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A NEW EXTRAGRADIENT METHOD FOR SINGLE-VALUED VARIATIONAL INEQUALITY

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Abstract

We propose a new extragradient method for the single-valued variational inequality problem. Our method is proven to be globally convergent to a solution of the variational inequality problem, provided the mapping is continuous and pseudomonotone. Convergence analysis is also presented.

1. Introduction

We consider the variational inequality problem (VIP), which is to find a vector $x^* \in C$ such that

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$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1}$$

where C is a nonempty closed convex set in $\mathbb{R}^n$, F is a single-valued mapping from $\mathbb{R}^n$ into itself, and $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and norm in $\mathbb{R}^n$, respectively.

Many methods for computing the solution of (1) are projection-type methods. Projection-type algorithms have been extensively studied in the literature, see [5, 6, 7, 8, 10] and the references therein. In [6, 8, 10], the next iterate is a projection of the current iterate onto the intersection of the feasible set C and the hyperplane. In [3], the next iterate is a projection onto a halfspace whose bounding hyperplane supports the feasible set C at a certain point; see also [2]. However, the mapping is required to be Lipschitz continuous in [3]. In this paper, we introduce an extragradient algorithm for the VIP and obtain a global convergence theorem, assuming that F is continuous on C. As claimed in [3], our work is only a theoretical development although its potential numerical advantages are obvious.

The organization of this paper is as follows: Section 2 provides necessary concepts and lemmas. Section 3 presents the algorithm and main theorems. Convergence analysis is reported in Section 4.

2. Preliminaries

F is called *pseudomonotone* on C, if for any $x, y \in C$,

$$\langle F(y), x - y \rangle \geq 0 \Rightarrow \langle F(x), x - y \rangle \geq 0. \tag{2}$$

Let S be the solution set of (1), that is, those points $x^* \in C$ satisfying (1). Throughout this paper, we assume that the solution set S of the problem (1) is nonempty and F is pseudomonotone on C with respect to the solution set S, i.e.,

$$\langle F(y), y - x \rangle \geq 0, \quad \forall y \in C, \ \forall x \in S. \tag{3}$$

The property (3) holds if F is pseudomonotone on C.

Let P_C denote the projector onto C and let $\mu > 0$ be a parameter.

Lemma 2.1. $x \in C$ *solves the problem* (1) *if and only if*

$$r_\mu(x) := x - P_C(x - \mu F(x)) = 0. \tag{4}$$

Lemma 2.2. *Let C be a closed convex subset of* $\mathbb{R}^n$. *For any* $x, y \in \mathbb{R}^n$ *and* $z \in C$, *the following statements hold*:

(i) $\langle P_C(x) - x, z - P_C(x)\rangle \geq 0.$

(ii) $\| P_C(x) - P_C(y)\|^2 \leq \| x - y\|^2 - \| P_C(x) - x + y - P_C(y)\|^2.$

Proof. See [11].

The proof of the following lemma is easy and we omit it (see Lemma 3.1 in [1], for example).

Lemma 2.3. *For any* $x \in \mathbb{R}^n$ *and* $\mu > 0$,

$$\min\{1, \mu\}\| r_1(x)\| \leq \|r_\mu(x)\| \leq \max\{1, \mu\}\| r_1(x)\|.$$

3. Main Results

Algorithm 3.1. Choose $x_0 \in C$ and two parameters $\gamma, \sigma \in (0, 1)$. Set $i = 0$.

Step 1. Let k_i is the smallest nonnegative integer satisfying

$$\gamma^{k_i}\| F(x_i) - F(P_C(x_i - \gamma^{k_i} F(x_i)))\| \leq \sigma\| r_\gamma k_i(x_i)\|. \tag{5}$$

Set $\rho_i = \gamma^{k_i}$ and

$$y_i = P_C(x_i - \rho_i F(x_i)). \tag{6}$$

If $r_{\rho_i}(x_i) = 0$, stop.

Step 2. Compute $x_{i+1} := P_{H_i \cap C}(x_i - \rho_i F(y_i))$, where

$$H_i := \{x \in \mathbb{R}^n : \langle (x_i - \rho_i F(x_i)) - y_i \;\; x - y_i\rangle \leq 0\}. \tag{7}$$

Let $i := i + 1$ and go to Step 1.

Remark 3.1. H_i in Step 2 is the halfspace whose bounding hyperplane supports C at y_i.

Remark 3.2. $C \subseteq H_i$. Indeed, in view of Lemma 2.2(i) and (6), we have

$$\langle (x_i - \rho_i F(x_i)) - y_i \ \ x - y_i \rangle \leq 0, \quad \forall x \in C.$$

Therefore, $C \subseteq H_i$.

We first show that Algorithm 3.1 is well defined.

Proposition 3.1. *If* x_i *is not a solution of problem* (1), *then there exists a nonnegative integer* k_i *satisfying* (5).

Proof. Suppose that for all k, we have

$$\gamma^k \| F(x_i) - F(P_C(x_i - \gamma^k F(x_i))) \| > \sigma \| r_{\gamma^k}(x_i) \|.$$

Therefore,

$$\begin{aligned} \| F(x_i) - F(P_C(x_i - \gamma^k F(x_i))) \| &> \frac{\sigma}{\gamma^k} \| r_{\gamma^k}(x_i) \| \\ &\geq \frac{\sigma}{\gamma^k} \min\{1, \gamma^k\} \| r_1(x_i) \| \\ &= \sigma \| r_1(x_i) \|, \end{aligned} \tag{8}$$

where the second inequality follows from Lemma 2.3 and the equality follows from $\gamma \in (0, 1)$ and $k \geq 0$. Since $P_C(\cdot)$ is continuous and $x_i \in C$, $P_C(x_i - \gamma^k F(x_i)) \to x_i$ $(k \to \infty)$. Let $k \to \infty$ in (8), we have

$$0 = \| F(x_i) - F(x_i) \| \geq \sigma \| r_1(x_i) \| > 0,$$

being F continuous on C. This contradiction completes the proof.

Now we obtain the following auxiliary result that will be used for proving the convergence of Algorithm 3.1.

Theorem 3.1. *Let* $\{x_i\}$ *be the sequence generated by Algorithm* 3.1 *and let* $x^* \in S$. *Suppose that the assumption* (3) *holds*, *then*

$$\| x_{i+1} - x^* \|^2 \le \| x_i - x^* \|^2 - (1 - \sigma^2)\rho_i^2 \| r_1(x_i) \|^2. \tag{9}$$

Proof. Since $x^* \in S$, it follows from assumption (3) that

$$\langle F(y_i), y_i - x^* \rangle \ge 0. \tag{10}$$

Therefore,

$$\langle F(y_i), x_{i+1} - x^* \rangle \ge \langle F(y_i), x_{i+1} - y_i \rangle. \tag{11}$$

By the definition of H_i, we have

$$\langle x_{i+1} - y_i, (x_i - \rho_i F(x_i)) - y_i \rangle \le 0.$$

Thus,

$$\begin{aligned} &\langle x_{i+1} - y_i, (x_i - \rho_i F(y_i)) - y_i \rangle \\ &= \langle x_{i+1} - y_i, x_i - \rho_i F(x_i) - y_i \rangle + \rho_i \langle x_{i+1} - y_i, F(x_i) - F(y_i) \rangle \\ &\le \rho_i \langle x_{i+1} - y_i, F(x_i) - F(y_i) \rangle. \end{aligned} \tag{12}$$

Denoting $z_i = x_i - \rho_i F(y_i)$,

$$\begin{aligned} &\| x_{i+1} - x^* \|^2 \\ &= \| P_{H_i}(z_i) - x^* \|^2 \\ &= \langle P_{H_i}(z_i) - z_i + z_i - x^*, P_{H_i}(z_i) - z_i + z_i - x^* \rangle \\ &= \| z_i - x^* \| + \| z_i - P_{H_i}(z_i) \|^2 + 2\langle P_{H_i}(z_i) - z_i, z_i - x^* \rangle. \end{aligned} \tag{13}$$

Since

$$\begin{aligned} &2\| z_i - P_{H_i}(z_i) \|^2 + 2\langle P_{H_i}(z_i) - z_i, z_i - x^* \rangle \\ &= 2\langle z_i - P_{H_i}(z_i), x^* - P_{H_i}(z_i) \rangle \le 0, \end{aligned} \tag{14}$$

we get

$$\| z_i - P_{H_i}(z_i) \|^2 + 2\langle P_{H_i}(z_i) - z_i,\, z_i - x^* \rangle \leq -\| z_i - P_{H_i}(z_i) \|^2, \quad (15)$$

Hence,

$$\begin{aligned}
\| x_{i+1} - x^* \|^2 &\leq \| z_i - x^* \|^2 - \| z_i - P_{H_i}(z_i) \|^2 \\
&= \| (x_i - \rho_i F(y_i)) - x^* \|^2 - \| (x_i - \rho_i F(y_i)) - P_{H_i}(z_i) \|^2 \\
&= \| x_i - x^* \|^2 - \| x_i - x_{i+1} \|^2 + 2\rho_i \langle x^* - x_{i+1},\, F(y_i) \rangle \\
&\leq \| x_i - x^* \|^2 - \| x_i - x_{i+1} \|^2 + 2\rho_i \langle y_i - x_{i+1},\, F(y_i) \rangle, \quad (16)
\end{aligned}$$

where the last inequality follows from (11). Therefore,

$$\begin{aligned}
\| x_{i+1} - x^* \|^2 &\leq \| x_i - x^* \|^2 - \| x_i - x_{i+1} \|^2 + 2\rho_i \langle y_i - x_{i+1},\, F(y_i) \rangle \\
&= \| x_i - x^* \|^2 - \langle x_i - y_i + y_i - x_{i+1},\, x_i - y_i + y_i - x_{i+1} \rangle \\
&\quad + 2\rho_i \langle y_i - x_{i+1},\, F(y_i) \rangle \\
&= \| x_i - x^* \|^2 - \| x_i - y_i \|^2 - \| y_i - x_{i+1} \|^2 \\
&\quad + 2\langle x_{i+1} - y_i,\, x_i - \rho_i F(y_i) - y_i \rangle \\
&\leq \| x_i - x^* \|^2 - \| x_i - y_i \|^2 - \| y_i - x_{i+1} \|^2 \\
&\quad + 2\rho_i \langle x_{i+1} - y_i,\, F(x_i) - F(y_i) \rangle \\
&\leq \| x_i - x^* \|^2 - \| x_i - y_i \|^2 - \| y_i - x_{i+1} \|^2 \\
&\quad + 2\sigma \| x_{i+1} - y_i \| \| r_{\rho_i}(x_i) \| \\
&= \| x_i - x^* \|^2 - \| x_i - y_i \|^2 - \| y_i - x_{i+1} \|^2 \\
&\quad + 2\sigma \| x_{i+1} - y_i \| \| x_i - y_i \|, \quad (17)
\end{aligned}$$

where the second inequality follows from (12) and the third one follows from Cauchy-Schwarz inequality and (5).

In addition,

$$0 \le (\sigma\| x_i - y_i \| - \| x_{i+1} - y_i \|)^2$$

$$= \sigma^2\| x_i - y_i \|^2 - 2\sigma\| x_{i+1} - y_i \|\| x_i - y_i \| + \| y_i - x_{i+1} \|^2. \quad (18)$$

Therefore,

$$2\sigma\| x_{i+1} - y_i \|\| x_i - y_i \| \le \sigma^2\| x_i - y_i \|^2 + \| y_i - x_{i+1} \|^2. \quad (19)$$

Combining (17) and (19), we have

$$\| x_{i+1} - x^* \|^2 \le \| x_i - x^* \|^2 - (1 - \sigma^2)\| x_i - y_i \|^2. \quad (20)$$

By Lemma 2.3,

$$\begin{aligned}\| x_i - y_i \| &= \| r_{\rho_i}(x_i) \| \\ &\ge \min\{1, \rho_i\}\| r_i(x_i) \| \\ &= \rho_i\| r_i(x_i) \|. \quad (21)\end{aligned}$$

It follows from (20) and (21) that

$$\| x_{i+1} - x^* \|^2 \le \| x_i - x^* \|^2 - (1 - \sigma^2)\rho_i^2\| r_1(x_i) \|^2. \quad (22)$$

This completes the proof.

Theorem 3.2. *If* $F : C \to \mathbb{R}^n$ *is continuous on* C *and the assumption* (3) *holds*, *then the sequence* $\{x_i\}$ *generated by Algorithm* 3.1 *converges to a solution* $\bar{x}$ *of* (1).

Proof. Let $x^* \in S$. Since $0 < \sigma < 1$, we have $1 - \sigma^2 \in (0, 1)$. It follows from Theorem 3.1 that

$$(1 - \sigma^2)\rho_i^2\| r_1(x_i) \|^2 \le \| x_i - x^* \|^2 - \| x_{i+1} - x^* \|^2. \quad (23)$$

It follows that the sequence $\{\| x_{i+1} - x^* \|^2\}$ is nonincreasing, and hence is a convergent sequence. Therefore, $\{x_i\}$ is bounded and

$$0 \le (1-\sigma^2)\rho_i^2 \| r_1(x_i) \|^2 \le \| x_i - x^* \|^2 - \| x_{i+1} - x^* \|^2 \to 0 \text{ as } i \to \infty,$$

which implies that

$$\lim_{i\to\infty} \rho_i \| r_i(x_i) \| = 0. \tag{24}$$

We consider two possible cases. Suppose first that $\limsup_{i\to\infty} \rho_i > 0$. Then, by (24), it must be the case that $\liminf_{i\to\infty} \| r_i(x_i) \| = 0$. Since $r_1(\cdot)$ is continuous and $\{x_i\}$ is bounded, there exists an accumulation point $\bar{x}$ of $\{x_i\}$ such that $r_1(\bar{x}) = 0$. It follows that $\bar{x}$ is a solution of the problem (1). We show next that the whole sequence $\{x_i\}$ converges to $\bar{x}$. Replacing x^* by $\bar{x}$ in the preceding argument, we obtain that the sequence $\{\| x_i - \bar{x} \|\}$ is nonincreasing and hence converges. Since $\bar{x}$ is an accumulation point of $\{x_i\}$, some subsequence of $\{\| x_i - \bar{x} \|\}$ converges to zero. This shows that the whole sequence $\{\| x_i - \bar{x} \|\}$ converges to zero, hence $\lim_{i\to\infty} x_i = \bar{x}$.

Suppose now that $\lim_{i\to\infty} \rho_i = 0$. By the choice of ρ_i, we have, for all $k_i \ge 1$,

$$\begin{aligned} \gamma^{k_i-1} \| F(x_i) - F(P_C(x_i - \gamma^{k_i-1} F(x_i))) \| &> \sigma \| r_{\gamma^{k_i-1}}(x_i) \| \\ &\ge \sigma \gamma^{k_i-1} \| r_1(x_i) \|, \end{aligned} \tag{25}$$

where the second inequality follows from Lemma 2.3. Therefore,

$$\| F(x_i) - F(P_C(x_i - \gamma^{-1}\rho_i F(x_i))) \| > \sigma \| r_i(x_i) \|, \tag{26}$$

Let $\bar{x}$ be any accumulation point of $\{x_i\}$ and $\{x_{i_j}\}$ is the corresponding subsequence converging to $\bar{x}$. It follows from (26) that

$$\| F(x_{i_j}) - F(P_C(x_{i_j} - \gamma^{-1}\rho_i F(x_{i_j}))) \| > \sigma \| r_1(x_{i_j}) \|. \tag{27}$$

Letting $j \to \infty$ in (27), we have

$$0 = \| F(\bar{x}) - F(\bar{x}) \| \geq \sigma \| r_1(\bar{x}) \|, \tag{28}$$

being F and $P_C(\cdot)$ continuous. Therefore, $r_1(\bar{x}) = 0$. This implies that $\bar{x}$ solves the variational inequality (1). Similar to the preceding proof, we obtain that $\lim_{i\to\infty} x_i = \bar{x}$.

4. Convergence Rate

Now we provide a result on the convergence rate of the iterative sequence generated by Algorithm 3.1. To establish this result, we need a certain error bound to hold locally (see (29) below). The research on error bound is a large topic in mathematical programming. One can refer to the survey [9] for the roles played by error bounds in the convergence analysis of iterative algorithms; more recent developments on this topic are included in Chapter 6 in [4].

We say that F is *Lipschitz continuous* on C if there exists a constant $L > 0$ such that, for all $x, y \in C$, $\| F(x) - F(y) \| \leq L \| x - y \|$.

Theorem 4.1. *In addition to the assumptions in Theorem* 3.2, *if F is Lipschitz continuous with modulus $L > 0$ and if there exist positive constants c and δ such that*

$$\mathrm{dist}(x, S) \leq c \| r_1(x) \|, \textit{ for all } x \textit{ satisfying } \| r_1(x) \| \leq \delta. \tag{29}$$

Then any sequence $\{x_i\}$ generated by Algorithm 3.1 *converges strongly to a solution $\bar{x}$ of* (1) *and the rate of convergence is R-linear*.

Proof. Put $\rho := \min\{1/2, L^{-1}\gamma\sigma\}$. We first prove that $\rho_i > \rho$ for all i. By the construction of ρ_i, we have $\rho_i \in (0, 1]$. If $\rho_i = 1$, then clearly $\rho_i > \frac{1}{2} \geq \rho$. Now we assume that $\rho_i < 1$. Since $\rho_i = \gamma^{k_i}$, it follows that the nonnegative integer $k_i \geq 1$. Thus the construction of k_i implies that

$$\sigma \| r_{\gamma^{k_i - 1}}(x_i) \| < \gamma^{k_i - 1} \| F(x_i) - F(P_C(x_i - \gamma^{k_i - 1} F(x_i))) \|. \tag{30}$$

It follows from the Lipschitz continuity of F that

$$\sigma\| r_{\gamma^{k_i-1}}(x_i) \| < L\gamma^{k_i-1}\| x_i - P_C(x_i - \gamma^{k_i-1}F(x_i)) \|$$

$$= L\rho_i\gamma^{-1}\| r_{\gamma^{k_i-1}}(x_i) \|. \tag{31}$$

Therefore $\rho_i > L^{-1}\gamma\sigma \geq \rho.$

Let $x^* \in P_S(x_i)$. By (9) and (29), we obtain that for sufficiently large i,

$$\begin{aligned}\operatorname{dist}^2(x_{i+1}, S) &\leq \| x_{i+1} - x^* \|^2 \leq \| x_i - x^* \|^2 - (1-\sigma^2)\| r_{\rho_i}(x_i) \|^2 \\ &\leq \| x_i - x^* \|^2 - (1-\sigma^2)\rho_i^2\| r_1(x_i) \|^2 \\ &\leq \| x_i - x^* \|^2 - (1-\sigma^2)\rho^2\| r_1(x_i) \|^2 \\ &\leq \operatorname{dist}^2(x_i, S) - (1-\sigma^2)\rho^2 c^{-2}\operatorname{dist}^2(x_i, S) \\ &= (1-(1-\sigma^2)\rho^2 c^{-2})\operatorname{dist}^2(x_i, S),\end{aligned} \tag{32}$$

where the second inequality follows from Lemma 2.3 and the third one follows from $\rho_i > \rho$. Therefore the sequence $\{\operatorname{dist}(x_i, S)\}$ converges Q-linearly to zero, and hence $\{x_i\}$ converges R-linearly to $\bar{x} \in S$.

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