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A STUDY ON FRIDAY AND PATIL BIVARIATE EXPONENTIAL DISTRIBUTION

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Abstract

In this paper, the statistical properties of the Friday and Patil bivariate exponential distribution are discussed. Estimators of the parameters of the distribution are obtained using an intuitive approach and their properties are studied. Estimates of the parameters of the Freund bivariate exponential distribution are obtained as a special case. Moreover, using the relations between the parameters of the Marshall-Olkin and the Friday and Patil bivariate exponential distributions, estimators of the parameters of Marshall-Olkin bivariate exponential distribution are derived. Finally, numerical illustrations are performed to highlight the theoretical results.

2010 Mathematics Subject Classification: 62E, 62F.

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1. Introduction

The exponential distribution plays an important role in the reliability theory. It represents the lifetimes of systems or components. For two component systems, the components are not necessarily independent. In such situation, it is relevant considering bivariate distribution for the component lifetimes. In the literature, there are different models of bivariate exponential distributions, see [6]. One of these models is the Friday and Patil bivariate exponential distribution [5], which is interpreted as three different models with the names threshold, gestation and warm-up. For the threshold model, two components, say *A* and *B* with lifetimes X_1 and X_2, respectively, are considered. There are two shocks, say S_1 and S_2, described by Poisson processes, that can destroy the two components, respectively. These shocks are assumed to have varying intensity and the intensity is independent randomly of the time at which the shocks occur. The intensity of S_1 is always sufficient to destroy *A* and the intensity of S_2 is always sufficient to destroy *B*. For each component *A* or *B*, there is a fixed intensity threshold. If the intensity of a shock is below this threshold, then it could destroy its component only. If the intensity exceeds this threshold, then both components are destroyed simultaneously. The probability of each shock exceeding its threshold is $1 - \alpha_0,\ \ 0 < \alpha_0 \leq 1,$ see [5]. Examples of this model are two engine plane, person's kidneys, eyes, ears or other paired organs.

The joint density function of X_1 and X_2, is

$$f(x_1, x_2) = \begin{cases} \alpha_0\alpha_1\alpha_2' e^{-(\alpha_1+\alpha_2-\alpha_2')x_1-\alpha_2' x_2}, & 0 \leq x_1 < x_2, \\ \alpha_0\alpha_2\alpha_1' e^{-(\alpha_1+\alpha_2-\alpha_1')x_2-\alpha_1' x_1}, & 0 \leq x_2 < x_1, \\ (1-\alpha_0)(\alpha_1+\alpha_2)e^{-(\alpha_1+\alpha_2)x}, & 0 \leq x_1 = x_2 = x, \end{cases}$$

$$\alpha_1,\ \alpha_2,\ \alpha_1',\ \alpha_2' > 0,\ 0 < \alpha_0 \leq 1. \tag{1.1}$$

Friday and Patil [5] called this bivariate distribution BEE. Notice that the first two terms in (1.1) are densities with respect to two dimensional Lebesgue measure and the third term is a density with respect to one dimensional Lebesgue measure. Clearly, the joint density in (1.1) can be rewritten as

$$f(x_1, x_2) = \begin{cases} \alpha_2'\phi_1(\alpha_1 + \alpha_2 - \alpha_2')e^{-(\alpha_1+\alpha_2-\alpha_2')x_1-\alpha_2'x_2}, & 0 \le x_1 < x_2, \\ \alpha_1'\phi_2(\alpha_1 + \alpha_2 - \alpha_1')e^{-(\alpha_1+\alpha_2-\alpha_1')x_2-\alpha_1'x_1}, & 0 \le x_2 < x_1, \\ (1-\alpha_0)(\alpha_1 + \alpha_2)e^{-(\alpha_1+\alpha_2)x}, & 0 \le x_1 = x_2 = x, \end{cases} \tag{1.2}$$

where $\phi_i = \dfrac{\alpha_0\alpha_i}{(\alpha_1 + \alpha_2 - \alpha_{3-i}')}$, provided that

$$\alpha_1 + \alpha_2 \ne \alpha_{3-i}' \ (i = 1, 2). \tag{1.3}$$

The joint survival function is

$$\bar{F}(x_1, x_2) = \begin{cases} \phi_1 e^{-(\alpha_1+\alpha_2-\alpha_2')x_1-\alpha_2'x_2} + (1-\phi_1)e^{-(\alpha_1+\alpha_2)x_2}, & x_1 \le x_2, \\ \phi_2 e^{-(\alpha_1+\alpha_2-\alpha_1')x_2-\alpha_1'x_1} + (1-\phi_2)e^{-(\alpha_1+\alpha_2)x_1}, & x_2 \le x_1. \end{cases} \tag{1.4}$$

The joint survival function can be written as a mixture of an absolutely continuous function

$$\bar{F}_F(x_1, x_2) = \begin{cases} 1/(\alpha_1 + \alpha_2 - \alpha_2')[\alpha_1 e^{-(\alpha_1+\alpha_2-\alpha_2')x_1-\alpha_2'x_2} & \\ + (\alpha_2 - \alpha_2')e^{-(\alpha_1+\alpha_2)x_2}], & 0 \le x_1 < x_2, \\ 1/(\alpha_1 + \alpha_2 - \alpha_1')[\alpha_2 e^{-(\alpha_1+\alpha_2-\alpha_1')x_2-\alpha_1'x_1} & \\ + (\alpha_1 - \alpha_1')e^{-(\alpha_1+\alpha_2)x_1}], & 0 \le x_2 < x_1 \end{cases} \tag{1.5}$$

and a singular survival function

$$\bar{F}_s(x_1, x_2) = e^{-(\alpha_1+\alpha_2)\max(x_1, x_2)} \tag{1.6}$$

in the form

$$\bar{F}(x_1, x_2) = \alpha_0 \bar{F}_F(x_1, x_2) + (1-\alpha_0)\bar{F}_s(x_1, x_2), \quad x_1, x_2 \geq 0. \tag{1.7}$$

The BEE distribution includes as special cases Freund [4], Marshall-Olkin [7], Block-Basu [3], and Proschan-Sullo [11] bivariate models, see [5]. Despite the importance of the BEE distribution, few work, to our knowledge, has been written on its properties. Mokhlis [8] has discussed the reliability of stress-strength models with BEE distributions. Nadarajah and Gupta [10] obtained the distribution of some relations of X_1 and X_2, when X_1 and X_2 have the joint density in (1.1) in the special case when $\alpha_0 = 1$.

In the present paper, we study the properties of the BEE distribution. In Section 2, we introduce the marginal densities and the cumulative function of the distribution. The correlation between X_1 and X_2 is also discussed. The conditional distributions are obtained and the regression equations are derived. In Section 3, we apply an intuitive approach for estimating the parameters of the distribution. The estimators obtained are the same as the maximum likelihood estimators in [8]. The properties of the estimators are also discussed. The estimators of the parameters of the Freund bivariate exponential distribution [4] are obtained as a special case. Moreover, we obtain estimates of the parameters of the Marshall-Olkin bivariate exponential distribution [7] by using relations between the parameters of the later and the BEE distribution. Finally, in Section 4, numerical illustration is performed to highlight the results obtained.

2. Properties of the Distribution

The BEE is the same as Freund's distribution [4] with additional condition that the two components *A* and *B* may fail together with probability $1-\alpha_0$, $0 < \alpha_0 \leq 1$, and with probability α_0 that one of the components may fail before the other.

2.1. The cumulative function

The joint cumulative distribution for BEE distributed rv's X_1, X_2 is

$$F(x_1, x_2) = \begin{cases} 1 - \phi_1 e^{-(\alpha_2^{/} x_2)} - \phi_2 e^{-(\alpha_1^{/} x_1)} - (1 - \phi_2) e^{-(\alpha_1 + \alpha_2) x_1} & \\ + \phi_1 e^{-(\alpha_1 + \alpha_2 - \alpha_2^{/}) x_1 - \alpha_2^{/} x_2}, & x_1 \leq x_2, \\ 1 - \phi_1 e^{-(\alpha_2^{/} x_2)} - (1 - \phi_1) e^{-(\alpha_1 + \alpha_2) x_2} - \phi_2 e^{-(\alpha_1^{/} x_1)} & \\ + \phi_2 e^{-(\alpha_1 + \alpha_2 - \alpha_1^{/}) x_2 - \alpha_1^{/} x_1}, & x_2 \leq x_1, \end{cases} \tag{2.1}$$

where ϕ_1 and ϕ_2 are given in (1.3).

2.2. The marginal densities

Noticing that X_1 represents the lifetime of component *A*, the marginal density of X_1 can be derived arguing as follows: Component *A* fails before component *B* with probability $\frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2}$, fails simultaneously with component *B* with probability $1 - \alpha_0$, or fails after component *B* with probability $\frac{\alpha_0 \alpha_2}{\alpha_1 + \alpha_2}$. If *A* fails before *B* or simultaneously with *B*, the lifetime of *A* will be exponential with parameter $\alpha_1 + \alpha_2$, while if it fails after *B*, then its lifetime will be the sum of two independent exponential random variables with parameters $\alpha_1 + \alpha_2$ and $\alpha_1^{/}$.

Thus the marginal density function of X_1 is

$$f_{X_1}(x_1) = \left(\frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2}\right)(\alpha_1 + \alpha_2) e^{-(\alpha_1 + \alpha_2) x_1} + (1 - \alpha_0)(\alpha_1 + \alpha_2) e^{-(\alpha_1 + \alpha_2) x_1}$$

$$+ \left(\frac{\alpha_0 \alpha_2}{\alpha_1 + \alpha_2}\right) \frac{\alpha_1^{/}(\alpha_1 + \alpha_2)}{(\alpha_1 + \alpha_2 - \alpha_1^{/})} [e^{-\alpha_1^{/} x_1} - e^{-(\alpha_1 + \alpha_2) x_1}], \tag{2.2}$$

which is equivalent to

$$f_{X_1}(x_1) = \alpha_1^{/}\phi_2 e^{-\alpha_1^{/}x_1} + (\alpha_1 + \alpha_2)(1 - \phi_2)e^{-(\alpha_1+\alpha_2)x_1},$$

$$x_1 \geq 0, \ \ \alpha_1 + \alpha_2 - \alpha_1^{/} \neq 0, \tag{2.3}$$

where ϕ_2 is given by (1.3).

Arguing in a similar manner, we have

$$f_{X_2}(x_2) = \left(\frac{\alpha_0\alpha_2}{\alpha_1 + \alpha_2}\right)(\alpha_1 + \alpha_2)e^{-(\alpha_1+\alpha_2)x_2} + (1 - \alpha_0)(\alpha_1 + \alpha_2)e^{-(\alpha_1+\alpha_2)x_2}$$

$$+ \left(\frac{\alpha_0\alpha_1}{\alpha_1 + \alpha_2}\right)\frac{\alpha_2^{/}(\alpha_1 + \alpha_2)}{(\alpha_1 + \alpha_2 - \alpha_2^{/})}[e^{-\alpha_2^{/}x_2} - e^{-(\alpha_1+\alpha_2)x_2}], \tag{2.4}$$

which is equivalent to

$$f_{X_2}(x_2) = \alpha_2^{/}\phi_1 e^{-\alpha_2^{/}x_2} + (\alpha_1 + \alpha_2)(1 - \phi_1)e^{-(\alpha_1+\alpha_2)x_2},$$

$$x_2 > 0, \ \ \alpha_1 + \alpha_2 - \alpha_2^{/} \neq 0, \tag{2.5}$$

where ϕ_1 is given by (1.3).

We can see from equations (2.3) and (2.5) that the marginal densities of X_1 and X_2 are mixtures of two exponential densities whenever $0 < \phi_2 < 1$ and $0 < \phi_1 < 1$, respectively. Otherwise, the marginal densities are weighted sums of exponential densities.

The marginal density of X_1 corresponding to special cases of the parameters:

Case 1. If $\alpha_1 < \alpha_1^{/} < \alpha_1 + \alpha_2$, then $\frac{\alpha_1 + \alpha_2 - \alpha_1^{/}}{\alpha_2} < 1$. Since $0 < \alpha_0 \leq 1$, α_0 may take four different values.

1. If $\alpha_0 < 1$ and $\alpha_0 < \frac{\alpha_1 + \alpha_2 - \alpha_1^{/}}{\alpha_2}$, then $\phi_2 < 1$, and the marginal density of X_1 will be a mixture of two exponential densities with parameters $(\alpha_1^{/})$ and $(\alpha_1 + \alpha_2)$ and their coefficients are ϕ_2 and $(1 - \phi_2)$, respectively.

2. If $\frac{\alpha_1 + \alpha_2 - \alpha_1^{/}}{\alpha_2} < \alpha_0 \le 1,$ then $\phi_2 > 1,$ and the marginal density of X_1 will be weighted sum of two exponential densities with parameters $(\alpha_1^{/})$ and $(\alpha_1 + \alpha_2)$ and their coefficients are ϕ_2 and $(1 - \phi_2)$, respectively, where the first coefficient ϕ_2 is greater than unity.

3. If $\alpha_0 < 1$ and $\alpha_0 = \frac{\alpha_1 + \alpha_2 - \alpha_1^{/}}{\alpha_2},$ then $\phi_2 = 1$ and the marginal density of X_1 will be exponential with parameter $(\alpha_1^{/})$.

Case 2. If $\alpha_1 + \alpha_2 < \alpha_1^{/},$ then $(1 - \phi_2) > 1,$ and the marginal density of X_1 is weighted sum of two exponential densities with parameters $(\alpha_1^{/})$ and $(\alpha_1 + \alpha_2)$ and their coefficients are ϕ_2 and $(1 - \phi_2)$, respectively, where the second coefficient $(1 - \phi_2)$ is greater than unity.

Case 3. If $\alpha_1^{/} < \alpha_1,$ then $0 < \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1'} < 1,$ and $0 < \phi_2 < 1,$ and the marginal density of X_1 is a mixture of two exponential densities with parameters $(\alpha_1^{/})$ and $(\alpha_1 + \alpha_2)$ and their coefficients are ϕ_2 and $(1 - \phi_2)$, respectively.

Case 4. If $\alpha_1^{/} = \alpha_1,$ then the failure of component *B* does not affect component *A*, and the marginal density of X_1 is a mixture of two exponential densities with parameters (α_1) and $(\alpha_1 + \alpha_2)$ and their coefficients are α_0 and $(1 - \alpha_0),$ respectively.

Case 5. When $\alpha_1 + \alpha_2 = \alpha_1^{/},$ the joint density of X_1 and X_2 will be

$$f(x_1, x_2) = \begin{cases} \alpha_0\alpha_1\alpha_2^{/}e^{-(\alpha_1+\alpha_2-\alpha_2^{/})x_1-\alpha_2^{/}x_2}, & x_1 < x_2, \\ \alpha_0\alpha_2\alpha_1^{/}e^{-\alpha_1^{/}x_1}, & x_2 < x_1, \\ (1-\alpha_0)(\alpha_1 + \alpha_2)e^{-(\alpha_1+\alpha_2)x}, & x_1 = x_2 = x, \end{cases}$$

and the marginal density of X_1 will be

$$f_{X_1}(x_1) = \left(1 - \frac{\alpha_0\alpha_2}{\alpha_1 + \alpha_2}\right)(\alpha_1 + \alpha_2)e^{-(\alpha_1+\alpha_2)x_1}$$

$$+ \frac{\alpha_0\alpha_2}{\alpha_1 + \alpha_2}(\alpha_1 + \alpha_2)^2 x_1 e^{-(\alpha_1+\alpha_2)x_1},$$

which is a mixture of an exponential density with parameter $(\alpha_1 + \alpha_2)$ and a gamma density with parameters $(2, (\alpha_1 + \alpha_2))$ and their coefficients are $\left(1 - \frac{\alpha_0\alpha_2}{\alpha_1 + \alpha_2}\right)$ and $\left(\frac{\alpha_0\alpha_2}{\alpha_1 + \alpha_2}\right)$, respectively.

Similarly, for the marginal density of X_2, we have:

Case 1. If $\alpha_2 < \alpha_2^{/} < \alpha_1 + \alpha_2$, then $\frac{\alpha_1 + \alpha_2 - \alpha_2^{/}}{\alpha_1} < 1$. Since $0 < \alpha_0 \leq 1$, α_0 may take different values as follows:

1. If $\alpha_0 < 1$ and $\alpha_0 < \frac{\alpha_1 + \alpha_2 - \alpha_2^{/}}{\alpha_1}$, then $\phi_1 < 1$, and the marginal density of X_2 will be a mixture of two exponential densities with parameters $(\alpha_2^{/})$ and $(\alpha_1 + \alpha_2)$ and their coefficients are ϕ_1 and $(1 - \phi_1)$, respectively.

2. If $\frac{\alpha_1 + \alpha_2 - \alpha_2^{/}}{\alpha_1} < \alpha_0 \leq 1$, then $\phi_1 > 1$ and the marginal density of X_2 is weighted sum of two exponential densities with parameters $(\alpha_2^{/})$ and $(\alpha_1 + \alpha_2)$ and their coefficients are ϕ_1 and $(1 - \phi_1)$, respectively, where the first coefficient ϕ_1 is greater than unity.

3. If $\alpha_0 < 1$ and $\alpha_0 = \frac{\alpha_1 + \alpha_2 - \alpha_2^{/}}{\alpha_1}$, then $\phi_1 = 1$ and the marginal density of X_2 will be exponential distribution with parameter $(\alpha_2^{/})$.

Case 2. If $\alpha_1 + \alpha_2 < \alpha_2^{/}$, then $f_{X_2}(x_2)$ is a weighted sum of two exponential densities with parameters $(\alpha_2^{/})$ and $(\alpha_1 + \alpha_2)$ and their coefficients are ϕ_1 and $(1 - \phi_1)$, respectively, where the second coefficient $(1 - \phi_1)$ is greater than unity.

Case 3. If $\alpha_2^{/} < \alpha_2$, then $f_{X_2}(x_2)$ is a mixture of two exponential densities with parameters $(\alpha_2^{/})$ and $(\alpha_1 + \alpha_2)$ and their coefficients are ϕ_1 and $(1 - \phi_1)$, respectively.

Case 4. If $\alpha_2^{/} = \alpha_2$, then this means that the failure of component *A* does not affect component *B*. In this case, $f_{X_2}(x_2)$ is a mixture of two exponential densities with parameters (α_2) and $(\alpha_1 + \alpha_2)$ and their coefficients are α_0 and $(1 - \alpha_0)$, respectively.

Case 5. If $\alpha_1 + \alpha_2 = \alpha_2^{/}$, then $f_{X_2}(x_2)$ is a mixture of an exponential density with parameter $(\alpha_1 + \alpha_2)$ and a gamma density with parameters $(2, \alpha_1 + \alpha_2)$ and their coefficients are $\left(1 - \frac{\alpha_0\alpha_1}{\alpha_1 + \alpha_2}\right)$ and $\left(\frac{\alpha_0\alpha_1}{\alpha_1 + \alpha_2}\right)$, respectively.

2.3. The correlation between X_1 and X_2

The correlation coefficient between X_1 and X_2 is given by

$$\rho = \frac{\alpha_1^{/}\alpha_2^{/} - \alpha_0^2\alpha_1\alpha_2}{\sqrt{(\alpha_1^{/2} + 2\alpha_0\alpha_2(\alpha_1 + \alpha_2) - (\alpha_0\alpha_2)^2)(\alpha_2^{/2} + 2\alpha_0\alpha_1(\alpha_1 + \alpha_2) - (\alpha_0\alpha_1)^2)}}.$$

For finding the range of this correlation coefficient, we argue as follows:

Case 1. If $\alpha_0 \to 0$, then $\rho \to 1$. This means, the two components *A* and *B* always fail together. In this sense, they are in close proximity and are highly positively correlated.

Case 2. If $\alpha_0 = 1$ and $\alpha_1\alpha_2 = \alpha_1^{/}\alpha_2^{/}$, then $\rho = 0$, in this case, X_1 and X_2 are uncorrelated. On the other hand, if $\alpha_0 = 1$, $\alpha_1 = \alpha_1^{/}$ and $\alpha_2 = \alpha_2^{/}$, then X_1 and X_2 are independent and hence $\rho = 0$.

Case 3. If $0 < \alpha_0 \le 1$, $\alpha_1 = \alpha_2 = \alpha$ and $\alpha_1^{/} \to 0$ and $\alpha_2^{/} \to 0$, then $\rho = \frac{-\alpha_0}{(4 - \alpha_0)}$. This corresponds to the case if either component fails, then the other component does not fail ever. In this case, ρ is a decreasing function in α_0, and if $\alpha_0 = 1$, then $\rho = \frac{-1}{3}$.

Case 4. If $0 < \alpha_0 \le 1$, $\alpha_1^{/} \to \infty$ and $\alpha_2^{/} \to \infty$, then $\rho \to 1$. This corresponds to the case where the two components cannot function if either component fails.

From the above cases and excluding the non-realistic situations, when $\alpha_1^{/} \to 0$, $\alpha_2^{/} \to 0$ and $\alpha_1^{/} \to \infty$, $\alpha_2^{/} \to \infty$, we have

$$\frac{-1}{3} < \rho < 1.$$

2.4. The conditional distributions

The conditional density of X_1, given $X_2 = x_2$, is given by

$$f(x_1 | x_2) = \begin{cases} \dfrac{\alpha_2^{/}\phi_1(\alpha_1 + \alpha_2 - \alpha_2^{/})e^{-(\alpha_1+\alpha_2-\alpha_2^{/})x_1-\alpha_2^{/}x_2}}{\alpha_2^{/}\phi_1 e^{-\alpha_2^{/}x_2} + (\alpha_1 + \alpha_2)(1-\phi_1)e^{-(\alpha_1+\alpha_2)x_2}}; & x_1 < x_2, \\[2ex] \dfrac{\alpha_1^{/}\phi_2(\alpha_1 + \alpha_2 - \alpha_1^{/})e^{-(\alpha_1+\alpha_2-\alpha_1^{/})x_2-\alpha_1^{/}x_1}}{\alpha_2^{/}\phi_1 e^{-\alpha_2^{/}x_2} + (\alpha_1 + \alpha_2)(1-\phi_1)e^{-(\alpha_1+\alpha_2)x_2}}; & x_2 < x_1, \\[2ex] \dfrac{(1-\alpha_0)(\alpha_1 + \alpha_2)e^{-(\alpha_1+\alpha_2)x_2}}{\alpha_2^{/}\phi_1 e^{-\alpha_2^{/}x_2} + (\alpha_1 + \alpha_2)(1-\phi_1)e^{-(\alpha_1+\alpha_2)x_2}}; & x_1 = x_2. \end{cases}$$

The regression for X_1 on $X_2 = x_2$ is given by

$$E(X_1 | X_2 = x_2) = \frac{\alpha_0 \alpha_1 \alpha_2'}{f_{X_2}(x_2)} \left(\frac{e^{-\alpha_2' x_2} - e^{-(\alpha_1+\alpha_2)x_2}}{(\alpha_1 + \alpha_2 - \alpha_2')^2} - \frac{x_2 e^{-(\alpha_1+\alpha_2)x_2}}{\alpha_1 + \alpha_2 - \alpha_2'} \right)$$

$$+ \frac{\alpha_0 \alpha_2 e^{-(\alpha_1+\alpha_2)x_2}}{f_{X_2}(x_2)} \left(x_2 + \frac{1}{\alpha_1'} \right)$$

$$+ \frac{(1-\alpha_0)(\alpha_1+\alpha_2) x_2 e^{-(\alpha_1+\alpha_2)x_2}}{f_{X_2}(x_2)}$$

$$= \frac{x_2(\alpha_1+\alpha_2)(1-\phi_1) e^{-(\alpha_1+\alpha_2)x_2}}{f_{X_2}(x_2)}$$

$$+ \frac{\alpha_0 e^{-(\alpha_1+\alpha_2)x_2}}{f_{X_2}(x_2)} \left(\frac{\alpha_2}{\alpha_1'} - \frac{\alpha_1 \alpha_2'}{(\alpha_1+\alpha_2-\alpha_2')^2} \right)$$

$$+ \frac{\alpha_2' \phi_1 e^{-\alpha_2' x_2}}{f_{X_2}(x_2)(\alpha_1+\alpha_2-\alpha_2')}; \quad 0 \le x_2,$$

where $f_{X_2}(x_2)$ is given by (2.4).

Similarly, the conditional density of X_2, given $X_1 = x_1$ is

$$f(x_2 | x_1) = \begin{cases} \dfrac{\alpha_2' \phi_1 (\alpha_1+\alpha_2-\alpha_2') e^{-(\alpha_1+\alpha_2-\alpha_2')x_1 - \alpha_2' x_2}}{\alpha_1' \phi_2 e^{-\alpha_1' x_1} + (\alpha_1+\alpha_2)(1-\phi_2) e^{-(\alpha_1+\alpha_2)x_1}}; & x_1 < x_2, \\[2ex] \dfrac{\alpha_1' \phi_2 (\alpha_1+\alpha_2-\alpha_1') e^{-(\alpha_1+\alpha_2-\alpha_1')x_2 - \alpha_1' x_1}}{\alpha_1' \phi_2 e^{-\alpha_1' x_1} + (\alpha_1+\alpha_2)(1-\phi_2) e^{-(\alpha_1+\alpha_2)x_1}}; & x_2 < x_1, \\[2ex] \dfrac{(1-\alpha_0)(\alpha_1+\alpha_2) e^{-(\alpha_1+\alpha_2)x_1}}{\alpha_1' \phi_2 e^{-\alpha_1' x_1} + (\alpha_1+\alpha_2)(1-\phi_2) e^{-(\alpha_1+\alpha_2)x_1}}; & x_1 = x_2 \end{cases}$$

and the regression for X_2 on $X_1 = x_1$ is given by

$$E(X_2 \mid X_1 = x_1) = \frac{\alpha_0 \alpha_1 e^{-(\alpha_1+\alpha_2)x_1}}{f_{X_1}(x_1)} \left(x_1 + \frac{1}{\alpha_2^{/}} \right)$$

$$+ \frac{\alpha_0 \alpha_2 \alpha_1^{/}}{f_{X_1}(x_1)} \left(\frac{e^{-\alpha_1^{/} x_1} - e^{-(\alpha_1+\alpha_2)x_1}}{(\alpha_1 + \alpha_2 - \alpha_1^{/})^2} - \frac{x_1 e^{-(\alpha_1+\alpha_2)x_1}}{\alpha_1 + \alpha_2 - \alpha_1^{/}} \right)$$

$$+ \frac{(1 - \alpha_0)(\alpha_1 + \alpha_2) x_1 e^{-(\alpha_1+\alpha_2)x_1}}{f_{X_1}(x_1)}$$

$$= \frac{x_1(\alpha_1 + \alpha_2)(1 - \phi_2) e^{-(\alpha_1+\alpha_2)x_1}}{f_{X_1}(x_1)}$$

$$+ \frac{\alpha_0 e^{-(\alpha_1+\alpha_2)x_1}}{f_{X_1}(x_1)} \left(\frac{\alpha_1}{\alpha_2^{/}} - \frac{\alpha_2 \alpha_1^{/}}{(\alpha_1 + \alpha_2 - \alpha_1^{/})^2} \right)$$

$$+ \frac{\alpha_1^{/} \phi_2 e^{-\alpha_1^{/} x_1}}{f_{X_1}(x_1)(\alpha_1 + \alpha_2 - \alpha_1^{/})}; \; 0 \le x_2,$$

where $f_{X_1}(x_1)$ is given by (2.2).

$E(X_2 \mid X_1)$ is calculated for different values of α_0, α_i and $\alpha_i^{/}$ $(i = 1, 2)$, and are presented in Figures 2.1, 2.2 and 2.3.

From Figures 2.1, 2.2 and 2.3, we see that the different Cases 1-4 are verified.

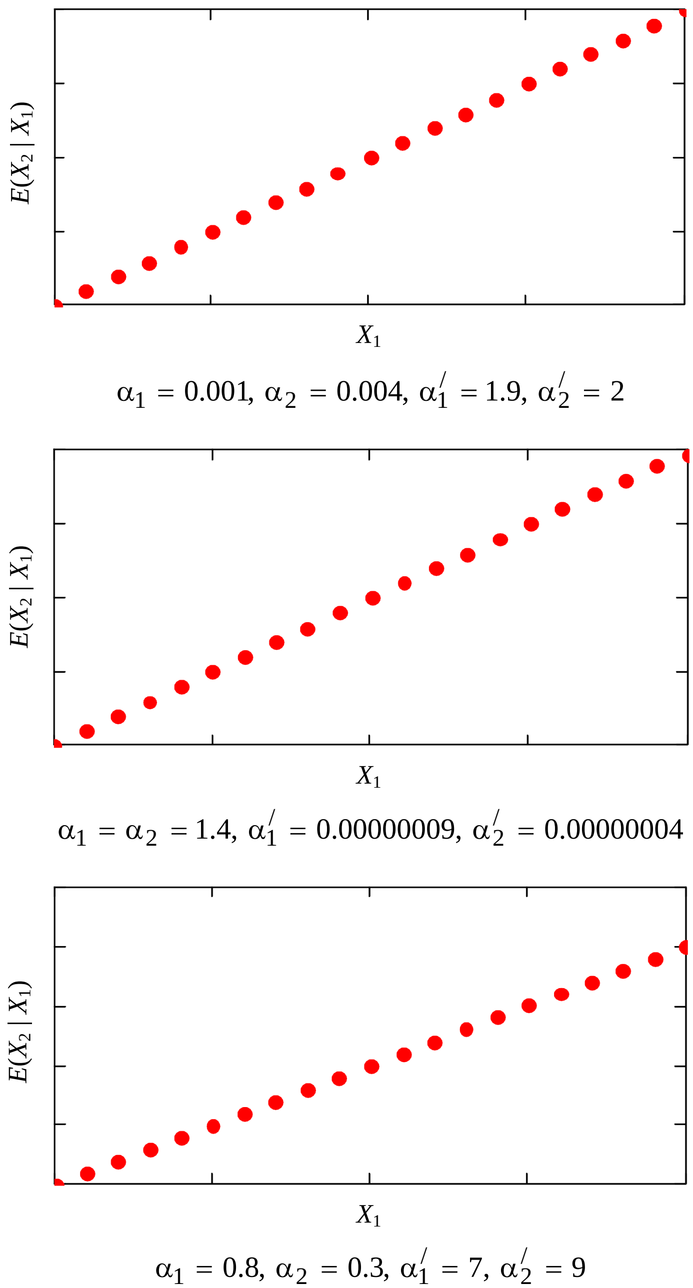
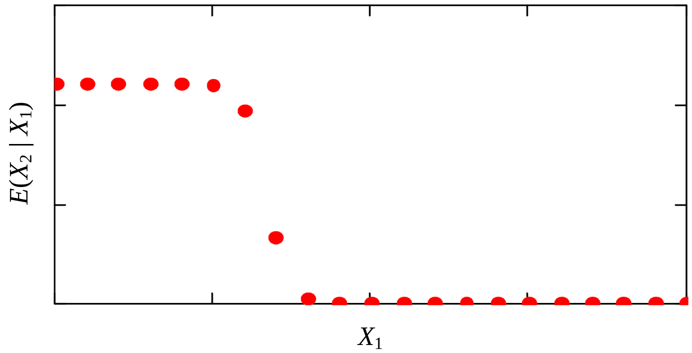
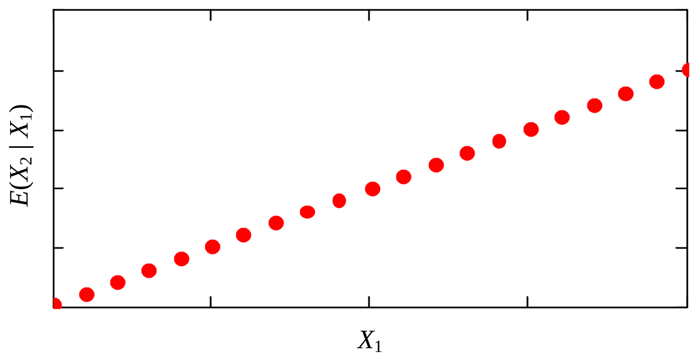


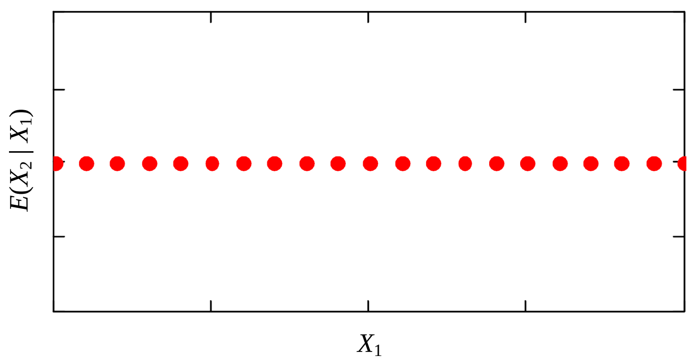
Figure 2.1. The regression of X_2 on $X_1 = x_1$, when $\alpha_0 \to 0$, α_1, α_2, α_1' and α_2' take different values.



$\alpha_1 = \alpha_2 = 1.4,\ \alpha_1' = 0.00000002,\ \alpha_2' = 0.000000009$

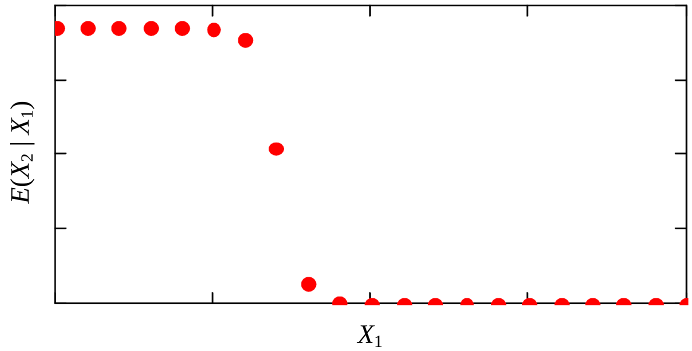


$\alpha_1 = 0.8,\ \alpha_2 = 0.3,\ \alpha_1' = 5.1,\ \alpha_2' = 4$

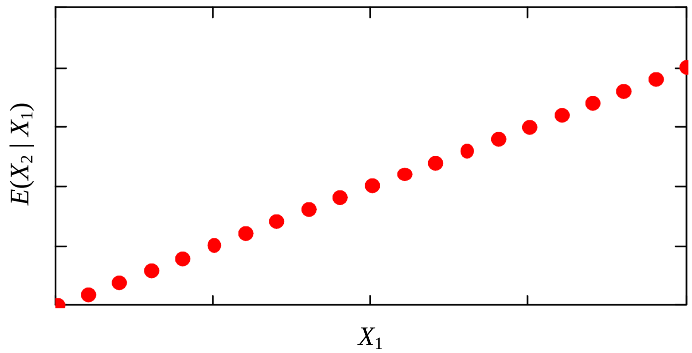


$\alpha_1 = \alpha_1' = 1,\ \alpha_2 = \alpha_2' = 2$

Figure 2.2. The regression of X_2 on $X_1 = x_1$, when $\alpha_0 = 1$, α_1, α_2, α_1' and α_2' take different values.



$\alpha_1 = \alpha_2 = 1.4,\ \alpha_1' = 0.00000002,\ \alpha_2' = 0.000000009$



$\alpha_1 = 0.8,\ \alpha_2 = 0.3,\ \alpha_1' = 5.1,\ \alpha_2' = 4$

Figure 2.3. The regression of X_2 on $X_1 = x_1$, when $\alpha_0 = 0.5$, α_1, α_2, α_1' and α_2' take different values.

3. Estimation of the Parameters

Suppose that a random sample of size n is drawn from a population having the BEE distribution given by (1.1). Suppose that the A component fails first n_1 times and the B component fails first n_2 times and both components A and B fail simultaneously n_3 times, where $n_i > 0$, $i = 1, 2, 3$.

Furthermore, let us write the sum of the lifetimes of the A components which failed first as $\sum_{i=1}^{n_1} x_{1i}$ and the sum of the lifetimes of the corresponding

B components as $\sum_{i=1}^{n_1} x_{2i}$, the sum of the lifetimes of the B components which failed first as $\sum_{i=1}^{\widetilde{n_2}} x_{2i}$ and the sum of the lifetimes of the corresponding A components as $\sum_{i=1}^{\widetilde{n_2}} x_{1i}$, and the sum of the lifetimes of both A and B components which failed together as $\sum_{i=1}^{n_3} x_i$. The likelihood function will be

$$L = (\alpha_0\alpha_1\alpha_2^{/})^{n_1}(\alpha_0\alpha_2\alpha_1^{/})^{n_2}((1-\alpha_0)(\alpha_1+\alpha_2))^{n_3}$$
$$\times \exp\left(-(\alpha_1+\alpha_2-\alpha_2^{/})\sum_{i=1}^{n_1} x_{1i} - \alpha_2^{/}\sum_{i=1}^{n_1} x_{2i} - \alpha_1^{/}\sum_{i=1}^{\widetilde{n_2}} x_{1i}\right.$$
$$\left. -(\alpha_1+\alpha_2-\alpha_1^{/})\sum_{i=1}^{\widetilde{n_2}} x_{2i} - (\alpha_1+\alpha_2)\sum_{i=1}^{n_3} x_i\right), \tag{3.1}$$

where $x_{1i} = x_{2i} = x_i$.

Mokhlis [8] obtained the maximum likelihood estimators of the parameters α_0, α_i and $\alpha_i^{/}$ $(i = 1, 2)$. Here we shall use another intuitive approach for obtaining the estimates of the parameters. We notice that:

1. Since the $\min(X_1, X_2)$ is exponential with parameter $(\alpha_1 + \alpha_2)$, the maximum likelihood estimate of $(\alpha_1 + \alpha_2)$ is given by

$$\widehat{(\alpha_1 + \alpha_2)} = \frac{n}{\sum_{i=1}^{n} \min(x_{1i}, x_{2i})}. \tag{3.2}$$

2. If the component B fails first n_2 times, then the residual lifetime of component A is $(X_1 - X_2)$, which is distributed exponentially with parameter

$\alpha_1^{/}$, and the maximum likelihood estimate of $\alpha_1^{/}$ is

$$\hat{\alpha}_1^{/} = \frac{n_2}{\sum_{i=1}^{\widetilde{n_2}} (x_{1i} - x_{2i})}. \tag{3.3}$$

3. If the component A fails first n_1 times, then the residual lifetime of component B is $X_2 - X_1$, which is exponentially distributed with parameter $\alpha_2^{/}$, and the maximum likelihood estimate of $\alpha_2^{/}$ is

$$\hat{\alpha}_2^{/} = \frac{n_1}{\sum_{i=1}^{n_1} (x_{2i} - x_{1i})}. \tag{3.4}$$

4. The parameter α_0 represents the probability that either the component A fails before B or the component B fails before A, i.e., the two components do not fail together. Since $(n_1 + n_2)$ is the number of times that A and B do not fail together, the maximum likelihood estimate of α_0 is

$$\hat{\alpha}_0 = \frac{(n_1 + n_2)}{n}. \tag{3.5}$$

Notice that $(n_1 + n_2)$ is a value of a binomial random variable with parameters (n, α_0).

5. The probability that the component A fails before B is $\left(\frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2}\right)$, and n_1 is the number of times the component A fails before B. Clearly, n_1 is a value of a binomial random variable with parameters $\left(n, \frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2}\right)$, then the maximum likelihood estimate of $\frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2}$ is

$$\widehat{\left(\frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2}\right)} = \frac{n_1}{n}.$$

Using the invariance property of maximum likelihood estimators, (3.2) and (3.5), we get

$$\hat{\alpha}_1 = \frac{n_1 n}{(n_1 + n_2)\sum_{i=1}^{n} \min(x_{1i}, x_{2i})}. \tag{3.6}$$

6. Similarly, since the probability that the component B fails before A is $\frac{\alpha_0\alpha_2}{(\alpha_1 + \alpha_2)}$, and n_2 is the number of times B fails before A, the maximum likelihood estimate of $\frac{\alpha_0\alpha_2}{(\alpha_1 + \alpha_2)}$ is

$$\widehat{\left(\frac{\alpha_0\alpha_2}{\alpha_1 + \alpha_2}\right)} = \frac{n_2}{n}.$$

Hence

$$\hat{\alpha}_2 = \frac{n n_2}{(n_1 + n_2)\sum_{i=1}^{n} \min(x_{1i}, x_{2i})}. \tag{3.7}$$

The estimates of the parameters of BEE in equations (3.3), (3.4), (3.5), (3.6) and (3.7) are the same as those obtained in [8].

3.1. Properties of the estimators

Now we study the properties of the estimators obtained. The properties discussed are sufficiency, unbiasedness and consistency.

1. Sufficiency

The likelihood function in (3.1) can be rewritten as

$$\begin{aligned} L = {} & (\alpha_0\alpha_1\alpha_2')^{n_1}(\alpha_0\alpha_2\alpha_1')^{n_2}((1-\alpha_0)(\alpha_1+\alpha_2))^{n_3} \\ & \times \exp\left\{-(\alpha_1+\alpha_2)\sum_{i=1}^{n}\min(x_{1i}, x_{2i})\right. \\ & \left. - \alpha_2'\sum_{i=1}^{n_1}(x_{2i} - x_{1i}) - \alpha_1'\sum_{i=1}^{\widetilde{n_2}}(x_{1i} - x_{2i})\right\}. \end{aligned} \tag{3.8}$$

We find that

$$\left\{ n_1,\ n_2,\ n_3,\ \sum_{i=1}^{n} \min(x_{1i},\ x_{2i}),\ \sum_{i=1}^{n_1} (x_{2i} - x_{1i}),\ \sum_{i=1}^{\widetilde{n_2}} (x_{1i} - x_{2i}) \right\}$$

are jointly sufficient. All estimators $\hat{\alpha}_0,\ \hat{\alpha}_1,\ \hat{\alpha}_2,\ \hat{\alpha}_1^{/}$ and $\hat{\alpha}_2^{/}$ are functions of the jointly sufficient statistics.

2. Unbiasedness and consistency

a. The number of times where the component *A* fails before *B* or the component *B* fails before *A* is $(n_1 + n_2)$ which is a value of a random variable having a binomial distribution with parameters $(n,\ \alpha_0),$ then

$$E(\hat{\alpha}_0) = \alpha_0, \text{ and } V(\hat{\alpha}_0) = \frac{\alpha_0(1-\alpha_0)}{n}. \tag{3.9}$$

Thus $\hat{\alpha}_0$ is unbiased and consistent.

b. Notice that $\sum_{i=1}^{\widetilde{n_2}} (x_{1i} - x_{2i})$ extends over the cases where the *B* component fails first. When *B* fails first, $(x_{1i} - x_{2i})$ is a value assumed by a random variable having the exponential distribution with parameter $\alpha_1^{/},$ then $M = \sum_{i=1}^{\widetilde{N_2}} (X_{1i} - X_{2i})$ is a gamma distributed random variable with parameters $(n_2,\ \alpha_1^{/}),$ where N_2 is the r.v. representing the number of times *B* fails first. Thus $\hat{\alpha}_1^{/}$ in (3.3) is the ratio of a value assumed by a binomial random variable with parameters $\left(n,\ \dfrac{\alpha_0\alpha_2}{\alpha_1 + \alpha_2}\right)$ to a value assumed by a random variable having a gamma distribution with parameters $(n_2,\ \alpha_1^{/}).$

Approximate formulas for the expectation and the variance of $\hat{\alpha}_1^{/}$ are given, respectively, by

$$E(\hat{\alpha}_1') = E\left(\frac{N_2}{M}\right) \simeq \frac{E(N_2)}{E(M)} - \frac{\operatorname{cov}(N_2, M)}{E^2(M)} + \frac{E(N_2)V(M)}{E^3(M)}, \tag{3.10}$$

$$V(\hat{\alpha}_1') \simeq \left(\frac{E(N_2)}{E(M)}\right)^2 \left[\frac{V(N_2)}{E^2(N_2)} + \frac{V(M)}{E^2(M)} - \frac{2\operatorname{cov}(N_2, M)}{E(N_2)E(M)}\right] \tag{3.11}$$

(see [9, p. 181]).

Noticing that

$$E(M) = E(E(M \mid N_2))$$

and

$$V(M) = E(V(M \mid N_2)) + V(E(M \mid N_2)),$$

we have

$$E(M) = \frac{n\alpha_0\alpha_2}{\alpha_1'(\alpha_1 + \alpha_2)}, \tag{3.12}$$

$$V(M) = \frac{n\alpha_0\alpha_2}{(\alpha_1')^2(\alpha_1 + \alpha_2)}\left(2 - \frac{\alpha_0\alpha_2}{\alpha_1 + \alpha_2}\right). \tag{3.13}$$

Also,

$$E(N_2) = \frac{n\alpha_0\alpha_2}{\alpha_1 + \alpha_2}. \tag{3.14}$$

Substituting with (3.12), (3.13) and (3.14) in (3.10) and (3.11), we get, respectively,

$$E(\hat{\alpha}_1') \simeq \alpha_1' - \frac{\operatorname{cov}(N_2, M)}{E^2(M)} + \frac{\alpha_1'(2(\alpha_1 + \alpha_2) - \alpha_0\alpha_2)}{n\alpha_0\alpha_2}, \tag{3.15}$$

$$V(\hat{\alpha}_1') \simeq (\alpha_1')^2\left[\frac{3(\alpha_1 + \alpha_2) - 2\alpha_0\alpha_2}{n\alpha_0\alpha_2} - \frac{2\operatorname{cov}(N_2, M)}{E(N_2)E(M)}\right]. \tag{3.16}$$

Since

$$|\operatorname{cov}(N_2, M)| \leq \sqrt{V(N_2)V(M)},$$

we have

$$\left|\frac{\operatorname{cov}(N_2, M)}{E^2(M)}\right| \leq \frac{\alpha_1' \sqrt{(\alpha_1 + \alpha_2(1-\alpha_0))(2(\alpha_1 + \alpha_2) - \alpha_0\alpha_2)}}{n\alpha_0\alpha_2}$$

and

$$\left|\frac{\operatorname{cov}(N_2, M)}{E(N_2)E(M)}\right| \leq \frac{\sqrt{(\alpha_1 + \alpha_2(1-\alpha_0))(2(\alpha_1 + \alpha_2) - \alpha_0\alpha_2)}}{n\alpha_0\alpha_2}.$$

Thus,

$$\lim_{n\to\infty} \frac{\operatorname{cov}(N_2, M)}{E^2(M)} = 0 \tag{3.17}$$

and

$$\lim_{n\to\infty} \frac{\operatorname{cov}(N_2, M)}{E(N_2)E(M)} = 0. \tag{3.18}$$

Taking the limit of both sides of equations (3.15) and (3.16) as $n \to \infty$, and using equations (3.17) and (3.18), we see that $\hat{\alpha}_1^{/}$ is asymptotically unbiased and consistent.

c. Similarly, $\hat{\alpha}_2^{/}$ in (3.4) is the ratio of a value assumed by a binomial random variable with parameters $\left(n, \dfrac{\alpha_0\alpha_1}{\alpha_1 + \alpha_2}\right)$ to a value assumed by a random variable having a gamma distribution with parameters $(n_1, \alpha_2^{/})$. So, we can show that $\hat{\alpha}_2^{/}$ is asymptotically unbiased and consistent.

d. From equation (3.6), we see that $E(\hat{\alpha}_1) = E\left(\dfrac{nR}{U}\right)$, where $U = \sum_{i=1}^{n} \min(X_{1i}, X_{2i})$ which is gamma distributed with parameters $(n, \alpha_1 + \alpha_2)$ and $R = \dfrac{N_1}{X}$, with N_1 is a binomial random variable with parameters

$\left(n, \frac{\alpha_0\alpha_1}{\alpha_1+\alpha_2}\right)$, and X is a binomial random variable with parameters (n, α_0).

Moreover, R and U are independent.

So,

$$E(\hat{\alpha}_1) = \frac{n}{n-1}(\alpha_1+\alpha_2)E(R) \tag{3.19}$$

and

$$V(\hat{\alpha}_1) \simeq (E(R)(\alpha_1+\alpha_2))^2\left[\frac{V(R)}{E^2(R)} + \frac{1}{n}\right]. \tag{3.20}$$

An approximation of $E(R)$ is given by

$$E(R) \simeq \frac{\alpha_1}{\alpha_1+\alpha_2} - \frac{\operatorname{cov}(N_1, X)}{E^2(X)} + \frac{\alpha_1(1-\alpha_0)}{n\alpha_0(\alpha_1+\alpha_2)}$$

and

$$\left|\frac{\operatorname{cov}(N_1, X)}{E^2(X)}\right| \le \frac{\sqrt{\alpha_1(\alpha_2+\alpha_1(1-\alpha_0))(1-\alpha_0)}}{n\alpha_0(\alpha_1+\alpha_2)}.$$

So we have

$$\lim_{n\to\infty} E(R) = \frac{\alpha_1}{\alpha_1+\alpha_2}.$$

The variance of R is given approximately by

$$V(R) \simeq \left(\frac{E(N_1)}{E(X)}\right)^2\left[\frac{V(N_1)}{E^2(N_1)} + \frac{V(X)}{E^2(X)} - \frac{2\operatorname{cov}(N_1, X)}{E(N_1)E(X)}\right]$$

(see [9, p. 181]).

So,

$$V(R) \simeq \left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^2\left[\frac{(\alpha_2+2\alpha_1(1-\alpha_0))}{n\alpha_0\alpha_1} - \frac{2\operatorname{cov}(N_1, X)}{E(N_1)E(X)}\right],$$

where

$$\left|\frac{\operatorname{cov}(N_1, X)}{E(N_1)E(X)}\right| \le \frac{\sqrt{\alpha_1(\alpha_2 + \alpha_1(1-\alpha_0))(1-\alpha_0)}}{n\alpha_0\alpha_1}.$$

Thus

$$\lim_{n\to\infty} V(R) = 0.$$

Taking the limit of both sides of equations (3.19) and (3.20), as $n \to \infty$, we see that $\hat{\alpha}_1$ is asymptotically unbiased and consistent.

e. Similarly, $E(\hat{\alpha}_2) = E\left(\frac{nQ}{U}\right)$, where $Q = \frac{N_2}{X}$, and N_2 is a binomial random variable with parameters $\left(n, \frac{\alpha_0\alpha_2}{\alpha_1 + \alpha_2}\right)$, and Q and U are independent. So, $\hat{\alpha}_2$ is asymptotically unbiased and consistent.

Notice that the estimates in (3.3)-(3.7) are derived by assuming that all $n_i > 0$, $i = 1, 2, 3.$ Now, we shall discuss the cases when one or more of the n_i 's are zeros.

(i) If $n_1 = 0$, then this means that the component A never fails first, and there will be no need for the parameter $\alpha_2^{/}$. Also, it is reasonable to set $\hat{\alpha}_1 = 0.$

In this case, the estimates will be

$$\hat{\alpha}_1 = \hat{\alpha}_2^{/} = 0,\ \hat{\alpha}_0 = \frac{n_2}{n},\ \hat{\alpha}_2 = \frac{n}{\sum_{i=1}^{n}\min(x_{1i}, x_{2i})} \text{ and } \hat{\alpha}_1^{/} = \frac{n_2}{\sum_{i=1}^{\widetilde{n_2}}(x_{1i} - x_{2i})}.$$

(ii) If $n_2 = 0$, then this means that the component B never fails first and in this case,

$$\hat{\alpha}_1^{/} = \hat{\alpha}_2 = 0,\ \hat{\alpha}_0 = \frac{n_1}{n},\ \hat{\alpha}_1 = \frac{n}{\sum_{i=1}^{n}\min(x_{1i}, x_{2i})} \text{ and } \hat{\alpha}_2^{/} = \frac{n_1}{\sum_{i=1}^{n_1}(x_{2i} - x_{1i})}.$$

(iii) If $n_3 = 0,$ then this means that the two components *A* and *B* do not fail together. Therefore,

$$\hat{\alpha}_0 = 1,\ \hat{\alpha}_1 = \frac{n_1}{\sum_{i=1}^{n} \min(x_{1i},\ x_{2i})}, \quad \hat{\alpha}_2 = \frac{n_2}{\sum_{i=1}^{n} \min(x_{1i},\ x_{2i})},$$

$$\hat{\alpha}_1' = \frac{n_2}{\sum_{i=1}^{\widetilde{n_2}} (x_{1i} - x_{2i})} \text{ and } \hat{\alpha}_2' = \frac{n_1}{\sum_{i=1}^{n_1} (x_{2i} - x_{1i})}.$$

(iv) If $n_1 = n_3 = 0,$ then this means that the component *B* always fails before component *A*. In this case $n_2 = n$ and

$$\hat{\alpha}_1 = \hat{\alpha}_2' = 0,\ \hat{\alpha}_0 = 1,\ \hat{\alpha}_2 = \frac{n}{\sum_{i=1}^{n} x_{2i}} \text{ and } \hat{\alpha}_1' = \frac{n}{\sum_{i=1}^{n} (x_{1i} - x_{2i})}.$$

(v) If $n_2 = n_3 = 0,$ then this means that the component *A* always fails before *B*. So, we have $n_1 = n$ and

$$\hat{\alpha}_2 = \hat{\alpha}_1' = 0,\ \hat{\alpha}_0 = 1,\ \hat{\alpha}_1 = \frac{n}{\sum_{i=1}^{n} x_{1i}} \text{ and } \hat{\alpha}_2' = \frac{n}{\sum_{i=1}^{n} (x_{2i} - x_{1i})}.$$

(vi) If $n_1 = n_2 = 0,$ then this means that the two components *A* and *B* always fail together. So it is reasonable to let $\hat{\alpha}_0 = 0,$ i.e., $\alpha_0 = 0.$ However, α_0 is defined such that $0 < \alpha_0 \leq 1,$ so this case could not happen.

Now, we shall obtain estimators of the parameters of Freund [4], and Marshall-Olkin [7] bivariate exponential distributions as special cases.

(a) Freund parameters

When $\alpha_0 = 1,$ the distribution in (1.1) is reduced to that of Freund [4], and the estimators in case (iii), when $n_3 = 0$ are exactly the same as those obtained by Freund [4].

(b) Marshall-Olkin parameters

The joint survival function of the Marshall-Olkin [7] bivariate exponential distribution is

$$\overline{F_M}(x_1, x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_0 \max(x_1, x_2)}, \; x_1, x_2 > 0, \lambda_0, \lambda_1, \lambda_2 > 0,$$

which can be written as a mixture of absolutely continuous survival function

$$\overline{F_{aM}}(x_1, x_2) = \frac{\lambda}{\lambda_1 + \lambda_2} e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_0 \max(x_1, x_2)} - \frac{\lambda_0}{\lambda_1 + \lambda_2} e^{-\lambda \max(x_1, x_2)},$$

where $\lambda = \lambda_0 + \lambda_1 + \lambda_2$, and a singular survival function

$$\overline{F_{sM}}(x_1, x_2) = e^{-\lambda \max(x_1, x_2)}$$

in the form

$$\overline{F_M}(x_1, x_2) = \frac{1}{\lambda}\{(\lambda_1 + \lambda_2)\overline{F_{aM}}(x_1, x_2) + \lambda_0 \overline{F_{sM}}(x_1, x_2)\} \qquad (3.21)$$

(see [6, p. 364]).

It is known that there are no exact solutions for the maximum likelihood estimators of the parameters λ_0, λ_1 and λ_2. However, in the literature, there are various approaches for estimating these parameters, for example, estimators obtained by Arnold [1], Proschan-Sullo [11], Bemis et al. [2] (see [6]).

Here, we shall obtain estimators of λ_0, λ_1 and λ_2 by using the relations between these parameters and the parameters of the BEE.

Friday and Patil [5] showed that the Marshall-Olkin distribution in [7] is obtained by replacing α_0, α_i and $\alpha_i^{/}$ $(i = 1, 2)$ in (1.7) by $(\lambda_1 + \lambda_2)\lambda^{-1}$, $\lambda_i(1 + \lambda_0(\lambda_1 + \lambda_2)^{-1})$ and $\lambda_i + \lambda_0$, respectively.

Writing $\lambda_0,\ \lambda_1$ and λ_2 in terms of $\alpha_0,\ \alpha_1$ and $\alpha_2,$ we have

$$\lambda_0 = (\alpha_1 + \alpha_2)(1 - \alpha_0),$$

$$\lambda_i = \alpha_i \alpha_0 \ (i = 1,\ 2).$$

Using the invariance property of the maximum likelihood estimators and the maximum likelihood estimators of $\alpha_0,\ \alpha_1$ and $\alpha_2,$ given by equations (3.5), (3.6) and (3.7), respectively, we get

$$\hat{\lambda}_0 = \frac{n_3}{\sum_{i=1}^{n} \min(x_{1i},\ x_{2i})}, \tag{3.22}$$

$$\hat{\lambda}_i = \frac{n_i}{\sum_{i=1}^{n} \min(x_{1i},\ x_{2i})} \quad (i = 1,\ 2). \tag{3.23}$$

It can be easily shown the estimators in (3.22) and (3.23) are consistent.

Notice that $\hat{\lambda}_0,\ \hat{\lambda}_1$ and $\hat{\lambda}_2$ in (3.22) and (3.23) are the same as those obtained by Arnold [1] except for the constant $\frac{n-1}{n},$ which appears in [1].

4. Numerical Illustration

Two hundred random samples of size 30 are generated from the BEE distribution in (1.1) by applying Theorems (3.1) and (3.2) in [5], for different values of the parameters $\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_1^{/}$ and $\alpha_2^{/}.$ The estimates of the parameters $\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_1^{/}$ and $\alpha_2^{/}$ are computed by using equations (3.5), (3.6), (3.7), (3.3) and (3.4), respectively. The average of each estimate is calculated by computing the mean of the two hundred replicates. The estimates of the bias and mean squared error (MSE) of each estimator are also calculated. The estimate of the bias is calculated by computing the difference between the average estimate and the true value of the parameter, while the estimate of the MSE is calculated by computing the mean of the squares of the differences of the two hundred replicates of the estimates from the true value of the parameters.

The average estimates, estimates of bias, estimates of MSE of $\hat{\alpha}_0$, $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_1^{/}$ and $\hat{\alpha}_2^{/}$ are given in Tables 4.1-4.3. We have fixed $\hat{\alpha}_0$ and changed the values of the other parameters, in order to cover the different cases considered in Section 2.

Table 4.1. Friday-Patil distribution: estimate, bias and MSE (when $\alpha_1 + \alpha_2 > \alpha_1^{/} + \alpha_2^{/}$)

	True	Avg. estimate	Bias	MSE
α_0	0.5	0.500833333	0.000833333	4.9E-05
α_1	0.3	0.298013899	–0.001986101	0.000139076
α_2	0.29	0.296488321	0.006488321	0.000246566
$\alpha_1^{/}$	0.1	0.112779415	0.012779415	0.000303213
$\alpha_2^{/}$	0.2	0.23763206	0.03763206	0.002859907

Table 4.2. Friday-Patil distribution: estimate, bias and MSE (when $\alpha_1 + \alpha_2 = \alpha_1^{/} + \alpha_2^{/}$)

	True	Avg. estimate	Bias	MSE
α_0	0.5	0.504	0.004	4.69444E-06
α_1	0.1	0.106190196	0.006190196	9.75456E-05
α_2	0.2	0.204582529	0.004582529	1.46205E-05
$\alpha_1^{/}$	0.14	0.151089483	0.011089483	0.000623954
$\alpha_2^{/}$	0.16	0.191004754	0.031004754	0.001997982

Table 4.3. Friday-Patil distribution: estimate, bias and MSE (when $\alpha_1 + \alpha_2 < \alpha_1^{/} + \alpha_2^{/}$)

	True	Avg. estimate	Bias	MSE
α_0	0.5	0.506	0.006	4.9E-05
α_1	0.1	0.103321806	0.003321806	8.51114E-07
α_2	0.2	0.199941412	–5.85878E-05	6.90681E-05
$\alpha_1^{/}$	0.3	0.344685505	0.044685505	0.001468237
$\alpha_2^{/}$	0.23	0.323476795	0.093476795	0.013828941

From Tables 4.1-4.3, we see that the performance of the estimators is good.

In order to investigate the performance of the estimates of the Marshall-Olkin parameters obtained in (3.22) and (3.23), 200 random samples of size 30 are generated from the Marshall-Olkin distribution in [7] for different values of the parameters, the estimates in (3.22) and (3.23) are calculated. Also the well-known estimates proposed by Arnold [1], Proschan-Sullo [11], and Bemis et al. [2] are calculated. The estimates proposed by Arnold [1] are

$$\hat{\lambda}_i = \frac{(n-1)n_i}{n\sum_{j=1}^{n}\min(x_{1j},\, x_{2j})} \quad (i = 1,\, 2) \text{ and } \hat{\lambda}_0 = \frac{(n-1)n_3}{n\sum_{i=1}^{n}\min(x_{1i},\, x_{2i})},$$

where n_i $(i = 1,\, 2)$ is the number of observations for which $X_{ij} < X_{3-i,\, j}$, $j = 1,\, 2,\, \ldots,\, n$, n_3 is the number of observations for which $X_{1j} = X_{2j}$. The estimates proposed by Proschan-Sullo [11] are $\hat{\lambda}_i = \dfrac{nn_i}{(n_1 + n_3)S_i}$ $(i = 1,\, 2)$, and

$$\hat{\lambda}_0 = \frac{n_3}{\sum_{i=1}^{n}\max(x_{1i},\, x_{2i})}\left(1 + \frac{n_1}{n_2 + n_3} + \frac{n_2}{n_1 + n_3}\right),$$

where

$$S_i = \sum_{j=1}^{n} X_{ij}, \quad i = 1, 2 \tag{4.1}$$

and the estimates proposed by Bemis et al. [2], are $\hat{\lambda}_i = \dfrac{\dfrac{n}{S_i} - \dfrac{n_3}{S_{3-i}}}{1 + \dfrac{n_3}{n}}$ $(i = 1, 2)$, and $\hat{\lambda}_0 = \dfrac{n_3\left(\dfrac{1}{S_1} + \dfrac{1}{S_2}\right)}{1 + \dfrac{n_3}{n}}$, where S_i is given by (4.1). The average estimates, estimates of the bias and the MSE of the parameters $\hat{\lambda}_0$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are calculated using the four approaches mentioned above, and are given in Tables 4.4(a, b) and 4.5(a, b).

Table 4.4(a). Marshall-Olkin distribution: estimates and bias (when $\lambda_0 = 0.225$, $\lambda_1 = 0.4$ and $\lambda_2 = 0.8$)

Parameter	True	I	Bias	II	Bias	III	Bias	IV	Bias
λ_0	0.225	0.224082	0.00092	0.21661	0.00839	0.22322	0.00178	0.22109	0.00391
λ_1	0.4	0.429365	0.02936	0.41505	0.01505	0.42847	0.02847	0.43071	0.03071
λ_2	0.8	0.81586	0.015858	0.78866	0.01134	1.403173	0.60317	0.81535	0.01535

Table 4.4(b). Marshall-Olkin distribution: MSE (when $\lambda_0 = 0.225$, $\lambda_1 = 0.4$ and $\lambda_2 = 0.8$)

Parameter	True	I	II	III	IV
λ_0	0.225	0.01307	0.01212	0.00896	0.01037
λ_1	0.4	0.01599	0.01482	0.01481	0.01417
λ_2	0.8	0.03889	0.03601	0.03601	0.0297

Table 4.5(a). Marshall-Olkin distribution: estimates and bias (when $\lambda_0 = 0.1$, $\lambda_1 = 0.04$ and $\lambda_2 = 0.2$)

Parameter	True	I	Bias	II	Bias	III	Bias	IV	Bias
λ_0	0.1	0.10115	0.00114	0.09777	0.00222	0.10222	0.00222	0.10126	0.00126
λ_1	0.04	0.03953	0.00047	0.03821	0.00179	0.04002	1.703E-05	0.04071	0.0007
λ_2	0.2	0.20006	5.6E-05	0.19339	0.00661	0.46528	0.39445	0.20127	0.00123

Table 4.5(b). Marshall-Olkin distribution: MSE (when $\lambda_0 = 0.1$, $\lambda_1 = 0.04$ and $\lambda_2 = 0.2$)

Parameter	True	I	II	III	IV
λ_0	0.1	0.00134	0.00126	0.00078	0.0001
λ_1	0.04	0.00061	0.00057	0.00056	0.00055
λ_2	0.2	0.00358	0.00323	0.00323	0.00345

In Tables 4.4(a, b) and 4.5(a, b), I stands for estimates calculated using equations (3.22) and (3.23), II stands for estimates calculated using Arnold [1] estimators, III stands for estimates calculated using Proshan-Sullo [11] estimators, and IV stands for estimates calculated using Bemis et al. [2] estimators.

From Tables 4.4(a, b) and 4.5(a, b), we see that the proposed estimates in (3.22) and (3.23) perform well.

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