)$$

$$= (1+2x-2y)^{-\alpha} {}_2F_1\left(\alpha, \delta - \frac{1}{2}; 2\delta - 1; \frac{4x}{1+2x-2y}\right). \tag{5.40}$$

9. The following results are the Exton's results, which are obtained by Exton in his paper (cf. [11]). We recover these results by substituting $n = 0$ into equations (3.1), (3.4), (3.5), (3.6), (3.8), (3.9), (3.12), (3.14), (3.15), (3.16), (3.18), (3.19), (3.20), (3.21), (3.23), (3.27), (3.28), (3.29) and (3.30):

$$X_1(\alpha, \beta; \gamma, \delta; x^2, y, z)$$

$$= (1+2x)^{-\alpha} X_6\left(\alpha, \beta, \gamma - \frac{1}{2}; \delta, 2\gamma - 1; \frac{y}{(1+2x)^2}, \frac{z}{1+2x}, \frac{4x}{1+2x}\right), \tag{5.41}$$

$$X_2(\alpha, \beta; \gamma, \delta, \lambda; x, y^2, z)$$

$$= (1+2y)^{-\alpha} X_8\left(\alpha, \beta, \delta - \frac{1}{2}; \gamma, \lambda, 2\delta - 1; \frac{x}{(1+2y)^2}, \frac{z}{1+2y}, \frac{4y}{1+2y}\right), \tag{5.42}$$

$$X_2(\alpha, \beta; \gamma, \delta, \lambda; x^2, y^2, z) = (1 + 2x + 2y)^{-\alpha}$$

$$\cdot F_A^{(3)}\left(\alpha, \gamma - \frac{1}{2}, \delta - \frac{1}{2}, \beta; 2\gamma - 1, 2\delta - 1, \lambda;\right.$$

$$\left.\frac{4x}{1 + 2x + 2y}, \frac{4y}{1 + 2x + 2y}, \frac{z}{1 + 2x + 2y}\right), \tag{5.43}$$

$$X_2(\alpha, \beta; \gamma, \delta, \lambda; x, y, z)$$

$$= (1 - z)^{-\alpha} X_2\left(\alpha, \lambda - \beta; \gamma, \delta, \lambda; \frac{x}{(1 - z)^2}, \frac{y}{(1 - z)^2}, \frac{z}{z - 1}\right), \tag{5.44}$$

$$X_4(\alpha, \beta; \gamma, \delta, \lambda; x^2, y, z)$$

$$= (1 + 2x)^{-\alpha} F_E\left(\alpha, \alpha, \alpha, \gamma - \frac{1}{2}, \beta, \beta; 2\gamma - 1, \delta, \lambda; \frac{4x}{1 + 2x}, \frac{y}{1 + 2x}, \frac{z}{1 + 2x}\right), \tag{5.45}$$

$$X_4(\alpha, \beta; \gamma, \beta, \beta; x, y, z)$$

$$= (1 - y - z)^{-\alpha} F_4\left(\frac{\alpha}{2}, \frac{\alpha + 1}{2}; \gamma, \beta; \frac{4x}{(1 - y - z)^2}, \frac{4yz}{(1 - y - z)^2}\right), \tag{5.46}$$

$$X_6(\alpha, \beta, \gamma; \varepsilon, \delta; x, y, z)$$

$$= (1 - z)^{-\alpha} X_6\left(\alpha, \beta, \delta - \gamma; \varepsilon, \delta; \frac{x}{(1 - z)^2}, \frac{y}{1 - z}, \frac{z}{z - 1}\right), \tag{5.47}$$

$$X_6(\alpha, \beta, \gamma; \delta, \gamma; x, y, z) = (1 - z)^{-\alpha} H_3\left(\alpha, \beta; \delta; \frac{x}{(1 - z)^2}, \frac{y}{1 - z}\right), \tag{5.48}$$

$$X_7(\alpha, \beta, \gamma; \delta, \lambda; x^2, y, z)$$

$$= (1 + 2x)^{-\alpha} F_G\left(\alpha, \alpha, \alpha, \delta - \frac{1}{2}, \beta, \gamma; 2\delta - 1, \lambda, \lambda; \frac{4x}{1 + 2x}, \frac{y}{1 + 2x}, \frac{z}{1 + 2x}\right), \tag{5.49}$$

$$X_7(\alpha,\ \beta,\ \delta-\beta;\ \gamma,\ \delta;\ x,\ y,\ z) = (1-z)^{-\alpha} H_4\left(\alpha,\ \beta;\ \gamma,\ \delta;\ \frac{x}{(1-z)^2},\ \frac{y-z}{1-z}\right), \tag{5.50}$$

$$X_8(\alpha,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \varepsilon;\ x^2,\ y,\ z)$$

$$= (1+2x)^{-\alpha} F_A^{(3)}\left(\alpha,\ \delta-\frac{1}{2},\ \beta,\ \gamma;\ 2\delta-1,\ \lambda,\ \varepsilon;\ \frac{4x}{1+2x},\ \frac{y}{1+2x},\ \frac{z}{1+2x}\right), \tag{5.51}$$

$$X_8(\alpha,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \varepsilon;\ x,\ y,\ z)$$

$$= (1-y)^{-\alpha} X_8\left(\alpha,\ \lambda-\beta,\ \gamma;\ \delta,\ \lambda,\ \varepsilon;\ \frac{x}{(1-y)^2},\ \frac{y}{y-1},\ \frac{z}{1-y}\right), \tag{5.52}$$

$$X_8(\alpha,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \varepsilon;\ x,\ y,\ z)$$

$$= (1-z)^{-\alpha} X_8\left(\alpha,\ \beta,\ \varepsilon-\gamma;\ \delta,\ \lambda,\ \varepsilon;\ \frac{x}{(1-z)^2},\ \frac{y}{1-z},\ \frac{z}{z-1}\right), \tag{5.53}$$

$$X_8(\alpha,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \varepsilon;\ x,\ y,\ z) = (1-y-z)^{-\alpha}$$

$$\cdot\ X_8\left(\alpha,\ \lambda-\beta,\ \varepsilon-\gamma;\ \delta,\ \lambda,\ \varepsilon;\ \frac{x}{(1-y-z)^2},\ \frac{y}{y+z-1},\ \frac{z}{y+z-1}\right), \tag{5.54}$$

$$X_8(\alpha,\ \beta,\ \gamma;\ \delta,\ \lambda,\ \gamma;\ x,\ y,\ z) = (1-z)^{-\alpha} H_4\left(\alpha,\ \beta;\ \delta,\ \lambda;\ \frac{x}{(1-z)^2},\ \frac{y}{1-z}\right), \tag{5.55}$$

$$X_9(\alpha,\ \beta;\ \gamma;\ x^2,\ 2xy,\ y^2)$$

$$= (1+2x)^{-\alpha}(1+2y)^{-\beta} F_1\left(\gamma-\frac{1}{2},\ \alpha,\ \beta;\ 2\gamma-1;\ \frac{4x}{1+2x},\ \frac{4y}{1+2y}\right), \tag{5.56}$$

$$X_9(\alpha,\ \beta;\ \gamma;\ x^2,\ 2x^2,\ x^2)$$

$$= (1+2x)^{-(\alpha+\beta)}\ {}_2F_1\left(\gamma-\frac{1}{2},\ \alpha+\beta;\ 2\gamma-1;\ \frac{4x}{1+2x}\right), \tag{5.57}$$

$$X_9(\alpha,\ 2\gamma-\alpha-1;\ \gamma;\ x^2,\ 2x^2,\ x^2) = (1+2x)^{\frac{1}{2}-\gamma}(1-2x)^{\frac{1}{2}-\gamma}, \tag{5.58}$$

$$X_{10}(\alpha,\ \beta;\ \gamma,\ \delta;\ x,\ y,\ z^2)$$

$$= (1+2z)^{-\beta} X_{14}\left(\alpha,\ \beta,\ \delta-\frac{1}{2};\ \gamma,\ 2\delta-1;\ x,\ \frac{y}{1+2z},\ \frac{4z}{1+2z}\right). \tag{5.59}$$

6. Conclusion

We used the Laplace integral representations of Exton's functions given by (1.1) to (1.10) to determine the new generating relations (3.1) to (3.30). Many of these results are the relations between Appell's functions $F_1,\ F_2,\ F_3,\ F_4$; Lauricella function $F_A^{(3)}$; Horn's functions $H_3,\ H_4$; Saran's functions $F_E,\ F_G$ and Gaussian hypergeometric function ${}_2F_1$. If we look at the special cases (5.1) and (5.2), we see that these are the generating relations between Horn's and Appell's functions. The relation (5.3) is the generating relation between Horn's and Gaussian hypergeometric functions. (5.4) and (5.5) are the generating relations between Appell's functions. We have seen in the section of special cases many of the results from Section 3 reduce to Exton's functions for $n = 0$.

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