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STOCHASTIC VOLATILITY CORRECTIONS FOR BOND PRICING IN THE FRACTIONAL VASICEK MODEL

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Abstract

Our purpose of this paper is to show how the asymptotic method developed in Cotton et al. [2] and Fouque et al. [7] in the context of the Black-Scholes theory is applied to interest rates. We do this by considering simple models of short rates (such as the Vasicek model) and computing corrections that come from a fast mean-reverting stochastic volatility given by a nonnegative function of a fractional Ornstein-Uhlenbeck (fOU) process. Here, fOU process is driven by a fractional Brownian motion (fBm) with arbitrary Hurst parameter $H \in (0, 1)$. What is important for the asymptotic results we present is that fOU process is characterized by α, the rate of mean-reversion, and

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has a unique invariant probability distribution $N(m, \nu_H^2)$, that is, the normal distribution with mean m and variance ν_H^2.

The asymptotic approximations we present are in the limit $\alpha \to \infty$ with ν_H^2 fixed, which we refer to as fast mean-reversion. We assume that volatility shocks and interest rate shocks are independent.

Then we obtain the corrected price for zero-coupon bond so that it is expanded around the usual Vasicek one-factor bond pricing function with the averaged parameters related to stochastic volatility model parameters.

Since for $H \neq 1/2$ fractional Brownian motion is neither a Markov process, nor a semimartingale, usual stochastic calculus cannot be applied to our model. Therefore, instead of the probabilistic approaches such as the use of conditional expectation, standard Ito formula and the Feynman-Kac representation, our research is made by the fractional integration theory which is due to Hu [10] and partial differential equation approach.

More precisely, we derive bond pricing partial differential equation by fractional Ito formula, introduce fast-scale to model fast mean-reversion in stochastic volatility, and hence obtain the expression for corrected price for zero-coupon bond so that it is characterized by the averaged parameters which are computed as the averaged values with respect to the invariant probability distribution of the fOU process.

Here, asymptotics in the fast-scale is made by singular perturbation expansion, analogous to these in Fouque et al. [5-7] and Narita [14-16]. This again leads to a leading order term which is the usual Vasicek one-factor bond pricing function with the corrected mean level related to fOU process. Our theorem is an extension of the results in Cotton et al. [2] and Fouque et al. [7] to a fractional Vasicek model in the case that fluctuations in price and volatility have zero correlation.

1. Introduction

A one-dimensional *fractional Brownian motion* (fBm) with Hurst parameter $H \in (0, 1)$ is a Gaussian stochastic process with $B_H(0) = 0$ such that

$$E[B_H(t)] = 0, \quad E[B_H(t)B_H(s)] = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}$$

for all $s, t \in \mathbb{R}$. Here, $E[\cdot]$ denotes the mathematical expectation with respect to the probability law μ_H for B_H.

The fBm B_H is self-similar with self-similar index H, that is, for every $c > 0$, the process $\{B_H(ct); t \in \mathbb{R}\}$ is identical in distribution to $\{c^H B_H(t); t \in \mathbb{R}\}$. Since for $H \neq 1/2$, fBm B_H is neither a Markov process, nor a semimartingale, usual stochastic calculus cannot be applied to the field of the network traffic analysis and mathematical finance. If $H = 1/2$, then B_H is one-dimensional *standard Brownian motion* (sBm).

In our Vasicek model, the short rate is modeled as a mean-reverting Gaussian stochastic process $(r(t))_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with an increasing filtration $(\mathcal{F}_t)_{t \geq 0}$. Under the subjective probability measure $\mathbb{P}$, this process is influenced by a stochastic volatility $\sigma(t)$ given by a nonnegative function $f(Y(t))$ of a *fractional Ornstein-Uhlenbeck* (fOU) process $(Y(t))_{t \geq 0}$ as follows:

$$dr(t) = \hat{a}(\bar{r}_\infty - r(t))dt + f(Y(t))dW(t), \tag{1.1}$$

$$dY(t) = \alpha(m - Y(t))dt + \beta dB_H(t) \tag{1.2}$$

with a family $\{\hat{a}, \bar{r}_\infty, m, \alpha, \beta\}$ of positive constants, where f is a positive suitably regular function on $\mathbb{R}$. Here, $(W(t))$ is a one-dimensional sBm, and $(B_H(t))$ is a one-dimensional fBm with Hurst parameter $H \in (0, 1)$. $(Y(t))$ is called the *volatility-driving process*, and the factor $(\sigma(t))$, where $\sigma(t) :=$ $f(Y(t))$ is called the *stochastic volatility process*.

Assumption 1.1. We assume the following:

(i) Throughout this paper, let the Hurst parameter H be arbitrary in $(0, 1)$ and fixed.

(ii) $(W(t))$ and $(B_H(t))$ are independent.

(iii) $f : \mathbb{R} \to \mathbb{R}$ is smooth and bounded above and below:

$$0 < c_1 \leq f \leq c_2 < \infty$$

for constants $c_1 > 0$ and $c_2 > 0$, and f has bounded derivatives.

The process $(Y(t))$ is mean-reverting fOU process; α measures the characteristic speed of *mean-reversion* of $(Y(t))$.

In (1.2), $(Y(t))$ is a process with a unique invariant probability distribution, modeling volatility mean-reversion. In this case, the invariant probability distribution is the normal distribution $N(m, \nu_H^2)$ with mean m and variance ν_H^2, where we define

$$\nu_H^2 = \beta^2 H \left(\frac{1}{\alpha}\right)^{2H} \Gamma(2H),$$

and $\Gamma(\cdot)$ is the Gamma function, i.e., $\Gamma(x) = \int_0^\infty e^{-\xi}\xi^{x-1}d\xi$.

Remark 1.2 (Fast-scale volatility factor driven by fBm)**.** The asymptotic approximations we present are in the limit $\alpha \to \infty$ with ν_H^2 fixed, which we refer to *fast mean-reversion*. Namely, we shall consider the fOU process $(Y(t))$ given by (1.2) under the scaling

$$\alpha = 1/\varepsilon \text{ and } \beta = O(1/\varepsilon^H); \quad 0 < \varepsilon \ll 1.$$

Then, for ε small enough, we shall derive a partial differential equation (PDE) satisfied by the no-arbitrage bond price (Lemma 3.1 in Section 3 and Lemma 4.4 in Section 4). Further, appealing to singular perturbation method, we shall obtain asymptotics of correction for bond prices such that it is expanded around the usual Vasicek one-factor bond pricing function with the averaged parameters related to stochastic volatility model parameters (Theorems 4.6 and 4.7 in Section 4).

Remark 1.3 (Fast-scale volatility factor driven by sBm)**.** Consider the one-factor Vasicek model where the short rate process $(r(t))$ is influenced by the stochastic volatility $\sigma(t) := f(Y(t))$ as follows:

$$dr(t) = \hat{a}(\bar{r}_\infty - r(t))dt + \sigma(t)dW(t),$$

$$dY(t) = \alpha(m - Y(t))dt + \beta d(\rho dW(t) + \sqrt{1-\rho^2}dZ(t))$$

with a family $\{\hat{a}, \bar{r}_\infty, \alpha, m, \beta\}$ of positive constants. Here, $(W(t))$ and $(Z(t))$ independent standard Brownian motions (sBms). The parameter $\rho \in (-1, 1)$ allows a correlation between the sBm $(W(t))$ driving the short rate and its volatility. In this case, $(Y(t))$ has the long-run distribution which is the normal distribution $N(m, \nu^2)$ with mean m and variance ν^2, where we define $\nu^2 = \beta^2/(2\alpha)$. Then, by singular perturbation method, Cotton et al. [2] and Fouque et al. [7] studied asymptotic analysis and obtained corrections for bond prices and bond option prices under the scaling

$$\alpha = 1/\varepsilon, \quad \beta = (\sqrt{2}\nu)/\sqrt{\varepsilon},$$

where $0 < \varepsilon \ll 1$ and $\nu = O(1)$ (fixed). Their result is summarized as Theorem 2.1 in the following Section 2.

As follows from Theorem 2.1, the generality of the class of models is in not having to specify a function *f*; the features of this function that are needed for the theory are captured by *group parameters* derived in the asymptotics and easily calibrated from data. Such group parameters will be extended to these for the case of the fractional Vasicek model (Theorems 4.6 and 4.7 in Section 4).

In the present paper, we investigate a bond pricing problem by the singular perturbation method as in Fouque et al. [5-7] and Narita [14-16].

In order to proceed to asymptotics of corrected price, we shall apply *fractional Ito formula* in Appendices B and C to the total value of the portfolio and then derive pricing partial differential equation (pricing PDE).

Our stochastic integral is in the sense of Hu [10], where stochastic integration theory is developed for *algebraically integrable* integrands by Wiener chaos expansions; a brief comparison among several definitions of stochastic integrals is introduced in Appendix A.

2. Stochastic Volatility Vasicek Models

For a clear understanding of the comparison between the fBm model and the sBm model, we first describe the simple one-factor Vasicek model (Vasicek [26]), introduce the two-factor Vasicek model (the stochastic volatility Vasicek model) and hence review how bonds are priced under these models in the sBm environment. We quote the following from Cotton et al. [2] and Fouque et al. [7, Chapter 11].

In the Vasicek model, the short rate is modeled as a mean-reverting Gaussian stochastic process $(\bar{r}(t))_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$. Under the real-world probability measure $\mathbb{P}$, it satisfies the linear stochastic differential equation (SDE):

$$d\bar{r}(t) = \hat{a}(\bar{r}_\infty - \bar{r}(t))dt + \bar{\sigma}d\overline{W}(t) \tag{2.1}$$

with constants $\hat{a} > 0$, $\bar{r}_\infty > 0$ and $\bar{\sigma} > 0$, where $(\overline{W}(t))_{t\geq 0}$ is a standard $\mathbb{P}$-Brownian motion. Here, $\bar{\sigma}$ is its constant volatility, $\hat{a}$ is the rate of mean-reversion and $\bar{r}_\infty$ is the long-run mean.

Under an equivalent martingale (pricing) measure $\mathbb{P}^\star$, it also follows a linear SDE:

$$d\bar{r}(t) = \hat{a}(r^\star - \bar{r}(t))dt + \bar{\sigma}dW^\star(t), \tag{2.2}$$

where $(W^\star(t))_{t\geq 0}$ is a standard $\mathbb{P}^\star$-Brownian motion, if we assume that *market price of interest rate risk*, denoted by λ, to be a constant. It is included in $r^\star = \bar{r}_\infty - (\lambda\bar{\sigma})/\hat{a}$. In other words, in the risk-neutral world $\mathbb{P}^\star$, the short rate process is an Ornstein-Uhlenbeck (OU) process fluctuating around its mean level $r^\star$ with a rate of mean-reversion $\hat{a}$.

The no-arbitrage price at time t of a zero-coupon bond maturing at time T, denoted by $\mathcal{B}(t, T)$, is given by

$$\mathcal{B}(t, T) = E^{\star}\left\{ e^{-\int_t^T \bar{r}(s)\,ds} \,\middle|\, \mathcal{F}_t \right\} = \bar{P}(t, \bar{r}(t); T), \tag{2.3}$$

where $E^{\star}$ denotes the mathematical expectation with respect to $\mathbb{P}^{\star}$, and the bond pricing function $\bar{P}(t, x; T)$ satisfies the partial differential equation (PDE)

$$\frac{\partial \bar{P}}{\partial t} + \frac{1}{2}\bar{\sigma}^2 \frac{\partial^2 \bar{P}}{\partial x^2} + \hat{a}(r^{\star} - x)\frac{\partial \bar{P}}{\partial x} - x\bar{P} = 0 \tag{2.4}$$

with the terminal condition $\bar{P}(T, x; T) = 1$.

The solution is

$$\bar{P}(t, x; T) = A(T - t)e^{-B(T-t)x}, \tag{2.5}$$

where A and B satisfy the system of ordinary differential equations

$$B' + \hat{a}B - 1 = 0, \tag{2.6}$$

$$-\frac{A'}{A} + \frac{1}{2}\bar{\sigma}^2 B^2 - \hat{a}r^{\star}B = 0 \tag{2.7}$$

with $A(0) = 1$ and $B(0) = 0$. This gives explicitly

$$B(\tau) = \frac{1 - e^{-\hat{a}\tau}}{\hat{a}}, \tag{2.8}$$

$$A(\tau) = \exp\left\{ -\left[R_{\infty}\tau - R_{\infty}\frac{1 - e^{-\hat{a}r}}{\hat{a}} + \frac{\bar{\sigma}^2}{4\hat{a}^3}(1 - e^{-\hat{a}\tau})^2 \right] \right\}, \tag{2.9}$$

where

$$R_{\infty} = r^{\star} - \frac{\bar{\sigma}^2}{2\hat{a}^2} = \bar{r}_{\infty} - \frac{\lambda\bar{\sigma}}{\hat{a}} - \frac{\bar{\sigma}^2}{2\hat{a}^2}. \tag{2.10}$$

This leads to an explicit formula for the zero-coupon bond price $\mathcal{B}(t, T) = \bar{P}(t, r(t); T)$:

$$\mathcal{B}(t, T)$$

$$= \exp\left\{-\left[R_\infty(T-t) - (R_\infty - \bar{r}(t))\frac{1-e^{-\hat{a}(T-t)}}{\hat{a}} + \frac{\bar{\sigma}^2}{4\hat{a}^3}(1-e^{-\hat{a}(T-t)})^2\right]\right\}. \tag{2.11}$$

We observe that the solution $\bar{r}(t)$ of the SDE (2.2) of the Vasicek type is given by

$$\bar{r}(t) = \bar{r}(0)e^{-\hat{a}t} + r^\star(1-e^{-\hat{a}t}) + \bar{\sigma}\int_0^t e^{-\hat{a}(t-s)}dW^\star(s). \tag{2.12}$$

Its integral is

$$\int_t^T \bar{r}(s)ds = r^\star(T-t) + (r^\star - \bar{r}(t))\frac{1-e^{-\hat{a}(T-t)}}{\hat{a}}$$

$$+ \bar{\sigma}\int_t^T \frac{1-e^{-\hat{a}(T-s)}}{\hat{a}}dW^\star(s), \tag{2.13}$$

which, conditional on $\bar{r}(t)$, is normally distributed with mean

$$E^\star\left\{\int_t^T \bar{r}(s)ds \,\middle|\, \bar{r}(t)\right\} = r^\star(T-t) + (r^\star - \bar{r}(t))\frac{1-e^{-\hat{a}(T-t)}}{\hat{a}} \tag{2.14}$$

and variance

$$V^\star\left\{\int_t^T \bar{r}(s)ds \,\middle|\, \bar{r}(t)\right\} = \bar{\sigma}^2\int_t^T \left(\frac{1-e^{-\hat{a}(T-s)}}{\hat{a}}\right)^2 ds. \tag{2.15}$$

Thus, using the moment generating function of a normal random variable, we can compute the expectation in (2.3) as

$$\mathcal{B}(t, T) = E^\star\left\{\exp\left(-\int_t^T \bar{r}(s)ds\right) \,\middle|\, \bar{r}(t)\right\}$$

$$= \exp\left(-E^\star\left\{\int_t^T \bar{r}(s)ds \,\middle|\, \bar{r}(t)\right\} + \frac{1}{2}V^\star\left\{\int_t^T \bar{r}(s)ds \,\middle|\, \bar{r}(t)\right\}\right).$$

Hence we derive (2.11) using explicit formulas for the conditional mean and variance.

Applying Ito's formula to $\mathcal{B}(t, T)$ given by (2.11), we deduce

$$d\mathcal{B}(t, T) = \mathcal{B}(t, T)\left(\bar{r}(t)dt + \bar{\sigma}\frac{1 - e^{-\hat{a}(T-t)}}{\hat{a}}dW^{\star}(t)\right). \tag{2.16}$$

This shows that the *bond price volatility* is given by

$$\frac{\bar{\sigma}}{\hat{a}}(1 - e^{-\hat{a}(T-t)}),$$

which is independent of $r^{\star}$.

The *yield curve* is defined by

$$R(t, \tau) = -\frac{1}{\tau}\log\mathcal{B}(t, t + \tau)$$

as a function of τ, and we deduce the explicit formula

$$R(t, \tau) = R_{\infty} - (R_{\infty} - \bar{r}(t))\frac{1 - e^{-\hat{a}\tau}}{\hat{a}\tau} + \frac{\bar{\sigma}^2}{4\hat{a}^3\tau}(1 - e^{-\hat{a}\tau})^2,$$

which shows that $R(t, \tau)$ is an affine function of the current rate $\bar{r}(t)$ and so justifies the name *affine model of term structure*. Observe that, for all t, $R(t, \tau)$ converges to

$$R_{\infty} = r^{\star} - \frac{\bar{\sigma}^2}{2\hat{a}^2}$$

as $\tau \to \infty$.

We next consider the case where the short rate process $(r(t))$ is influenced by the volatility driving process $(Y(t))$, under the real-world measure $\mathbb{P}$, as follows:

$$dr(t) = \hat{a}(\bar{r}_{\infty} - r(t))dt + f(Y(t))dW(t),$$

$$dY(t) = \alpha(m - Y(t))dt + \beta d(\rho dW(t) + \rho' dZ(t))$$

with a family $\{\hat{a}, r_{\infty}, \alpha, m, \beta\}$ of positive constants. Here, $(W(t))$ and $(Z(t))$

are independent sBms, and $\rho' = \sqrt{1-\rho^2}$. The parameter ρ with $|\rho| < 1$ allows a correlation between the sBm $(W(t))$ driving the short rate and its volatility. The volatility function f is assumed to be nonnegative, smooth and bounded above and below, and also assumed to have bounded derivatives. In this case, $(Y(t))$ has the invariant probability distribution which is the normal distribution $N(m, \nu^2)$ with mean m and variance ν^2, where we define $\nu^2 = \beta^2/(2\alpha)$.

In the risk-neutral world $\mathbb{P}^{\star(\lambda,\gamma)}$, the model for the short rate $(r(t))$ becomes

$$dr(t) = (\hat{a}(\bar{r}_\infty - r(t)) - \lambda(Y(t))\, f(Y(t)))dt + f(Y(t))dW^\star(t), \tag{2.17}$$

$$\begin{aligned} dY(t) = (\alpha(m - Y(t)) - \beta[\rho\lambda(Y(t)) + \rho'\gamma(Y(t))])dt \\ + \beta(\rho dW^\star(t) + \rho' dZ^\star(t)), \end{aligned} \tag{2.18}$$

where $(W^\star(t))$ and $(Z^\star(t))$ are two independent standard $\mathbb{P}^{\star(\lambda,\gamma)}$-Brownian motions.

The market price of risk $\lambda(Y(t))$ may depend on $Y(t)$, but we assume that it does not depend on the short rate $r(t)$. Similarly, the market price of volatility risk $\gamma(Y(t))$ associated with the second source of randomness $(Z(t))$ may depend on $Y(t)$ or may simply be a constant. The risk premium processes $\lambda(Y(t))$ and $\gamma(Y(t))$ are assumed to be bounded; the functions $\lambda(y)$ and $\gamma(y)$ are also assumed to have bounded derivatives.

The no-arbitrage price $L(t, T)$ of a zero-coupon bond maturing at time T is given by

$$P(t, x, y; T) = E^{\star(\lambda,\gamma)}\left\{ e^{-\int_t^T r(s)ds} \,\middle|\, r(t) = x, Y(t) = y \right\}, \tag{2.19}$$

where the expectation $E^{\star(\lambda,\gamma)}$ is taken with respect to the distribution of

$(r(t), Y(t))$ solution of (2.17)-(2.18) starting at time t from (x, y). Then P is a classical solution of Feynman-Kac partial differential equation

$$\frac{\partial P}{\partial t} + \frac{1}{2} f(y)^2 \frac{\partial^2 P}{\partial x^2} + (\hat{a}(\bar{r}_\infty - x) - \lambda(y) f(y))\frac{\partial P}{\partial x} - xP$$

$$+ \beta\rho f(y)\frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2}\beta^2 \frac{\partial^2 P}{\partial y^2} + (\alpha(m - y) - \beta\Lambda(y))\frac{\partial P}{\partial y} = 0 \quad (2.20)$$

with the terminal condition $P(T, x, y; T) = 1$ for every x and y, where we define

$$\Lambda(y) = \rho\lambda(y) + \rho'\gamma(y),$$

the *combined market price of risk.*

In the model of fast mean-reverting stochastic volatility, the rate of mean-reversion α in OU process $(Y(t))$ driving the volatility is assumed to be large. Hence, we need to find asymptotic behavior of $P(t, x, y; T)$ in the limit $\alpha \to \infty$ with $\nu = \beta/\sqrt{2\alpha}$ remaining constant. The asymptotic analysis is carried out as $1/\alpha$ becomes small. For this purpose, we make the following scaling:

$$\varepsilon = \frac{1}{\alpha}, \quad (2.21)$$

$$\beta = \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}, \quad (2.22)$$

$$P(t, x, y; T) = M^{\varepsilon}(T - t, y)e^{-B(T-t)x}, \quad (2.23)$$

where B is given by (2.8). The function $M^{\varepsilon}(t, y)$ is defined by $M^{\varepsilon}(t, y) = M(t, y)$ with M as given in the following. In (2.23), we re-label M as M^{ε} to stress the dependence on the small parameter ε, while we notice that $e^{-B(T-t)x}$ does not depend on ε:

$$M(t,\ y) := E\left\{ e^{\int_t^T c(s, \overline{Y}(s))\, ds} \,\middle|\, \overline{Y}(t) = y \right\},$$

where $\overline{Y}$ is defined by

$$d\overline{Y}(t) = (\alpha(m - \overline{Y}(t)) - \beta b(t,\ \overline{Y}(t)))dt + \beta d\overline{W}(t),$$

on some probability space, where $\overline{W}$ is a standard Brownian motion, and we define

$$b(t,\ y) = \Lambda(y) + \rho f(y) B(T - t), \tag{2.24}$$

$$c(t,\ y) = \frac{1}{2} f(y)^2 B(T - t)^2 - B(T - t)(\hat{a}\overline{r}_\infty - \lambda(y) f(y)) \tag{2.25}$$

with $B(\tau)$ as given by (2.8), and with $f(y)$, $\lambda(y)$, $\gamma(y)$ and $\Lambda(y) = \rho\lambda(y) + \rho'\gamma(y)$ as given in (2.20); the same assumptions on these functions are imposed as in the preceding. Then the coefficients of $\overline{Y}$ are bounded with smooth derivatives, and hence M is the unique classical solution of the Feynman-Kac partial differential equation:

$$\frac{\partial M}{\partial t} + \frac{1}{2}\beta^2 \frac{\partial^2 M}{\partial y^2} + (\alpha(m - y) - \beta b(t,\ y))\frac{\partial M}{\partial y} + c(t,\ y)M = 0, \tag{2.26}$$

$$M(T,\ y) = 1. \tag{2.27}$$

Moreover, we define P_M by

$$P_M(t,\ x,\ y;\ T) = M(t,\ y)e^{-B(T-t)x} \tag{2.28}$$

with B as defined in (2.8). Then it follows from direct calculation that P_M is a classical solution of (2.20) with terminal condition 1, and hence $P_M(t,\ x,\ y;\ T) = P(t,\ x,\ y;\ T)$, that is, the bond price (2.19) is given by formula (2.28):

$$E^{\star(\lambda,\ \gamma)}\left\{ e^{-\int_t^T r(s)\, ds} \,\middle|\, r(t) = x,\ Y(t) = y \right\} = M(t,\ y)e^{-B(T-t)x}. \tag{2.29}$$

Hence, validity of formula (2.23) is verified.

Then, by singular perturbation method, Cotton et al. [2] and Fouque et al. [7] studied asymptotic analysis and obtained corrections for bond prices under the scaling (2.21) and (2.22) as the following theorem:

Theorem 2.1. *Introduce the expressions*

$$\bar{\sigma}^2 = \langle f^2 \rangle, \tag{2.30}$$

$$r^\star = \bar{r}_\infty - \langle \lambda f \rangle / \hat{a}, \tag{2.31}$$

$$V_3 = \frac{\nu\sqrt{\varepsilon}}{\sqrt{2}} \rho \langle f \phi' \rangle, \tag{2.32}$$

$$V_2 = -\frac{\nu\sqrt{\varepsilon}}{\sqrt{2}} \langle \Lambda \phi' \rangle - \nu\rho\sqrt{2\varepsilon} \langle f \psi' \rangle, \tag{2.33}$$

$$V_1 = \nu\sqrt{2\varepsilon} \langle \Lambda \psi' \rangle, \tag{2.34}$$

where $\Lambda(y) = \rho\lambda(y) + \sqrt{1-\rho^2}\,\gamma(y),$ *and* $\langle \cdot \rangle$ *denotes the averaging with respect to the density* $n(y)$ *of the normal distribution* $N(m, \nu^2),$ *and* ϕ *and* ψ *are specific solutions of the Poisson equations*

$$\mathcal{L}_0 \phi = f^2 - \langle f^2 \rangle, \tag{2.35}$$

$$\mathcal{L}_0 \psi = \lambda f - \langle \lambda f \rangle, \tag{2.36}$$

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y}. \tag{2.37}$$

Here, writing $\mathcal{L}_0 = \frac{\nu^2}{n}\frac{\partial}{\partial y}\left(n \frac{\partial}{\partial y}\right),$ *under the boundedness assumptions on* f *and* λ, *we can choose* ϕ *and* ψ *such that their first derivatives given by*

$$\phi' = \frac{1}{\nu^2 n(y)} \int_{-\infty}^{y} (f(z)^2 - \langle f^2 \rangle) n(z) dz,$$

$$\psi' = \frac{1}{\nu^2 n(y)} \int_{-\infty}^{y} (\lambda(z) f(z) - \langle \lambda f \rangle) n(z) dz,$$

are bounded and ϕ *and* ψ *themselves are at most linearly growing in* $|\, y \,|$.

Further, *let*

$$D(\tau) = \frac{V_3}{\hat{a}^3}\left(\tau - B(\tau) - \frac{1}{2}\hat{a}B(\tau)^2 - \frac{1}{3}\hat{a}^2 B(\tau)^3\right)$$

$$- \frac{V_2}{\hat{a}^2}\left(\tau - B(\tau) - \frac{1}{2}\hat{a}B(\tau)^2\right) + \frac{V_1}{\hat{a}}(\tau - B(\tau)), \tag{2.38}$$

$$P_0(t, x; T) = \overline{P}(t, x; T) = A(T-t)e^{-B(T-t)x}, \tag{2.39}$$

$$\widetilde{P}_1(t, x; T) = D(T-t)A(T-t)e^{-B(T-t)x}, \tag{2.40}$$

where A and B are defined in (2.9) *and* (2.8), *respectively*. *We notice that* P_0 *is exactly the Vasicek one-factor bond pricing function* $\overline{P}$ *with the* "*averaged parameters*" $(\hat{a}, r^\star, \overline{\sigma})$ *related to the stochastic volatility parameters in* (2.17)-(2.18) *by* (2.30)-(2.31); *see* (2.4)-(2.5).

Let $P(t, x, y; T)$ *be the model's bond price given by* (2.19). *Then the corrected bond price is given by*

$$P(t, x, y; T) \approx P_0(t, x; T) + \widetilde{P}_1(t, x; T)$$

$$= A(T-t)[1 + D(T-t)]e^{-B(T-t)x} \text{ as } \varepsilon \to 0, \tag{2.41}$$

where D is a small factor of order $\sqrt{\varepsilon} = 1/\sqrt{\alpha}$. *The error in approximation* (2.41) *is of order* $\varepsilon = 1/\alpha$. *Thus*, *for any fixed* $t < T$, $x, y \in \mathbb{R}$,

$$|\, P(t, x, y; T) - (P_0(t, x; T) + \widetilde{P}_1(t, x; T))\,| = O(\varepsilon). \tag{2.42}$$

Theorems 4.6 and 4.7 in Section 4 will show an extension of group parameters $\{V_3, V_2, V_1\}$ as given by (2.32)-(2.34) to the case of the fractional Vasicek model.

3. Pricing PDE

Let us consider the interest rate model as given by (1.1) and (1.2). Then we shall derive bond pricing PDE by heuristic method, using fractional Ito formula in fractional integration theory. For simplicity of notation, we rewrite equations (1.1) and (1.2), omitting the variables t, as follows:

$$dr = b_r dt + f(Y)dW, \tag{3.1}$$

$$dY = b_Y dt + \beta dB_H, \tag{3.2}$$

where

$$b_r := \hat{a}(\bar{r}_\infty - r), \quad b_Y := \alpha(m - Y). \tag{3.3}$$

Then equations (3.1) and (3.2) can be simplified as follows:

$$d\begin{pmatrix} r \\ Y \end{pmatrix} = \begin{pmatrix} b_r \\ b_Y \end{pmatrix} dt + \begin{pmatrix} f(Y) & 0 \\ 0 & \beta \end{pmatrix} d\begin{pmatrix} W \\ B_H \end{pmatrix}. \tag{3.4}$$

By Assumption 1.1(ii), since the random sources W and B_H are independent, we can apply both standard Ito formula and fractional Ito formula to equation (3.4).

Given a time-t short rate r and time-t random sources (W, B_H), the expected change and the variance of the change in interest rate and volatility factor are assumed to be functions of both r and Y. Let

$$V := V(t, r, Y; T)$$

be the price of the T-maturity discount bond with unit par. Then, by standard Ito formula and fractional Ito formula (Theorem B.3 in Appendix B and Lemma C.2 in Appendix C), the stochastic process of the price of the discount bond is given by dV/V, where

$$\begin{aligned} dV &= \left\{ \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial r} dr + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} f^2(Y) dt \right\} \\ &\quad + \left\{ \frac{\partial V}{\partial Y} dY + \frac{1}{2} \frac{\partial^2 V}{\partial Y^2} \left[e^{-2\alpha t} \frac{d}{dt} \| g_t \|^2_{\Theta_{H,t}} \right] dt \right\} \\ &= \left\{ \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r} b_r + \frac{1}{2} \frac{\partial^2 V}{\partial r^2} f^2(Y) \right) \right. \end{aligned}$$

$$+\left(\frac{\partial V}{\partial Y}b_Y + \frac{1}{2}\frac{\partial^2 V}{\partial Y^2}\left[e^{-2\alpha t}\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right]\right)\Bigg\}dt$$

$$+\left\{\frac{\partial V}{\partial r}f(Y)dW + \frac{\partial V}{\partial Y}\beta dB_H\right\}. \tag{3.5}$$

Here, $\|\cdot\|_{\Theta_{H,t}}$ is the norm defined on the Hilbert space $\Theta_{H,t}$ of integrands on $[0, t]$, $t \le T$, associated with the induced transformation of representation for $B_H(t)$ (see Remarks B.1 and B.2 and Lemma C.2), and

$$g_t(u) := \chi_{[0,t]}(u)g(u), \quad g(u) := e^{\alpha u}\beta, \quad 0 \le u \le t \le T.$$

Namely,

$$\frac{dV}{V} = \mu(t, r, Y; T)dt + \sigma_r(t, r, Y; T)dW + \sigma_Y(t, r, Y; T)dB_H, \tag{3.6}$$

where

$$\mu(t, r, Y; T) = \frac{1}{V}\left[\frac{\partial V}{\partial t} + b_r\frac{\partial V}{\partial r} + \frac{1}{2}f^2(Y)\frac{\partial^2 V}{\partial r^2}\right.$$

$$\left. + b_Y\frac{\partial V}{\partial Y} + \frac{1}{2}\left[e^{-2\alpha t}\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right]\frac{\partial^2 V}{\partial Y^2}\right], \tag{3.7}$$

$$\sigma_r(t, r, Y; T) = \frac{1}{V}f(Y)\frac{\partial V}{\partial r} \quad \text{and} \quad \sigma_Y(t, r, Y; T) = \frac{1}{V}\beta\frac{\partial V}{\partial Y}. \tag{3.8}$$

In the following, we shall take the same argument as that in Kwok [12, pp. 404-406].

Since the short rate is not a traded security, it cannot be used to hedge with bond, like the role of the underlying asset in an equity option. Instead we try to hedge bonds of different maturities. This is possible because the instantaneous returns on bonds of varying maturities are correlated as there exists the common underlying stochastic short rate that derives the bond prices. In fact, we construct a portfolio as follows: Since there are two stochastic factors in the model, we need bonds of three different maturities to form a riskless hedging portfolio. Suppose we construct a portfolio which contains $\Delta_1, \Delta_2, \Delta_3$ units of bonds with maturity dates T_1, T_2, T_3, respectively. Let Π denote the value of the portfolio. As usual, we follow the Black-Scholes approach of keeping the portfolio composition to instantaneously "frozen". By (3.6), the rate of return on the portfolio over time dt is given by

$$\begin{aligned} d\Pi &= [\Delta_1\mu(T_1) + \Delta_2\mu(T_2) + \Delta_3\mu(T_3)]dt \\ &\quad + [\Delta_1\sigma_r(T_1) + \Delta_2\sigma_r(T_2) + \Delta_3\sigma_r(T_3)]dW \\ &\quad + [\Delta_1\sigma_Y(T_1) + \Delta_2\sigma_Y(T_2) + \Delta_3\sigma_Y(T_3)]dB_H. \end{aligned} \tag{3.9}$$

Here, $\mu(T_i) := \mu(t, r, Y; T_i)$ denotes the drift rate of the bond with maturity T_i, $i = 1, 2, 3$, and similar notational interpretation for $\sigma_r(T_i) := \sigma_r(t, r, Y; T_i)$ and $\sigma_Y(T_i) := \sigma_Y(t, r, Y; T_i)$. Suppose we choose Δ_1, Δ_2 and Δ_3 such that the coefficients of the stochastic terms in (3.9) are zero, thus making the portfolio value to be instantaneously riskless. This leads to two equations for Δ_1, Δ_2 and Δ_3 such that

$$\Delta_1\sigma_r(T_1) + \Delta_2\sigma_r(T_2) + \Delta_3\sigma_r(T_3) = 0, \tag{3.10}$$

$$\Delta_1\sigma_Y(T_1) + \Delta_2\sigma_Y(T_2) + \Delta_3\sigma_Y(T_3) = 0. \tag{3.11}$$

Since the portfolio is now instantaneously riskless, it must earn the riskless short rate to avoid arbitrage, that is,

$$\begin{aligned} d\Pi &= [\Delta_1\mu(T_1) + \Delta_2\mu(T_2) + \Delta_3\mu(T_3)]dt \\ &= r[\Delta_1 + \Delta_2 + \Delta_3]dt. \end{aligned}$$

Thus,

$$\Delta_1[\mu(T_1) - r] + \Delta_2[\mu(T_2) - r] + \Delta_3[\mu(T_3) - r] = 0. \tag{3.12}$$

The simultaneous linear equations (3.10), (3.11) and (3.12) for Δ_1, Δ_2 and Δ_3 can be represented by the matrix equation as follows:

$$\begin{pmatrix} \sigma_r(T_1) & \sigma_r(T_2) & \sigma_r(T_3) \\ \sigma_Y(T_1) & \sigma_Y(T_2) & \sigma_Y(T_3) \\ \mu(T_1)-r & \mu(T_2)-r & \mu(T_3)-r \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.13}$$

Nontrivial solutions for Δ_1, Δ_2 and Δ_3 exist when the third row in the above coefficient matrix can be expressed as some linear combination of the first and second rows. Since the maturity dates T_1, T_2 and T_3 are arbitrary, this leads to the following relation between the drift rate and volatility functions:

$$\mu(t, r, Y; T) - r = \lambda_r \sigma_r(t, r, Y; T) + \lambda_Y \sigma_Y(t, r, Y; T), \tag{3.14}$$

where the multipliers λ_r and λ_Y are independent of maturity T; in general, λ_r and λ_Y should have dependence on r, Y and t. Here, λ_r and λ_Y are recognized as the respective *market price of risk* of the short rate r and volatility factor Y. Substituting the expressions for $\mu(t, r, Y; T)$, $\sigma_r(t, r, Y; T)$ and $\sigma_Y(t, r, Y; T)$, as given by (3.7) and (3.8), into (3.14), we obtain the following governing equation for the bond price:

$$\frac{1}{V}\left[\frac{\partial V}{\partial t} + b_r \frac{\partial V}{\partial r} + \frac{1}{2} f^2(Y) \frac{\partial^2 V}{\partial r^2} \right.$$
$$\left. + b_Y \frac{\partial V}{\partial Y} + \frac{1}{2}\left[e^{-2\alpha t} \frac{d}{dt} \| g_t \|^2_{\Theta_{H,t}}\right] \frac{\partial^2 V}{\partial Y^2}\right] - r$$
$$= \lambda_r \frac{1}{V} f(Y) \frac{\partial Y}{\partial r} + \lambda_Y \frac{1}{V} \beta \frac{\partial V}{\partial Y}.$$

Namely,

$$\left[\frac{\partial V}{\partial t} + b_r \frac{\partial V}{\partial r} + \frac{1}{2} f^2(Y) \frac{\partial^2 V}{\partial r^2} \right.$$
$$\left. + b_Y \frac{\partial V}{\partial Y} + \frac{1}{2}\left[e^{-2\alpha t} \frac{d}{dt} \| g_t \|^2_{\Theta_{H,t}}\right] \frac{\partial^2 V}{\partial Y^2}\right] - rV$$
$$= \lambda_r f(Y) \frac{\partial V}{\partial r} + \lambda_Y \beta \frac{\partial V}{\partial Y}.$$

Therefore,

$$\frac{\partial V}{\partial t} + (b_r - \lambda_r f(Y))\frac{\partial V}{\partial r} + \frac{1}{2} f^2(Y)\frac{\partial^2 V}{\partial r^2}$$

$$+ (b_Y - \lambda_Y \beta)\frac{\partial V}{\partial Y} + \frac{1}{2}\left[e^{-2\alpha t}\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right]\frac{\partial^2 V}{\partial Y^2} - rV$$

$$= 0.$$

Combining this with Lemma C.2 in Appendix C, we obtain the following lemma:

Lemma 3.1 (Bond pricing PDE)**.** *Suppose Assumption* 1.1. *Then the no-arbitrage bond price satisfies the following equation*:

$$\frac{\partial V}{\partial t} + (b_r - \lambda_r f(Y))\frac{\partial V}{\partial r} + \frac{1}{2} f^2(Y)\frac{\partial^2 V}{\partial r^2}$$

$$+ (b_Y - \lambda_Y \beta)\frac{\partial V}{\partial Y} + \frac{1}{2}\left[e^{-2\alpha t}\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right]\frac{\partial^2 V}{\partial Y^2} - rV$$

$$= 0 \tag{3.15}$$

with suitable scalars λ_r *and* λ_Y, *i.e.*, *the market prices of risk with respect to r and Y. Here,* $\|\cdot\|_{\Theta_{H,t}}$ *is the norm which is defined on the Hilbert space* $\Theta_{H,t}$ *of integrands on* $[0, t]$ *associated with the induced transformation of representation for* $B_H(t)$, *and*

$$b_r = \hat{a}(\bar{r}_\infty - r), \quad b_Y = \alpha(m - Y),$$

$$g_t(s) = \chi_{[0,t]}(s) g(s), \quad g(s) = e^{\alpha s}\beta, \quad 0 \le s \le t \le T.$$

Further, *the expression for*

$$e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right]$$

is obtained by applying (C.6) *and* (C.7) *of Lemma* C.2 *in Appendix* C, *according to* $H > 1/2$ *and* $0 < H < 1/2$, *respectively. The terminal condition for V is given by* $V(T, r, Y; T) = 1$.

The market price of risk λ_r may depend on *Y* but we assume that it does not depend on the short rate *r*. Similarly, the *market price of volatility risk* λ_Y associated with the second source of randomness $B_H(t)$ may depend on *Y* or simply be a constant. More precisely, we assume the following:

Assumption 3.2. In our model described by (1.1) and (1.2), we assume that the functions λ_r and λ_Y, the market prices of risk with respect to *r* and *Y*, respectively, are expressed by

$$\lambda_r = \lambda(Y), \tag{3.16}$$

$$\lambda_Y = \alpha^{\frac{1}{2}-H}\gamma(Y), \tag{3.17}$$

appealing to the Hurst parameter *H*. The risk premium processes $\lambda(Y(t))$ and $\gamma(Y(t))$ are bounded:

$$c_1' \leq \lambda, \gamma \leq c_2' < \infty$$

for some constants $c_1' > 0$ and $c_2' > 0$, and λ and γ have bounded derivatives.

4. Asymptotics in the Fast-scale

In order to model fast mean-reversion, we rescale α, rate of mean-reversion, under the following assumption:

Assumption 4.1. Let $0 < H < 1$. Consider $(Y(t))$ as given by (1.2). Then we introduce the scaling as follows:

(i) The rate of mean-reversion α or its inverse, the typical correlation time of $(Y(t))$, is characterized by a small parameter $0 < \varepsilon \ll 1$ such that

$$\varepsilon = \frac{1}{\alpha}.$$

(ii) Let ν_H^2 be the variance of the normal distribution $N(m, \nu_H^2)$ which is the invariant distribution of $(Y(t))$, as given by (C.4) of Lemma C.1 in Appendix C; ν_H^2 controls the long-run size of the volatility fluctuations. Then we assume this quantity remains fixed as we consider smaller and smaller values of ε such that

$$\beta = \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\left(\frac{1}{\alpha}\right)^{-H} = \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\frac{1}{\varepsilon^H}.$$

Remark 4.2. Under Assumption 3.2, define $k(t, r, Y)$ and $\widetilde{k}(t, r, Z)$ by

$$k(t, r, Y) := b_r - \lambda_r f(Y),$$

$$\widetilde{k}(t, r, Y) = b_Y - \lambda_Y \beta.$$

Then k and $\widetilde{k}$ are expressed as follows:

$$k(t, r, Y) = \hat{a}(\bar{r}_\infty - r) - \lambda(Y) f(Y), \tag{4.1}$$

$$\begin{aligned} \widetilde{k}(t, r, Y) &= \alpha(m - Y) - \alpha^{\frac{1}{2}-H}\gamma(Y)\beta \\ &= \frac{1}{\varepsilon}(m - Y) - \frac{1}{\sqrt{\varepsilon}}\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\gamma(Y). \end{aligned} \tag{4.2}$$

If $H = 1/2$, then equations (4.1) and (4.2) coincide with the expression for the corrected drift term as given by Cotton et al. [2] and Fouque et al. [7], and with uncorrelated case where correlation coefficient $\rho = 0$ such that volatility shocks and short rate shocks are independent; see PDE (2.20) in Section 2.

The following result is given in Narita [16, Lemmas 10.2 and 10.3].

Remark 4.3. Let $0 < H < 1$. Let $(Y(t))$ be given by (1.2). Define the function g_t by

$$g_t(u) = \chi_{[0,t]}(u) g(u), \quad g(u) = e^{\alpha u}\beta, \quad 0 \le u \le t \le T.$$

Let $\alpha = 1/\varepsilon$, where $0 < \varepsilon \ll 1$. Then, for $t > 0$,

$$e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right] = 2\left\{\left(\nu_H^2 \frac{1}{\varepsilon}\right) + o(1)\right\} \tag{4.3}$$

as $\varepsilon \to 0$; see Lemma D.2 in Appendix D.

Then, by Lemma 3.1 and Remarks 4.2 and 4.3, we obtain the following pricing PDE:

Lemma 4.4 (Bond pricing PDE in terms of ε)**.** *Let* $0 < H < 1$. *Suppose that Assumptions* 1.1, 3.2 *and* 4.1 *hold. Denote the no-arbitrage price at time* t *of a zero-coupon bond with maturity time* T *by* P_t^{ε}, *emphasizing dependence on the small parameter* ε. *Then* $\{P_t^{\varepsilon};\, 0 \le t \le T\}$ *is a function of current price of short rate and volatility*:

$$P_t^{\varepsilon} = P^{\varepsilon}(t,\, r(t),\, Y(t);\, T).$$

Moreover, *for* ε *small enough*, *the function* $P^{\varepsilon}(t,\, x,\, y;\, T)$ *when*

$$r(t) = x, \quad Y(t) = y$$

satisfies the following equation:

$$\left[\frac{\partial P^{\varepsilon}}{\partial t} + \frac{1}{2} f^2(y)\frac{\partial^2 P^{\varepsilon}}{\partial x^2} + (\hat{a}(\bar{r}_{\infty} - x) - \lambda(y) f(y))\frac{\partial P^{\varepsilon}}{\partial x} - xP^{\varepsilon}\right]$$

$$+ \left(\nu_H^2 \frac{1}{\varepsilon}\right)\frac{\partial^2 P^{\varepsilon}}{\partial y^2} + \left\{\frac{1}{\varepsilon}(m - y) - \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\frac{1}{\sqrt{\varepsilon}}\gamma(y)\right\}\frac{\partial P^{\varepsilon}}{\partial y}$$

$$= 0 \tag{4.4}$$

with the terminal condition

$$P^{\varepsilon}(T,\, x,\, y;\, T) = 1. \tag{4.5}$$

We shall find an asymptotic solution for PDE (4.4). For this purpose, we introduce the following operators:

$$\mathcal{L}_0 := \nu_H^2 \frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y}, \tag{4.6}$$

$$\mathcal{L}_1 := -\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\gamma(y)\frac{\partial}{\partial y}, \tag{4.7}$$

$$\mathcal{L}_2 := \frac{\partial}{\partial t} + \frac{1}{2} f^2(y)\frac{\partial^2}{\partial x^2} + (\hat{a}(\bar{r}_\infty - x) - \lambda(y) f(y))\frac{\partial}{\partial x} - x. \tag{4.8}$$

Remark 4.5. Introduce the usual one-factor Vasicek operator:

$$\mathcal{L}_{Vasicek}(\overline{\sigma}, r^\star) := \frac{\partial}{\partial t} + \frac{1}{2}\overline{\sigma}^2 \frac{\partial^2}{\partial x^2} + \hat{a}(r^\star - x)\frac{\partial}{\partial x} - x. \tag{4.9}$$

with deterministic parameters $\overline{\sigma} > 0$ and $r^\star > 0$, where $\hat{a}$ measures the characteristic speed of mean-reversion of OU process $(\bar{r}(t))$; see (2.1) and (2.2) in Section 2. Then $\mathcal{L}_2$ is viewed as

$$\mathcal{L}_2 = \mathcal{L}_{Vasicek}(\overline{\sigma}, r^\star)$$

with constant levels taken by

$$\overline{\sigma} = f(y), \quad r^\star = \bar{r}_\infty - (\lambda(y) f(y))/\hat{a}.$$

In Theorem 4.7 further, the y-dependent parameters are computed as the averaged values with respect to the density $n(y)$ of the invariant probability distribution of $(Y(t))$, and hence main operator $\langle \mathcal{L}_2 \rangle$ is obtained as follows:

$$\langle \mathcal{L}_2 \rangle = \mathcal{L}_{Vasicek}(\overline{\sigma}, r^\star)$$

with deterministic parameters

$$\overline{\sigma} = \langle f(y) \rangle, \quad r^\star = \bar{r}_\infty - (\langle \lambda(y) f(y) \rangle)/\hat{a}.$$

Here and hereafter, $\langle h(y) \rangle$ denotes the average of h on $\mathbb{R}$ with respect to the density $n(y)$ of the normal distribution $N(m, \nu_H^2)$ (see (C.3) of Lemma C.1

in Appendix C), the invariant distribution of fOU process $(Y(t))$:

$$n(y) = \frac{1}{\sqrt{2\pi\nu_H^2}} \exp\left(-\frac{(y-m)^2}{2\nu_H^2}\right),$$

$$\langle h(y)\rangle := \langle h\rangle := \int_{-\infty}^{\infty} h(y)n(y)dy = \frac{1}{\sqrt{2\pi\nu_H^2}} \int_{-\infty}^{\infty} h(y)\exp\left(-\frac{(y-m)^2}{2\nu_H^2}\right)dy.$$

PDE (4.4) is equivalent to the following singularly perturbed equation:

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P^{\varepsilon} = 0 \tag{4.10}$$

with the terminal condition (4.5).

In the following, we shall find asymptotic expansion for P^{ε} with respect to the small parameter ε. For this purpose, we shall assume that P^{ε} can be expanded by powers of $\sqrt{\varepsilon}$ as follows:

$$P^{\varepsilon}(t, x, y; T) = P_0(t, x, y; T) + \sqrt{\varepsilon}P_1(t, x, y; T) + \varepsilon P_2(t, x, y; T) + \cdots \tag{4.11}$$

for small ε, where $P_0, P_1, \ldots$ are functions of (t, x, y) to be determined by the terminal conditions

$$P_0(T, x, y; T) = 1, \quad P_i(T, x, y; T) = 0 \quad \text{for } i \geq 1. \tag{4.12}$$

Recall that the volatility factor associated with ε corresponds to the fast volatility factor $Y(t)$. In the case of the fast mean-reverting stochastic volatility, such an expansion in powers of $\sqrt{\varepsilon}$ is considered in Cotton et al. [2] and Fouque et al. [7], and the references therein, where the *singular perturbation* analysis for financial markets with stochastic volatilities is taken under the sBm environment; see (2.20)-(2.23) and Theorem 2.1 in Section 2.

Our theorem (Theorem 4.6 in Section 4) is an extension of the precedent result in Cotton et al. [2] and Fouque et al. [7] to the stochastic volatility interest rate model under the fBm environment.

In the following, we shall find the solution P^{ε} of the form (4.11). Substituting the expansion (4.11) into (4.10), we can write as follows:

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)(P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \cdots)$$

$$= \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P_0$$

$$+ \sqrt{\varepsilon}\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P_1 + \varepsilon\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P_2 + \cdots$$

$$= 0.$$

Therefore,

$$\frac{1}{\varepsilon}\mathcal{L}_0P_0 + \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_0P_1 + \mathcal{L}_1P_0) + (\mathcal{L}_0P_2 + \mathcal{L}_1P_1 + \mathcal{L}_2P_0)$$

$$+ \sqrt{\varepsilon}(\mathcal{L}_0P_3 + \mathcal{L}_1P_2 + \mathcal{L}_2P_1) + \cdots$$

$$= 0. \tag{4.13}$$

For equation above, step by step, the terms of order $1/\varepsilon, 1/\sqrt{\varepsilon}, \ldots$ will be studied.

Term of order $1/\varepsilon$. At order $1/\varepsilon,$ we have

$$\mathcal{L}_0P_0 = 0. \tag{4.14}$$

The operator $\mathcal{L}_0$ contains partial derivatives with respect to *y* but no derivatives with respect to *x*. Hence P_0 must be a constant with respect to the variable *y*, which implies

$$P_0 = P_0(t,\ x;\ T) \tag{4.15}$$

with terminal condition $P_0(T,\ x;\ T) = 1.$

Term of order $1/\sqrt{\varepsilon}$. At order $1/\sqrt{\varepsilon}$, we have

$$\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0. \tag{4.16}$$

Notice that P_0 only depends on t and x and that the operator $\mathcal{L}_1$ involves the derivative with respect to y. Then we have that $\mathcal{L}_1 P_0 = 0$, and hence (4.16) is reduced to $\mathcal{L}_0 P_1 = 0$. The operator $\mathcal{L}_0$ involves derivatives with respect to y. Thus, P_1 must be a constant with respect to y, which implies $P_1 = P_1(t, x; T)$ with the terminal condition $P_1(T, x; T) = 0$.

Thus, we note that the term $P_0 + \sqrt{\varepsilon} P_1$ in (4.11) will not depend on y.

Zeroth order-term. At order 1, we have

$$\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0. \tag{4.17}$$

The discussion above implies that P_0 and P_1 only depend on (t, x) and that $\mathcal{L}_1$ and $\mathcal{L}_0$ involve derivatives with respect to y. Thus, $\mathcal{L}_1 P_1 = 0$, and hence (4.17) is reduced to

$$\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0. \tag{4.18}$$

Here, P_0 only depends on t and x. When regarding x as fixed, $\mathcal{L}_2 P_0$ only depends on y. Hence equation (4.18) is a Poisson equation for P_2 with respect to $\mathcal{L}_0$, that is, $\mathcal{L}_0 P_2 = -\mathcal{L}_2 P_0$.

In order to have a solution to the Poisson equation (4.18), $\mathcal{L}_2 P_0$ must be in the orthogonal complement of the null space of $\mathcal{L}_0^*$, where $\mathcal{L}_0^*$ is the adjoint operator of $\mathcal{L}_0$ such that

$$\mathcal{L}_0^* p = -\frac{\partial}{\partial y}((m - y)p) + \nu_H^2 \frac{\partial^2 p}{\partial y^2} \tag{4.19}$$

for $p \in C^2(\mathbb{R})$. This solvability condition is equivalent to saying that $\mathcal{L}_0 P_2$ has mean zero with respect to the invariant distribution, which yields

$$E[\mathcal{L}_2 P_0] = \langle \mathcal{L}_2 P_0 \rangle = \int_{-\infty}^{\infty} \mathcal{L}_2 P_0 n(y) dy = 0, \tag{4.20}$$

where $n(y)$, the density of the normal distribution $N(m, \nu_H^2)$, solves $\mathcal{L}_0^* n = 0$. Namely, the solvability condition above implies that $\langle \mathcal{L}_2 P_0 \rangle = 0$. Since P_0 does not depend on y, the solvability condition is reduced to

$$\langle \mathcal{L}_2 \rangle P_0 = 0, \tag{4.21}$$

which is exactly the partial differential equation

$$\langle \mathcal{L}_2 \rangle P_0 = \frac{\partial P_0}{\partial t} + \frac{1}{2} \overline{\sigma}^2 \frac{\partial^2 P_0}{\partial x^2} + \hat{a}(r^* - x) \frac{\partial P_0}{\partial x} - x P_0 = 0 \tag{4.22}$$

with the coefficients

$$\overline{\sigma}^2 := \langle f^2 \rangle \quad \text{and} \quad r^\star := \bar{r}_\infty - \frac{1}{\hat{a}} \langle \lambda f \rangle \tag{4.23}$$

and with the same terminal condition $P_0(T, x; T) = 1$. We recall that equation (4.22) is the same as equation (2.4) for the one-factor Vasicek model. This implies that $P_0 = \overline{P}$ with the function $\overline{P}$ as given by (2.5), where constant parameters $\overline{\sigma}$ and $r^\star$ are replaced by (4.23).

In equations above and below, we notice that the averaged quantity $\langle \cdot \rangle$ does not depend on ε.

The change of variable $\tau = T - t$ is again convenient, and we obtain

$$P_0(T - \tau, x; T) = A(\tau) e^{-B(\tau) x}, \tag{4.24}$$

where $A(\tau)$ and $B(\tau)$ are given explicitly by formulas (2.9) and (2.8) in Section 2, respectively, except that $\overline{\sigma}$ and $r^\star$ are given by (4.23).

Observe the equation

$$\mathcal{L}_0 P_2 = -\mathcal{L}_2 P_0. \tag{4.25}$$

Apply the same solvability condition as given in equation (4.20). Then, since $\langle \mathcal{L}_2 P_0 \rangle = 0,$ we see that $\mathcal{L}_2 P_0$ in the right-hand side of (4.25) can be written as

$$\mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle (= \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 \rangle P_0)$$

$$= \frac{1}{2}(f(y)^2 - \langle f^2 \rangle)\frac{\partial^2 P_0}{\partial x^2} - (\lambda f - \langle \lambda f \rangle)\frac{\partial P_0}{\partial x}.$$

Recall that $\mathcal{L}_0 P_2 = -\mathcal{L}_2 P_0.$ Then we have

$$\mathcal{L}_0 P_2 = -\frac{1}{2}(f^2 - \langle f^2 \rangle)\frac{\partial^2 P_0}{\partial x^2} + (\lambda f - \langle \lambda f \rangle)\frac{\partial P_0}{\partial x}. \qquad (4.26)$$

The solution of the Poisson equation (4.26) is given by

$$P_2 = \mathcal{L}_0^{-1}\left\{-\frac{1}{2}(f^2 - \langle f^2 \rangle)\frac{\partial^2 P_0}{\partial x^2} + (\lambda f - \langle \lambda f \rangle)\frac{\partial P_0}{\partial x}\right\}$$

$$= -\frac{1}{2}\mathcal{L}_0^{-1}\{(f^2 - \langle f^2 \rangle)\}\frac{\partial^2 P_0}{\partial x^2} + \mathcal{L}_0^{-1}\{(\lambda f - \langle \lambda f \rangle)\}\frac{\partial P_0}{\partial x}$$

$$= -\frac{1}{2}\{\phi(y) + c(t, x)\}\frac{\partial^2 P_0}{\partial x^2} + \{\psi(y) + d(t, x)\}\frac{\partial^2 P_0}{\partial x}. \qquad (4.27)$$

Here, $\phi(y)$ and $\psi(y)$ are specific solutions of the Poisson equations

$$\mathcal{L}_0 \phi = f(y)^2 - \langle f^2 \rangle \text{ and } \mathcal{L}_0 \psi = \lambda(y) f(y) - \langle \lambda f \rangle, \qquad (4.28)$$

and $c(t, x)$ and $d(t, x)$ are constants that may depend on (t, x). In fact, $\mathcal{L}_0$ does only involve the variable y and P_0 does not depend on y, and hence we obtain (4.27). We shall go into detail about specific solutions ϕ and ψ in the proof of Theorem 4.7 in Section 4; see (4.48) and (4.49).

Term of order $\sqrt{\varepsilon}$. At order $\sqrt{\varepsilon},$ we have

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0. \qquad (4.29)$$

This is a Poisson equation for P_3 with respect to $\mathcal{L}_0$, which is written by

$$\mathcal{L}_0 P_3 = -(\mathcal{L}_1 P_2 + \mathcal{L}_2 P_1). \tag{4.30}$$

Again, applying the same solvability condition as given in equation (4.20), we obtain

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0.$$

Hence

$$\langle \mathcal{L}_2 P_1 \rangle = -\langle \mathcal{L}_1 P_2 \rangle, \tag{4.31}$$

where P_2 is already known by (4.27). In the following, we investigate $\langle \mathcal{L}_2 P_1 \rangle$. Notice that P_1 does not depend on y and consider that $\langle \mathcal{L}_2 \rangle$ is the one-factor Vasicek operator as given by (4.22) and (4.23). Then we have that the left-hand side of equation (4.31) is equal to $\langle \mathcal{L}_2 \rangle P_1$. Observe the right-hand side of (4.31) with P_2 replaced by (4.27). Then we find

$$\begin{aligned} -\langle \mathcal{L}_1 P_2 \rangle &= \frac{1}{2}\langle \mathcal{L}_1(\phi(y) + c(t, x))\rangle \frac{\partial^2 P_0}{\partial x^2} - \langle \mathcal{L}_1(\psi(y) + d(t, x))\rangle \frac{\partial P_0}{\partial x} \\ &= \frac{1}{2}\langle \mathcal{L}_1 \phi(y)\rangle \frac{\partial^2 P_0}{\partial x^2} - \langle \mathcal{L}_1 \psi(y)\rangle \frac{\partial P_0}{\partial x}, \end{aligned} \tag{4.32}$$

where ϕ and ψ are specific solutions of the Poisson equations (4.28), and we used that $\mathcal{L}_1 c(t, x) = 0$ and $\mathcal{L}_1 d(t, x) = 0$, since $\mathcal{L}_1$ does only involve the variable y. Hence, by (4.31) and (4.32), we obtain

$$\begin{aligned} \langle \mathcal{L}_2 \rangle P_1 &= -\langle \mathcal{L}_1 P_2 \rangle \\ &= \frac{1}{2}\langle \mathcal{L}_1 \phi(y)\rangle \frac{\partial^2 P_0}{\partial x^2} - \langle \mathcal{L}_1 \psi(y)\rangle \frac{\partial P_0}{\partial x} \\ &= -\frac{1}{2}\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\langle \gamma(y)\phi'(y)\rangle \frac{\partial^2 P_0}{\partial x^2} \\ &\quad + \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\langle \gamma(y)\psi'(y)\rangle \frac{\partial P_0}{\partial x}, \end{aligned} \tag{4.33}$$

where

$$\mathcal{L}_1 = -\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\gamma(y)\frac{\partial}{\partial y}.$$

According to Fouque et al. [7, Chapter 11], we denote the first correction by

$$\widetilde{P}_1(t, x; T) = \sqrt{\varepsilon}P_1(t, \varepsilon; T), \tag{4.34}$$

where $\varepsilon = 1/\alpha$, and introduce the expressions

$$V_2 = -\sqrt{\varepsilon}\,\frac{1}{2}\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\langle\gamma(y)\phi'(y)\rangle, \tag{4.35}$$

$$V_1 = \sqrt{\varepsilon}\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\langle\gamma(y)\psi'(y)\rangle. \tag{4.36}$$

Further, we define the operator $\mathcal{A}$ by

$$\mathcal{A} = V_2\frac{\partial^2}{\partial x^2} + V_1\frac{\partial}{\partial x}. \tag{4.37}$$

Then, by (4.33), we observe that the function $\widetilde{P}_1$ is a solution of

$$\langle L_2\rangle\widetilde{P}_1 = \mathcal{A}P_0 \tag{4.38}$$

with the terminal condition $\widetilde{P}_1(T, x; T) = 0.$

Finally, we want to find the explicit form of the solution $\widetilde{P}_1$ to (4.38). Let P_0 be given by $P_0 = \overline{P}$, where $\overline{P}$ has the form (2.5) in Section 2, except that constant parameters $\overline{\sigma}$ and $r^\star$ are expressed by (4.23). Then the change of variable $\tau = T - t$ for P_0 is again convenient; $P_0(T-\tau, x; T) = A(\tau)e^{-B(\tau)x}$ as shown by (4.24). Observe expression (4.37) for $\mathcal{A}$. Then equation (4.38) becomes

$$\frac{\partial\widetilde{P}_1}{\partial\tau} = \frac{1}{2}\overline{\sigma}^2\frac{\partial^2\widetilde{P}_1}{\partial x^2} + \hat{a}(r^\star - x)\frac{\partial\widetilde{P}_1}{\partial x} - x\widetilde{P}_1 + A(\tau)e^{-B(\tau)x}(-V_2B(\tau)^2 + V_1B(\tau)) \tag{4.39}$$

with the terminal condition $\widetilde{P}_1(T-0,\ x;\ T)=0;$ $A(\tau)$ and $B(\tau)$ are expressed by (2.9) and (2.8) in Section 2, respectively, where constant parameters $\overline{\sigma}$ and $r^{\star}$ are given by (4.23).

We now try to find a solution of the form

$$\widetilde{P}_1(T-\tau,\ x;\ T)=D(\tau)A(\tau)e^{-B(\tau)x}, \tag{4.40}$$

which leads to the following equation for $D(\tau)$:

$$D'(\tau)=-V_2B(\tau)^2+V_1B(\tau). \tag{4.41}$$

Here, we have used the differential equation (2.7) in Section 2, satisfied by $A(\tau)$. Since $D(0)=0$ and $A(0)=1$, we obtain an explicit expression for $D(\tau)$, written here as a function of $B(\tau)=(1-e^{-\hat{a}\tau})/\hat{a}$:

$$D(\tau)=-\frac{V_2}{\hat{a}^2}\left(\tau-B(\tau)-\frac{1}{2}\hat{a}B(\tau)^2\right)+\frac{V_1}{\hat{a}}(\tau-B(\tau)). \tag{4.42}$$

Substitute equations (4.24) and (4.40) together with (4.42) into the expansion (4.11). Then we get the bond price with stochastic correction as follows:

$$\begin{aligned}P(T-\tau,\ x,\ y;\ T)&\approx P_0(T-\tau,\ x;\ T)+\widetilde{P}_1(T-\tau,\ x;\ T)\\&=(1+D(\tau))A(\tau)e^{-B(\tau)x}\end{aligned}$$

for ε small enough. Therefore, the *corrected bond price* is given by

$$\begin{aligned}P(t,\ x,\ y;\ T)&\approx P_0(t,\ x;\ T)+\widetilde{P}_1(t,\ x;\ T)\\&=(1+D(T-t))A(T-t)e^{-B(T-t)x}\end{aligned} \tag{4.43}$$

for ε small enough. Here, D is a small factor of order $\sqrt{\varepsilon}=1/\sqrt{\alpha}$. The error in the approximation (4.43) is of order $\varepsilon=1/\alpha$.

Theorem 4.6. *Let* $0<H<1$. *Suppose Assumptions* 1.1, 3.2 *and* 4.1. *Let* P_0 *and* $\widetilde{P}_1$ *be given as follows*:

$$P_0(t,\ x;\ T)=A(T-t)e^{-B(T-t)x}, \tag{4.44}$$

$$\widetilde{P}_1(t, x; T) = D(T-t)A(T-t)e^{-B(T-t)x}, \tag{4.45}$$

where $A(\tau)$ *and* $B(\tau)$ *are given explicitly by formulas* (2.9) *and* (2.8) *in Section* 2, *respectively*, *and* $D(\tau)$ *is given by* (4.42). *Here*, *the* "*averaged*" *parameters* $(\overline{\sigma}, r^\star)$ *appearing in* $A(\tau)$, $B(\tau)$ *and* $D(\tau)$ *are redefined by* (4.23). *Further*, $D(\tau)$ *is a small factor of order* $\sqrt{\varepsilon} = 1/\sqrt{\alpha}$ *as* "*group parameters*" (V_2, V_1) *in equations* (4.35) *and* (4.36) *imply*. *Then*, *for any fixed* $t < T$, $x, y \in \mathbb{R}$,

$$|\, P(t, x, y; T) - (P_0(t, x; T) + \widetilde{P}_1(t, x; T))\,| = O(\varepsilon)$$

for ε *small enough*, *where* $P(t, x, y; T)$ *is the model's bond price*.

In order to illustrate how the fundamental parameters (V_2, V_1) in equations (4.35) and (4.36) are related to the model parameters, we can compute V_2 and V_1 by using the definition of $\phi'(y)$ and $\psi'(y)$, the derivatives of the functions $\phi(y)$ and $\psi(y)$ which are the solutions of the Poisson equations (4.28), i.e.,

$$\mathcal{L}_0\phi = f(y)^2 - \langle f^2\rangle, \quad \mathcal{L}_0\psi = \lambda(y)f(y) - \langle \lambda f\rangle,$$

where $\langle\cdot\rangle$ denotes the averaging with respect to the density

$$n(y) = \frac{1}{\sqrt{2\pi\nu_H^2}}\exp\left(-\frac{(y-m)^2}{2\nu_H^2}\right)$$

of the normal distribution $N(m, \nu_H^2)$. We first notice

$$n'(y) = \frac{1}{\sqrt{2\pi\nu_H^2}}\left(-\frac{(y-m)}{\nu_H^2}\right)\exp\left(-\frac{(y-m)^2}{2\nu_H^2}\right) \qquad \left(' = \frac{d}{dy}\right)$$

$$= -\frac{(y-m)}{\nu_H^2}n(y),$$

and hence

$$\frac{(m-y)}{\nu_H^2} = \frac{n'(y)}{n(y)}.$$

Since $\mathcal{L}_0$ has the form (4.6), the equation above implies

$$\begin{aligned}
\mathcal{L}_0\phi &= \nu_H^2\phi''(y) + (m-y)\phi'(y) \\
&= \nu_H^2\left(\phi''(y) + \frac{(m-y)}{\nu_H^2}\phi'(y)\right) \\
&= \nu_H^2\left(\phi''(y) + \frac{n'(y)}{n(y)}\phi'(y)\right) \\
&= \frac{\nu_H^2}{n(y)}(n(y)\phi''(y) + n'(y)\phi'(y)) \\
&= \frac{\nu_H^2}{n(y)}(n(y)\phi'(y))'.
\end{aligned}$$

Therefore, we have the following relations:

$$\begin{aligned}
\mathcal{L}_0\phi = f(y)^2 - \langle f^2\rangle &\Leftrightarrow \frac{\nu_H^2}{n(y)}(n(y)\phi'(y))' = f(y)^2 - \langle f^2\rangle \\
&\Leftrightarrow \frac{1}{n(y)}(n(y)\phi'(y))' = \frac{1}{\nu_H^2}(f(y)^2 - \langle f^2\rangle) \\
&\Leftrightarrow (n(y)\phi'(y))' = \frac{1}{\nu_H^2}(f(y)^2 - \langle f^2\rangle)n(y).
\end{aligned}$$

Integrating the both sides of the last equation above, we have

$$n(y)\phi'(y) = \frac{1}{\nu_H^2}\int_{-\infty}^{y}(f(z)^2 - \langle f^2\rangle)n(z)dz,$$

and hence

$$\phi'(y) = \frac{1}{\nu_H^2 n(y)}\int_{-\infty}^{y}(f(z)^2 - \langle f^2\rangle)n(z)dz. \tag{4.46}$$

Replacing ϕ and $\langle f^2 \rangle$ by ψ and $\langle \lambda f \rangle$, we take the same argument as taken in the preceding. Then we get

$$\psi'(y) = \frac{1}{\nu_H^2 n(y)} \int_{-\infty}^{y} (\lambda(z) f(z) - \langle \lambda f \rangle) n(z) dz. \tag{4.47}$$

Here, we notice that under the boundedness assumptions on f and λ (see Assumption 1.1(iii) and Assumption 3.2), we can choose ϕ and ψ such that their first derivatives are bounded and ϕ and ψ themselves are at most linearly growing in $|y|$.

The V's are complicated functions of the model parameters, including market prices of risk, and the volatility function *f*. By (4.36) and (4.47), we notice that a market price of risk such that λf is a constant implies $\psi' = 0$ and hence

$$V_1 = \sqrt{\varepsilon}\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\langle \gamma(y)\psi'(y) \rangle = 0.$$

Using (4.46), the formula of $\phi'(y)$, we can find the explicit expression for V_2 as given by (4.35):

$$V_2 = -\sqrt{\varepsilon}\,\frac{1}{2}\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\langle \gamma(y)\phi'(y) \rangle.$$

We first calculate the value of $\langle \gamma\phi' \rangle$:

$$\begin{aligned}
\langle \gamma\phi' \rangle &= \left\langle \gamma\left(\frac{1}{\nu_H^2 n}\right)\left(\int_{-\infty}^{\cdot} (f^2 - \langle f^2 \rangle) n\right)\right\rangle \\
&= \int_{-\infty}^{\infty} \gamma(y)\left(\frac{1}{\nu_H^2 n(y)}\right)\left(\int_{-\infty}^{y} (f(z)^2 - \langle f^2 \rangle) n(z) dz\right) n(y) dy \\
&= \int_{-\infty}^{\infty} \gamma(y)\left(\frac{1}{\nu_H^2}\right)\left(\int_{-\infty}^{y} (f(z)^2 - \langle f^2 \rangle) n(z) dz\right) dy
\end{aligned}$$

$$= \frac{1}{\nu_H^2} \int_{-\infty}^{\infty} \gamma(y) \left(\int_{-\infty}^{y} (f(z)^2 - \langle f^2 \rangle) n(z) dz \right) dy. \tag{4.48}$$

Denote by $G(y)$ the primitive function of $\gamma(y)$, that is,

$$G(y) = \int \gamma(y) dy.$$

Put

$$M(y) = \int_{-\infty}^{y} (f(z)^2 - \langle f^2 \rangle) n(z) dz,$$

where $\langle f^2 \rangle = \int_{-\infty}^{\infty} f(y)^2 n(y) dy$. Then, integrating by parts, from (4.48), we have

$$\begin{aligned}
\langle \gamma \phi' \rangle &= \frac{1}{\nu_H^2} \int_{-\infty}^{\infty} G'(y) M(y) dy \\
&= \frac{1}{\nu_H^2} \left\{ [G(y) M(y)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} G(y) M'(y) dy \right\} \\
&= \frac{1}{\nu_H^2} \left\{ -\int_{-\infty}^{\infty} G(y) M'(y) dy \right\} \quad (\because M(\infty) = M(-\infty) = 0) \\
&= -\frac{1}{\nu_H^2} \int_{-\infty}^{\infty} G(y)(f(y)^2 - \langle f^2 \rangle) n(y) dy \\
&= -\frac{1}{\nu_H^2} \langle G(f^2 - \langle f^2 \rangle) \rangle. \qquad (4.49)
\end{aligned}$$

Replacing $f(y)^2$ by $\lambda(y) f(y)$, we take the same argument as taken in the preceding. Then we get

$$\langle \gamma \psi' \rangle = \left\langle \gamma \left(\frac{1}{\nu_H^2 n} \right) \left(\int_{-\infty}^{\cdot} (\lambda f - \langle \lambda f \rangle) n \right) \right\rangle$$

$$= -\frac{1}{\nu_H^2}\langle G(\lambda f - \langle \lambda f \rangle)\rangle \tag{4.50}$$

with $G(y) = \int \gamma(y)dy,$ the primitive function of $\gamma(y)$.

Therefore, substituting (4.49) and (4.50) into (4.35) and (4.36), respectively, we obtain the following theorem:

Theorem 4.7. *Let* $0 < H < 1$. *Suppose that assumptions in Theorem* 4.6 *hold. Let* V_2 *be defined by* (4.35) *and* V_1 *by* (4.36). *Consider the approximated expansion*

$$P(t, x, y; T) \approx (1 + D(T - t))A(T - t)e^{-B(T-t)x}$$

as shown in Theorem 4.6. *Then the function* $D(\tau)$ *is given by* (4.42) *in terms of* V_2 *and* V_1. *Moreover*, V_2 *and* V_1 *have the explicit expressions as follows*:

$$V_2 = -\sqrt{\varepsilon}\,\frac{1}{2}\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\langle\gamma\phi'\rangle$$

$$= \sqrt{\varepsilon}\,\frac{1}{2\nu_H}\left(\frac{1}{\sqrt{H\Gamma(2H)}}\right)\langle G(f^2 - \langle f^2\rangle)\rangle, \tag{4.51}$$

$$V_1 = \sqrt{\varepsilon}\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\langle\gamma\psi'\rangle$$

$$= -\sqrt{\varepsilon}\,\frac{1}{\nu_H}\left(\frac{1}{\sqrt{H\Gamma(2H)}}\right)\langle G(\lambda f - \langle\lambda f\rangle)\rangle, \tag{4.52}$$

where $G(y) = \int \gamma(y)dy$ *and* $\varepsilon = 1/\alpha$.

Remark 4.8. Under standard Brownian motion environment, Cotton et al. [2] and Fouque et al. [7] obtained the corrected bond price in terms of the group market parameters (V_3, V_2, V_1) in the general case, where the parameter ρ with $\rho \in (-1, 1)$ allows a correlation between volatility shocks and interest rate shocks; see (2.32)-(2.34) of Theorem 2.1 in Section 2. If

$H = 1/2$ is formally substituted into (4.51) and (4.52), then the representations coincide with these of Theorem 2.1, where $\rho = 0$ and $V_3 = 0$.

Appendix

A. Stochastic integral

Here, we begin to introduce stochastic integration theory. For given $H \in (1/2, 1)$, define $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ by

$$\phi(s, t) := H(2H - 1)|s - t|^{2H-2}, \quad s, t \in \mathbb{R}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable such that

$$|f|_\phi^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s, t) ds dt < \infty.$$

Then the stochastic integral with respect to fBm B_H is well-defined to be a Gaussian random variable. It follows from Gripenberg and Norros [8] and Nualart [22] that for any deterministic integrand $f \in L^2(\mathbb{R}, \mathbb{R}) \cap L^1(\mathbb{R}, \mathbb{R})$,

$$E\left[\int_0^\infty f(t) dB_H(t)\right] = 0,$$

$$E\left[\left(\int_0^\infty f(t) dB_H(t)\right)^2\right] = \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s, t) ds dt.$$

In this paper, we shall follow the stochastic integration theory with respect to fBm B_H by Hu [10, Chapters 6-7]; Hu [10] extended the integral above to the general integrands under arbitrary Hurst parameter $H \in (0, 1)$.

Remark A.1 (Pathwise integral)**.** There are several definitions of stochastic integrals for general integrands with respect to fBm B_H. One of them is the fractional *pathwise integral* which is taken by the limit of the usual Riemann sum as defined using pointwise products. However, this integral does not have expectation zero. Further, Rogers [24] showed that *arbitrage* is possible when the risky asset has a log-normal price driven by an fBm if stochastic integrals are defined using pointwise product.

Remark A.2 (Wick-Ito integral)**.** In the white noise approach, the *Wick product* is used instead of the ordinary product in the Riemann sums in order to define the stochastic integrals. The Wick product for F and G is written by $F \lozenge G;$ here commutative law, associative law and distributive law hold. If at least one of F and G is deterministic, e.g., $F = a_0 \in \mathbb{R},$ then the Wick product coincides with the ordinary product in the deterministic case. Such an integral is called *Wick-Ito integral* or *fractional Ito-integral*.

In BS model, a risky asset is often formulated by a *geometric Brownian motion* (gBm) which is a solution of linear stochastic differential equation (SDE). For an application of Wick calculus to option pricing, for instance, we can refer to Necula [21] and Narita [17, 18]; here risky asset is formulated by a fractional geometric Brownian motion which is a solution of SDE driven by fBm B_H with Hurst parameter $H \in (1/2, 1).$ For the details of an application of the Wick calculus to SDEs, we can also refer to Biagini et al. [1], Holden et al. [9], Narita [19, 20] and the references therein.

Remark A.3 (Fractional calculus)**.** Another definition of stochastic integrals with respect to fBm B_H for general integrands is given by *fractional calculus* for arbitrary Hurst parameter $H \in (0, 1).$ In this case, the stochastic integration theory is based on both the left- and right-sided Riemann-Liouville *fractional integral* and the left- and right-sided Riemann-Liouville *fractional derivative*. A risky asset in BS model can be formulated by a fractional gBm which is a solution of SDE driven by fBm B_H under fractional calculus. We can refer to Mishura [13], Nualart [22] and the references therein for the existence of pathwise solutions and the uniqueness in law for SDEs driven by fBm B_H.

In general, quadratic variations of stochastic integrals with respect to fBm B_H for general integrands have abstract and complicated expression, and hence there is difficulty in application of Ito formula.

Remark A.4 (Stochastic integral in the sense of Hu [10]). In this paper, we shall take the stochastic integral in the sense of Hu [10, Chapters 6-7] (Hu integral, for short). This is the stochastic integral with respect to $B_H (H \in (0, 1))$ for *algebraically integrable* integrands; the integration theory is developed by using Wiener chaos expansion and an idea of creation operator from quantum field theory. If $1/2 < H < 1,$ then the Hu integral coincides with the Wick-Ito integral in the sense of Duncan et al. [3]. If $0 < H < 1,$ then the Hu integral for deterministic integrands coincides with the stochastic integral of variation in the sense of Hu [10, Definition 6.11] and Nualart-Pardoux [23].

The Hu integral with respect to fBm $B_H (H \in (0, 1))$ has expectation zero and can be concretely evaluated in the case of deterministic integrands. This enables us to apply Ito formula to linear SDE of the form

$$dX(t) = a(t)X(t)dt + b(t)X(t)dB_H(t)\ (H \in (0, 1))$$

with deterministic coefficients $a(t)$ and $b(t)$.

In the following, we shall introduce the Hilbert space, denoted by $\Theta_H = \Theta_H([0, T]),$ for the sake of understanding of the stochastic integrals.

Let $W(t)$ be a standard Brownian motion (sBm). Then a fractional Brownian motion (fBm) with Hurst parameter $H,$ $0 < H < 1,$ can be represented in terms of sBm $W(t)$ as follows:

$$B_H(t) = \int_0^t Z_H(t, s)dW(s), \quad 0 \le t < \infty, \tag{A.1}$$

where

$$Z_H(t, s)$$
$$= \kappa_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right]$$

and

$$\kappa_H = \sqrt{\frac{2H\Gamma\left(\frac{3}{2} - H\right)}{\Gamma\left(H + \frac{1}{2}\right)\Gamma(2 - 2H)}}$$

with the Gamma function $\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du.$ If we formally differentiate (A.1) with respect to t, then we have

$$\dot{B}_H(t) = \frac{d}{dt} B_H(t) = \frac{d}{dt}\int_0^t Z_H(t, s) dW(s) = \frac{d}{dt}\int_0^t Z_H(t, s)\dot{W}(s) ds.$$

Let $T > 0$ be arbitrary and fixed. Let us consider functions over the interval $[0, T]$. Then we can introduce the following integro-differential transformation:

$$\mathbf{\Gamma}_{H,T} f(t) = \frac{d}{dt}\int_0^t Z_H(t, s) f(s) ds, \quad 0 < t < T. \tag{A.2}$$

In the case of no ambiguity, we use $\mathbf{\Gamma}_H = \mathbf{\Gamma}_{H,T}$, and hence $\mathbf{\Gamma}_H \dot{W} = \dot{B}_H$.

The inverse operator of $\mathbf{\Gamma}_{H,T}$ can be defined by $\mathbb{B}_{H,T}$, satisfying

$$\mathbf{\Gamma}_{H,T}\mathbb{B}_{H,T} = I,$$

where I is the identity operator. The transpose of $\mathbf{\Gamma}_{H,T}$ and $\mathbb{B}_{H,T}$ is, respectively, denoted by $\mathbf{\Gamma}^*_{H,T}$ and $\mathbb{B}^*_{H,T}$.

Let $\mathbf{A} := \{\mathbb{B}^*_{H,T} f : f \in \mathbf{S}\}$, where $\mathbf{S}$ denotes the set of all functions on $[0, T]$ whose derivatives are bounded. If $\mathbb{B}^*_{H,T} f \equiv 0$, then $f = \mathbf{\Gamma}^*_{H,T}\mathbb{B}^*_{H,T} f = 0$. This means that $\mathbb{B}^*_{H,T}$ is a bijection from $\mathbf{S}$ to $\mathbf{A}$. For any two elements in $\mathbf{A}$, say $g_1 = \mathbb{B}^*_{H,T} f_1$ and $g_2 = \mathbb{B}^*_{H,T} f_2$, define

$$\langle g_1, g_2\rangle_{\Theta_H} := \int_0^T f_1(t) f_2(t) dt = \int_0^T \mathbf{\Gamma}^*_{H,T} g_1(t)\mathbf{\Gamma}^*_{H,T} g_2(t) dt. \tag{A.3}$$

Then **A** is a pre-Hilbert space with respect to the above scalar product. The completion of **A** with respect to the scalar product $\langle \cdot, \cdot \rangle_{\Theta_H}$ is denoted by $\Theta_H([0, T])$, which is a Hilbert space. Since $T > 0$ is fixed, we use Θ_H to denote $\Theta_H([0, T])$. $\mathbb{B}^*_{H,T} : \mathbf{S} \to \mathbf{A}$ can be extended to an isometry from the Hilbert space $L^2([0, T])$ to the Hilbert space Θ_H. Its inverse is the extension of the $\mathbf{\Gamma}^*_{H,T}$ to Θ_H. We continue to use $\mathbb{B}^*_{H,T}$ and $\mathbf{\Gamma}^*_{H,T}$ to denote their extensions.

Remark A.5 (Induced transformations)**.** The following is due to Hu [10]: $\mathbb{B}^*_{H,T} : L^2([0, T]) \to \Theta_H$ is an isometry from the Hilbert space $L^2([0, T])$ to the Hilbert space Θ_H. The value of $\mathbb{B}^*_{H,T}$ on smooth function space **S** is given by explicit formulas. The inverse of $\mathbb{B}^*_{H,T}$ is an isometry, denoted by $\mathbf{\Gamma}^*_{H,T} : \Theta_H \to L^2([0, T])$. The value of $\mathbf{\Gamma}^*_{H,T}$ on smooth function space **S** is also given by explicit formulas.

By (A.2), we observe

$$\dot{B}_H(t) = \mathbf{\Gamma}_{H,T}(\dot{W})(t), \quad 0 \le t \le T,$$

and hence

$$\int_0^T f(t)dB_H(t) = \int_0^T f(t)\dot{B}_H(t)dt = \int_0^T f(t)\mathbf{\Gamma}_{H,T}(\dot{W})(t)dt$$

$$= \int_0^T (\mathbf{\Gamma}^*_{H,T} f)(t)\dot{W}(t)dt = \int_0^T (\mathbf{\Gamma}^*_{H,T} f)(t)dW(t).$$

Therefore, we can use the identity above to define our stochastic integral for deterministic integrands.

Let $f \in \Theta_H$. Then it is known from (A.3) and Remark A.5 that $g = \mathbf{\Gamma}^*_{H,T} f \in L^2([0, T])$. Thus $\int_0^T g(t)dW(t)$ is well-defined. This implies that stochastic integral can be defined as follows: for $f \in \Theta_H$,

$$\int_0^T f(t) dB_H(t) := \int_0^T (\mathbf{\Gamma}^*_{H,T} f)(t) dW(t). \tag{A.4}$$

Hu [10] explained as follows: It seems that (A.4) suggests stochastic integral for general integrands. If we use formula (A.4), then the integrand on the right-hand side of (A.4) should be a functional of sBm. However, the probability laws of sBm and fBm are mutually singular and a functional of sBm may not be well-defined as a functional of fBm and vice versa. This means that a functional of fBm (a random variable on the probability space of fBm) may not be well-defined as a functional of sBm. Moreover, even if the right-hand side of (A.4), i.e., $\int_0^T (\mathbf{\Gamma}^*_{H,T} f)(t) dW(t)$ is well-defined, then it is not straightforward to consider it as a functional of fBm. Hence the definition of stochastic integral needs to be improved.

Hu [10] mainly used the integral kernels $Z_H(t, s)$ and $\eta_H(t, s)$ which are, respectively, related to expressions for operators $\mathbf{\Gamma}_{H,T}$ and $\mathbb{B}_{HT}$, i.e., the fact that the fBm can be represented by sBm and the sBm can also be represented by fBm by explicit formulas. Hu [10] extended this correspondence to between nonlinear functionals of fBm and nonlinear functionals of sBm, and hence obtained the stochastic integral for algebraically integrable integrands.

B. Ito formula

Let $\Theta_H = \Theta_H([0, T])$ be the Hilbert space as defined in Hu [10, Chapter 5]; Θ_H is the space of integrands associated with the induced transformation of representation for fBm $B_H(t)$. Let $f(s)$ be given over $[0, T]$. Let $0 \le s \le t \le T$. Then, considering the functions $f_t(s)$ restricted to $[0, t]$, that is, $f_t(s) = f(s)\chi_{[0,t]}(s)$, we shall use $\Theta_{H,t}$ to denote $\Theta_H([0, t])$, where the norm $\| f_t \|_{\Theta_{H,t}}$ is well-defined. According to Hu [10, pp. 102-103], we summarize expression for $\| f_t \|_{\Theta_{H,t}}$ in the following remarks:

Remark B.1. Let $H > 1/2$. Then

$$\| f_t \|^2_{\Theta_{H,t}} = H(2H-1)\int_0^t\int_0^t |v-u|^{2H-2} f(u) f(v) du dv.$$

If f is continuous in $[0, T]$, then $\| f_t \|_{\Theta_{H,t}}$ is differentiable and

$$\frac{d}{dt}\| f_t \|^2_{\Theta_{H,t}} = 2H(2H-1) f(t)\int_0^t |t-u|^{2H-2} f(u) du, \quad 0 \le t \le T. \tag{B.1}$$

Remark B.2. Let $0 < H < 1/2$ and let f be continuously differentiable. Then

$$\| f_t \|^2_{\Theta_{H,t}} = Hf(0)\int_0^t v^{2H-1} f(v) dv + Hf(t)\int_0^t |t-v|^{2H-1} f(v) dv$$

$$+ H\int_0^t\int_0^t |v-u|^{2H-1}\mathrm{sign}(v-u) f'(u) f(v) du dv.$$

Making substitution $u = t\xi$, we have

$$\frac{d}{dt}\left(\int_0^t (t-u)^{2H-1} f(u) du\right) = \frac{d}{dt}\left(t^{2H}\int_0^1 (1-\xi)^{2H-1} f(t\xi) d\xi\right)$$

$$= 2Ht^{2H-1}\int_0^1 (1-\xi)^{2H-1} f(t\xi) d\xi$$

$$+ t^{2H}\int_0^1 (1-\xi)^{2H-1}\xi f'(t\xi) d\xi.$$

Hence

$$\frac{d}{dt}\| f_t \|^2_{\Theta_{H,t}} = Ht^{2H-1} f(0) f(t) + 2H^2 t^{2H-1} f(t)\int_0^1 (1-\xi)^{2H-1} f(t\xi) d\xi$$

$$+ Ht^{2H} f(t)\int_0^1 (1-\xi)^{2H-1}\xi f'(t\xi) d\xi$$

$$+ Hf(t)\int_0^t (t-u)^{2H-1} f'(u) du. \tag{B.2}$$

Hu [10, p. 103] showed Ito formula for general deterministic *f* and Hurst parameter $H \in (0, 1)$ as follows:

Theorem B.3 (Ito formula)**.** *Let* $0 < H < 1$ *and let* $f \in \Theta_{H,T} \cap L^2([0, T])$ *be a deterministic function. Denote* $f_t(s) = f(s)\chi_{[0,t]}(s)$, $0 \le s \le t \le T$. *Suppose that* $f_t \in \Theta_{H,t}$ *and* $\| f_t \|_{\Theta_{H,t}}$ *is continuously differentiable as a function of* $t \in [0, T]$. *Denote*

$$X(t) = X(0) + \int_0^t g(s)ds + \int_0^t f(s)dB_H(s), \quad 0 \le t \le T, \tag{B.3}$$

where $X(0)$ *is a constant, g is deterministic with* $\int_0^T |g(s)|ds < \infty$. *Let F be an entire function of order less than* 2. *Namely*,

$$M_f(r) := \sup_{|z|=r} |f(z)| < Ce^{A_r K} \text{ for all } r,$$

where K is a positive number less than 2 *and C is a constant. Then*

$$F(t, X(t)) = F(0, X(0)) + \int_0^t \frac{\partial F}{\partial s}(s, X(s))ds + \int_0^t \frac{\partial F}{\partial x}(s, X(s))dX(s)$$

$$+ \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X(s))\left[\frac{d}{ds}\| f_s \|^2_{\Theta_{H,s}}\right]ds, \quad 0 \le t \le T. \tag{B.4}$$

Here, *the stochastic integral in* (B.4) *is in the sense of the Hu integral*.

Equation (B.4) is rewritten by the stochastic differentials as follows:

$$dF(t, X(t)) = \frac{\partial F}{\partial t}(t, X(t))dt + \frac{\partial F}{\partial x}(t, X(t))dX(t)$$

$$+ \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t, X(t))\left[\frac{d}{dt}\| f_t \|^2_{\Theta_{H,t}}\right]dt.$$

Let $f \equiv 1$. Then $\| f_t \|_{\Theta_{H,t}} = t^H$, hence $\frac{d}{dt}\| f_t \|_{\Theta_{H,t}} = Ht^{H-1}$. Therefore, we have the following:

Corollary B.4. *Let* $0 < H < 1$ *and let* F *satisfy the conditions of Theorem* B.3. *Then*

$$F(t, B_H(t)) = F(0, 0) + \int_0^t \frac{\partial F}{\partial s}(s, B_H(s))ds + \int_0^t \frac{\partial F}{\partial x}(s, B_H(s))dB(s)$$

$$+ H\int_0^t \frac{\partial^2 F}{\partial x^2}(s, B_H(s))s^{2H-1}ds, \quad 0 \le t \le T. \tag{B.5}$$

C. Fractional Ornstein-Uhlenbeck process

By Narita [16, Lemmas 8.1 and 8.8], we note the following:

Lemma C.1 (Property of fractional OU process)**.** *Let* $0 < H < 1$. *Let* $Y(t)$ *be the fractional OU process given by*

$$dY(t) = \alpha(m - Y(t))dt + \beta dB_H(t), \quad Y(0) = y_0 \in \mathbb{R} \tag{C.1}$$

with constants $\alpha > 0$, $\beta > 0$ *and* $m > 0$. *Then* $Y(t)$ *is the pathwise unique solution of* (C.1) *with the following form*:

$$Y(t) = m + e^{-\alpha t}(y_0 - m) + \beta e^{-\alpha t}\int_0^t e^{\alpha s} dB_H(s). \tag{C.2}$$

Further, $Y(t)$ *is a Gaussian stochastic process and has the long-run distribution which is the normal distribution* $N(m, \nu_H^2)$ *with mean m and variance* ν_H^2 *such that the density is given by*

$$n(y) = \frac{1}{\sqrt{2\pi\nu_H^2}} \exp\left(-\frac{(y-m)^2}{2\nu_H^2}\right), \tag{C.3}$$

where

$$\nu_H^2 = \beta^2 H\left(\frac{1}{\alpha}\right)^{2H} \Gamma(2H), \tag{C.4}$$

and $\Gamma(\cdot)$ *is the Gamma function*, *i.e.*, $\Gamma(x) = \int_0^\infty e^{-\xi}\xi^{x-1}d\xi.$

Let $Y(t)$ be the solution of (C.1). Let $F(t, y)$ be a function in $C^{1,2}([0, T] \times \mathbb{R})$. Define $X(t)$ by

$$X(t) = \int_0^t e^{\alpha s}(\alpha m)ds + \int_0^t e^{\alpha s}\beta dB_H(s) + y_0.$$

Then we notice that $Y(t) = e^{-\alpha t}X(t)$. For the function $F(t, y)$, define the function $G(t, x)$ by $G(t, x) := F(t, e^{-\alpha t}x)$. Then we notice that

$$F(t, Y(t)) = F(t, e^{-\alpha t}X(t)) = G(t, X(t)).$$

Applying Theorem B.3 to $G(t, x)$ and $X(t)$ and using the change of the variable such that $y = e^{-\alpha t}x$, Narita [16, Lemmas 8.3 and 8.7] obtained the following Ito formula for $Y(t)$:

Lemma C.2. *Let* $0 < H < 1$. *Let* $Y(t)$ *be the solution of* (C.1). *Let* $F(t, y)$ *be a function in* $C^{1,2}([0, T] \times \mathbb{R})$. *Then*

$$dF(t, Y(t)) = \frac{\partial F}{\partial t}(t, Y(t))dt + \frac{\partial F}{\partial y}(t, Y(t))dY(t)$$

$$+ \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t, Y(t))e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right]dt, \qquad \text{(C.5)}$$

where

$$g_t(u) = g(u)\chi_{[0,t]}(u), \quad g(u) = e^{\alpha u}\beta, \quad 0 \le u \le t \le T.$$

Further, *Remarks* B.1 *and* B.2 *yield the explicit form of*

$$e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right]$$

as follows:

(i) *If* $H > 1/2$, *then*

$$e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right] = 2\beta^2 H\left\{e^{-\alpha t}t^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha t)\right\}. \quad \text{(C.6)}$$

(ii) *If* $0 < H < 1/2$, *then*

$$\begin{aligned} e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right] &= H\beta^2 t^{2H-1}e^{-\alpha t} \\ &+ 2H^2\beta^2 t^{-1}\left(\frac{1}{\alpha}\right)^{2H} B(\alpha t) \\ &+ H\alpha\beta^2 t^{-1}\left(\frac{1}{\alpha}\right)^{2H}\left\{tB(\alpha t) - \left(\frac{1}{\alpha}\right)C(\alpha t)\right\} \\ &+ H\alpha\beta^2\left(\frac{1}{\alpha}\right)^{2H} B(\alpha t). \end{aligned} \quad \text{(C.7)}$$

Here,

$$B(x) = \int_0^x z^{2H-1}e^{-z}dz, \quad C(x) = \int_0^x z^{2H}e^{-z}dz.$$

D. Fast-scale

Let us consider the model described by (C.1) and (C.2).

Assumption D.1. Let $0 < H < 1$. Then we assume the following:

(i) The rate of mean-reversion α or its inverse, the typical correlation time of $(Y(t))$, is characterized by a small parameter ε such that

$$\varepsilon = \frac{1}{\alpha}.$$

(ii) Let ν_H^2 be given by (C.4), which controls the long-run size of the volatility fluctuations. Then we assume this quantity remains fixed as we consider smaller and smaller values of ε such that

$$\beta = \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\left(\frac{1}{\alpha}\right)^{-H} = \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\frac{1}{\varepsilon^H}.$$

The following result is given in Narita [16, Lemmas 10.2 and 10.3].

Lemma D.2. *Let* $0 < H < 1$. *Let* $(Y(t))$ *be given by* (C.2). *Define the function* g_t *by*

$$g_t(u) = \chi_{[0,t]}(u)g(u), \quad g(u) = e^{\alpha u}\beta, \quad 0 \le u \le t \le T.$$

Under Assumption D.1, *consider the multiplier of the second derivative* $\partial^2 F/\partial y^2$ *in* (C.5) *of Lemma* C.2. *Then, for* $t > 0$,

$$e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right] = 2\left\{\left(\nu_H^2 \frac{1}{\varepsilon}\right) + o(1)\right\} \tag{D.1}$$

as $\varepsilon \to 0$, *where* $\varepsilon = 1/\alpha$. *Hence, for* $t > 0$,

$$\frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t, Y(t))e^{-2\alpha t}\left[\frac{d}{dt}\| g_t \|^2_{\Theta_{H,t}}\right]$$

$$= \frac{\partial^2 F}{\partial y^2}(t, Y(t))\left[\left(\nu_H^2 \frac{1}{\varepsilon}\right) + o(1)\right] \tag{D.2}$$

as $\varepsilon \to 0$.

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