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A GENERALIZED BARTHOLDI ZETA FUNCTION FOR A HYPERGRAPH

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Abstract

We introduce a generalized Bartholdi zeta function of a bipartite graph, and define a generalized Bartholdi zeta function of a hypergraph H with three variables. Furthermore, we present three types of determinant expressions for the generalized Bartholdi zeta function of a hypergraph H.

1. Introduction

1.1. Zeta functions of graphs

Graphs and digraphs treated here are finite. Let G be a connected graph and D_G be the symmetric digraph corresponding to G. Set $D(G) = \{(u, v), (v, u) | uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$.

A *path P of length n* in G is a sequence $P = (e_1, ..., e_n)$ of n arcs such

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that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})(1 \le i \le n-1)$. If $e_i = (v_{i-1}, v_i)$ for $i = 1$, ..., n, then we write $P = (v_0, v_1, ..., v_{n-1}, v_n)$. Set $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an $(o(P), t(P))$*-path*. We say that a path $P = (e_1, ..., e_n)$ has a *backtracking* or a *bump* at $t(e_i)$ if $e_{i+1}^{-1} = e_i$ for some i $(1 \le i \le n-1)$. A (v, w)-path is called a v*-cycle* (or v*-closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (e_1, ..., e_n)$ is the cycle $C^{-1} = (e_n^{-1}, ..., e_1^{-1})$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, ..., e_m)$ and $C_2 = (f_1, ..., f_m)$ are called *equivalent* if $f_j = e_{j+k}$ for all j. The inverse cycle of C is in general not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G.

The *Ihara*(*-Selberg*) *zeta function* of G is defined by

$$\mathbf{Z}(G, t) = \prod_{[C]} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G. Ihara [6] defined Ihara zeta functions of graphs, and showed that the reciprocals of Ihara zeta functions of regular graphs are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [12, 13]. Hashimoto [5] generalized Ihara’s result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial by a determinant containing the edge matrix. Bass [3] presented another determinant expression for the Ihara zeta function of an irregular graph by using its adjacency matrix.

Let G be a connected graph with n vertices and m edges. Then two $2m \times 2m$ matrices

$$\mathbf{B} = \mathbf{B}(G) = (\mathbf{B}_{e,f})_{e,f \in D(G)} \text{ and } \mathbf{J}_0 = \mathbf{J}_0(G) = (\mathbf{J}_{e,f})_{e,f \in D(G)}$$

are defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} 1 & \text{if } t(e) = o(f), \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{J}_{e,f} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1 (Hashimoto; Bass)**.** *Let G be a connected graph with n vertices and m edges. Then the reciprocal of the Ihara zeta function of G is given by*

$$\mathbf{Z}(G, t)^{-1} = \det(\mathbf{I}_{2m} - t(\mathbf{B} - \mathbf{J}_0))$$
$$= (1 - t^2)^{m-n} \det(\mathbf{I}_n - t\mathbf{A}(G) + t^2(\mathbf{D}_G - \mathbf{I}_n)),$$

where $\mathbf{D}_G = (d_{ij})$ is the diagonal matrix with

$$d_{ii} = \deg_G v_i (V(G) = \{v_1, \ldots, v_n\}).$$

The first identity in Theorem 1 was also obtained by Hashimoto [5]. Bass [3] proved the second identity by using a linear algebraic method.

Stark and Terras [10] gave an elementary proof of this formula, and discussed three different zeta functions of any graph. Various proofs of Bass' Theorem were given by Kotani and Sunada [7], and Foata and Zeilberger [4].

Let G be a connected graph. Then the *cyclic bump count* $cbc(\pi)$ of a cycle $\pi = (\pi_1, \ldots, \pi_n)$ is

$$cbc(\pi) = |\{i = 1, \ldots, n \mid \pi_i = \pi_{i+1}^{-1}\}|,$$

where $\pi_{n+1} = \pi_1$.

Bartholdi [2] introduced the Bartholdi zeta function of a graph. The *Bartholdi zeta function* of G is defined by

$$\zeta(G, u, t) = \prod_{[C]} (1 - u^{cbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G, and u, t are complex variables with $|u|$, $|t|$ sufficiently small.

Bartholdi [2] gave a determinant expression of the Bartholdi zeta function of a graph.

Theorem 2 (Bartholdi)**.** *Let G be a connected graph with n vertices and m unoriented edges. Then the reciprocal of the Bartholdi zeta function of G is given by*

$$\begin{aligned}\zeta(G, u, t)^{-1} &= \det(\mathbf{I}_{2m} - t(\mathbf{B} - (1-u)\mathbf{J}_0)) \\ &= (1 - (1-u)^2 t^2)^{m-n} \det(\mathbf{I}_n - t\mathbf{A}(G) \\ &\quad + (1-u)(\mathbf{D}_G - (1-u)\mathbf{I}_n)t^2).\end{aligned}$$

We state Amitsur Theorem which is used in the proof of Theorem 6. Foata and Zeilberger [4] gave a new proof of Bass' Theorem by using the algebra of Lyndon words. Let X be a finite nonempty set, $<$ be a total order in X, and X^* be the free monoid generated by X. Then the total order $<$ on X derives the lexicographic order $<^*$ on X^*. A *Lyndon word* in X is defined to a nonempty word in X^* which is prime, i.e., not the power l^r of any other word l for any $r \geq 2$, and which is also minimal in the class of its cyclic rearrangements under $<^*$ (see [8]). Let L denote the set of all Lyndon words in X.

Foata and Zeilberger [4] gave a short proof of Amitsur's identity [1].

Theorem 3 (Amitsur)**.** *For square matrices* $\mathbf{A}_1, \ldots, \mathbf{A}_k$,

$$\det(\mathbf{I} - (\mathbf{A}_1 + \cdots + \mathbf{A}_k)) = \prod_{l \in L} \det(\mathbf{I} - \mathbf{A}_l),$$

where the product runs over all Lyndon words in $\{1, \ldots, k\}$, *and* $\mathbf{A}_l = \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_p}$ *for* $l = i_1 \cdots i_p$.

1.2. Zeta functions of hypergraphs

Storm [11] defined the Ihara-Selberg zeta function of a hypergraph. A hypergraph $H = (V(H), E(H))$ is a pair of a set $V(H)$ of *hypervertices* and a set $E(H)$ of *hyperedges*, where the union of all hyperedges is $V(H)$. A hypervertex v is *incident* to a hyperedge e if $v \in e$. For a hypergraph H, its dual H^* is the hypergraph obtained by letting its hypervertex set be indexed by $E(H)$ and its hyperedge set by $V(H)$.

A bipartite graph B_H associated with a hypergraph H is defined as follows: $V(B_H) = V(H) \cup E(H)$ and $v \in V(H)$ and $e \in E(H)$ are *adjacent* in B_H if v is incident to e. Let $V(H) = \{v_1, \ldots, v_n\}$. Then an *adjacency matrix* $\mathbf{A}(H)$ of H is defined as a matrix whose rows and columns are parameterized by $V(H)$, and (i, j)-entry is the number of paths in B_H from v_i to v_j of length 2 with no backtracking.

For the bipartite graph B_H associated with a hypergraph H, let $V_1 = V(H)$ and $V_2 = E(H)$. Then, the *halved graph* $B_H^{[i]}$ of B_H is defined to be the graph with vertex set V_i and arc set $\{P : \text{reduced path} \mid |P| = 2;\ o(P), t(P) \in V_i\}$ for $i = 1, 2$.

Let H be a hypergraph. A *path* P *of length* n in H is a sequence $P = (v_1, e_1, v_2, e_2, \ldots, e_n, v_{n+1})$ of $n + 1$ hypervertices and n hyperedges such that $v_i \in V(H)$, $e_j \in E(H)$, $v_1 \in e_1$, $v_{n+1} \in e_n$ and $v_i \in e_i, e_{i-1}$ for $i = 2, \ldots, n-1$. Set $|P| = n$, $o(P) = v_1$ and $t(P) = v_{n+1}$. Also, P is called an $(o(P), t(P))$-*path*. We say that a path P has a *hyperedge backtracking* if there is a subsequence of P of the form (e, v, e), where $e \in E(H)$, $v \in V(H)$. A (v, w)-path is called a *v-cycle* (or *v-closed path*) if $v = w$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (v_1, e_1, v_2, ..., e_m, v_1)$ and $C_2 = (w_1, f_1, w_2, ..., f_m, w_1)$ are called *equivalent* if $w_j = v_{j+k}$ and $f_j = e_{j+k}$ for all j. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no hyperedge backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle.

The *Ihara-Selberg zeta function* of H is defined by

$$\zeta_H(t) = \prod_{[C]} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of H, and t is a complex variable with $|t|$ sufficiently small (see [11]).

Let H be a hypergraph with $E(H) = \{e_1, ..., e_m\}$, and let $\{c_1, ..., c_m\}$ be a set of m *colors*, where $c(e_i) = c_i$. Then an *edge-colored graph* GH_c is defined as a graph with vertex set $V(H)$ and edge set $\{vw | v, w \in V(H); v, w \in e \in E(H)\}$, where an edge vw is colored c_i if $v, w \in e_i$. Note that GH_c is the halved graph $B_H^{[1]}$ of B_H.

Let GH_c^o be the symmetric digraph corresponding to the edge-colored graph GH_c. Then the *oriented line graph* $H_L^o = (V_L, E_L^o)$ associated with GH_c^o by

$$V_L = D(GH_c^o)$$

and

$$E_L^o = \{(e_i, e_j) \in D(GH_c^o) \times D(GH_c^o) | c(e_i) \neq c(e_j), t(e_i) = o(e_j)\},$$

where $c(e_i)$ is the color assigned to the oriented edge $e_i \in D(GH_c^o)$ such that

$c(u, v) = c(uv)$, $(u, v) \in D(GH_c^o)$. Also, H_L^o is called the *oriented line graph* of GH_c. The *Perron-Frobenius operator* $T : C(V_L) \to C(V_L)$ is given by

$$(Tf)(x) = \sum_{e \in E_o(x)} f(t(e)),$$

where $E_o(x) = \{e \in E_L^o \mid o(e) = x\}$ is the set of all oriented edges with x as their origin vertex, and $C(V_L)$ is the set of functions from V_L to the complex number field $\mathbf{C}$.

Storm [11] gave two nice determinant expressions of the Ihara-Selberg zeta function of a hypergraph by using the results of Kotani and Sunada [7], and Bass [3].

Theorem 4 (Storm)**.** *Let H be a finite, connected hypergraph such that every hypervertex is in at least two hyperedges. Then*

$$\zeta_H(t)^{-1} = \det(\mathbf{I} - tT) = \mathbf{Z}(B_H, \sqrt{t})^{-1}$$

$$= (1-t)^{m-n} \det(\mathbf{I} - \sqrt{t}\mathbf{A}(B_H) + t\mathbf{Q}_{B_H}),$$

where $n = |V(B_H)|$, $m = |E(B_H)|$ *and* $\mathbf{Q}_{B_H} = \mathbf{D}_{B_H} - \mathbf{I}$.

Let H be a hypergraph. Then a path $P = (v_1, e_1, v_2, e_2, \dots, e_n, v_{n+1})$ has a (*broad*) *backtracking* or (*broad*) *bump* at e or v if there is a subsequence of P of the form (e, v, e) or (v, e, v), where $e \in H(H)$, $v \in V(H)$. Furthermore, the *cyclic bump count* $cbc(C)$ of a cycle $C = (v_1, e_1, v_2, e_2, \dots, e_n, v_1)$ is

$$cbc(C) = |\{i = 1, \dots, n \mid v_i = v_{i+1}\}| + |\{i = 1, \dots, n \mid e_i = e_{i+1}\}|,$$

where $v_{n+1} = v_1$ and $e_{n+1} = e_1$.

The *Bartholdi zeta function* of H is defined by

$$\zeta(H, u, t) = \prod_{[C]} (1 - u^{cbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of H, and u, t are complex variables with $|u|$, $|t|$ sufficiently small.

If $u = 0,$ then the Bartholdi zeta function of H is the Ihara-Selberg zeta function of H.

Sato [9] presented a determinant expression of the Bartholdi zeta function of a hypergraph.

Theorem 5 (Sato)**.** *Let H be a finite, connected hypergraph. Then*

$$\zeta(H, u, t)^{-1} = \zeta(B_H, u, \sqrt{t})^{-1}$$
$$= (1-(1-u)^2 t)^{m-n} \det(\mathbf{I} - \sqrt{t}\mathbf{A}(B_H) + (1-u)t(\mathbf{D}_{B_H} - (1-u)\mathbf{I})),$$

where $n = |V(B_H)|$ *and* $m = |E(B_H)|$.

Unfortunately, in the case that H is a graph G, the Bartholdi zeta function $\zeta(H, u, t) = \zeta(B_G, u, \sqrt{t})$ of G is not equal to the original Bartholdi zeta function $\zeta(G, u, t)$ of G.

In this paper, we present a three variable Bartholdi zeta function of a hypergraph H which is the original Bartholdi zeta function of a graph G in the case of $H = G$ and $s = 0$.

In Section 2, we introduce a generalized Bartholdi zeta function of a bipartite graph with three variables, and present a determinant expression of it. In Section 3, we introduce a generalized Bartholdi zeta function of a hypergraph with three variables, and present a determinant expression of it. In Section 4, we give a determinant expression for the generalized Bartholdi zeta function of a hypergraph by using a modified Perron-Frobenius operator. In Section 5, we present a decomposition formula for the generalized Bartholdi zeta function of a semiregular bipartite graph. As a corollary, we obtain a decomposition formula for the generalized Bartholdi zeta function of a (d, r)-regular hypergraph.

2. A Generalized Bartholdi Zeta Function of a Bipartite Graph

Let $G = (V_1, V_2)$ be a connected bipartite graph. For $j = 1, 2,$ the *cyclic* *bump count* $cbc_j(\pi)$ of a cycle $\pi = (\pi_1, \dots, \pi_n)$ in G is

$$cbc_j(\pi) = |\{i = 1, \dots, n \mid \pi_i = \pi_{i+1}^{-1},\ t(\pi_i) \in V_j\}|,$$

where $\pi_{n+1} = \pi_1$. Then the *generalized Bartholdi zeta function* of a bipartite graph G is defined by

$$\zeta(G, u, s, t) = \prod_{[C]} (1 - u^{cbc_1(C)} s^{cbc_2(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G, and u, s, t are complex variables with $|u|$, $|s|$, $|t|$ sufficiently small.

Let $G = (V_1, V_2)$ be a connected bipartite graph with ν vertices and ε edges. Then three $2\varepsilon \times 2\varepsilon$ matrices $\mathbf{B} = \mathbf{B}(G) = ((\mathbf{B})_{e,f})_{e,f \in D(G)}$ and $\mathbf{J}_i = \mathbf{J}_i(G) = ((\mathbf{J}_i)_{e,f})_{e,f \in D(G)}$ $(i = 1, 2)$ are defined as follows:

$$(\mathbf{B})_{e,f} = \begin{cases} 1 & \text{if } t(e) = o(f), \\ 0 & \text{otherwise,} \end{cases} \qquad (\mathbf{J}_i)_{e,f} = \begin{cases} 1 & \text{if } f = e^{-1} \text{ and } t(e) \in V_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

where i is treated modulo 2.

A determinant expression for the generalized Bartholdi function of G is given as follows.

Theorem 6. *Let $G = (V_1, V_2)$ be a connected bipartite graph with ν vertices and ε unoriented edges, $|V_1| = n$ and $|V_2| = m$. Then the reciprocal of the generalized Bartholdi zeta function of G is*

$$\begin{aligned} \zeta(G, u, s, t)^{-1} &= \det(\mathbf{I}_{2\varepsilon} - t(\mathbf{B} - (1-s)\mathbf{J}_1 - (1-u)\mathbf{J}_2)) \\ &= (1 - (1-s)(1-u)t^2)^{\varepsilon - \nu} \det(\mathbf{I}_\nu - t\mathbf{A}(G) \\ &\quad + t^2((1-s)(\mathbf{D}_{V_1} - (1-u)\mathbf{I}_n) \oplus (1-u)(\mathbf{D}_{V_2} - (1-s)\mathbf{I}_m))), \end{aligned}$$

where $\mathbf{D}_W = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg_G v_i$ $(W = \{v_1, \dots, v_p\})$ for any subset W of $V(G)$.

Proof. The argument is an analogue of Bass' method [3].

Let $V_1 = \{v_1, ..., v_n\}$ and $V_2 = \{w_1, ..., w_m\}$. Furthermore, let $D(G) = \{f_1, ..., f_\varepsilon, f_1^{-1}, ..., f_\varepsilon^{-1}\}$ such that $o(f_i) \in V_1$ $(1 \le i \le n)$.

Arrange arcs of G as follows: $f_1, ..., f_\varepsilon, f_1^{-1}, ..., f_\varepsilon^{-1}$. Furthermore, arrange vertices of G as follows: $v_1, ..., v_n, w_1, ..., w_m$.

Now, we define two $2\varepsilon \times \nu$ matrices $\mathbf{K} = (\mathbf{K}_{fv})_{f \in D(G); v \in V(G)}$ and $\mathbf{L} = (\mathbf{L}_{fv})_{f \in D(G); v \in V(G)}$ as follows:

$$\mathbf{K}_{fv} := \begin{cases} 1 & \text{if } t(f) = v, \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{L}_{fv} := \begin{cases} 1 & \text{if } o(f) = v, \\ 0 & \text{otherwise.} \end{cases}$$

Here we consider two matrices **K** and **L** under the above order.

Now, let

$$\mathbf{K} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_1 \\ \mathbf{K}_2 & \mathbf{0} \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2 \end{bmatrix},$$

where $\mathbf{K}_1$, $\mathbf{L}_2$ are $\varepsilon \times m$ matrices, and $\mathbf{K}_2$, $\mathbf{L}_1$ are $\varepsilon \times n$ matrices. By the definitions of **K** and **L**,

$$\mathbf{K}_1 = \mathbf{L}_2 \text{ and } \mathbf{K}_2 = \mathbf{L}_1.$$

Thus,

$$\mathbf{L} = \begin{bmatrix} \mathbf{K}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_1 \end{bmatrix}.$$

But, we have

$$\mathbf{K}^t\mathbf{L} = \mathbf{B}(G) = \begin{bmatrix} \mathbf{0} & \mathbf{K}_1 {}^t\mathbf{K}_1 \\ \mathbf{K}_2 {}^t\mathbf{K}_2 & \mathbf{0} \end{bmatrix} \tag{1}$$

and

$${}^t\mathbf{L}\mathbf{K} = \mathbf{A}(G) = \begin{bmatrix} \mathbf{0} & {}^t\mathbf{K}_2\mathbf{K}_1 \\ {}^t\mathbf{K}_1\mathbf{K}_2 & \mathbf{0} \end{bmatrix}. \tag{2}$$

Furthermore,

$$ {}^t\mathbf{K}\mathbf{K} = \mathbf{D}_G = \begin{bmatrix} {}^t\mathbf{K}_2\mathbf{K}_2 & \mathbf{0} \\ \mathbf{0} & {}^t\mathbf{K}_1\mathbf{K}_1 \end{bmatrix}. \tag{3} $$

Note that

$$ {}^t\mathbf{K}_2\mathbf{K}_2 = \begin{bmatrix} \deg_G v_1 & & 0 \\ & \ddots & \\ 0 & & \deg_G v_n \end{bmatrix} = \mathbf{D}_{V_1} \tag{4} $$

and

$$ {}^t\mathbf{K}_1\mathbf{K}_1 = \begin{bmatrix} \deg_G w_1 & & 0 \\ & \ddots & \\ 0 & & \deg_G w_m \end{bmatrix} = \mathbf{D}_{V_2}. \tag{5} $$

We introduce two $(\nu + 2\varepsilon) \times (\nu + 2\varepsilon)$ matrices as follows:

$$ \mathbf{P} = \begin{bmatrix} (1-(1-u)(1-s)t^2)\mathbf{I}_n & \mathbf{0} & -{}^t\mathbf{K}_2 & (1-s)t\,{}^t\mathbf{K}_2 \\ \mathbf{0} & (1-(1-u)(1-s)t^2)\mathbf{I}_m & (1-u)t\,{}^t\mathbf{K}_1 & -{}^t\mathbf{K}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_\varepsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_\varepsilon \end{bmatrix} $$

and

$$ \mathbf{Q} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} & {}^t\mathbf{K}_2 & -(1-s)t\,{}^t\mathbf{K}_2 \\ \mathbf{0} & \mathbf{I}_m & -(1-u)t\,{}^t\mathbf{K}_1 & {}^t\mathbf{K}_1 \\ \mathbf{0} & t\mathbf{K}_1 & (1-(1-u)(1-s)t^2)\mathbf{I}_\varepsilon & \mathbf{0} \\ t\mathbf{K}_2 & \mathbf{0} & \mathbf{0} & (1-(1-u)(1-s)t^2)\mathbf{I}_\varepsilon \end{bmatrix}. $$

By (3), (4) and (5), we have

$$ \mathbf{PQ} = \begin{bmatrix} a\mathbf{I}_n + (1-s)t^2\,{}^t\mathbf{K}_2\mathbf{K}_2 & -t\,{}^t\mathbf{K}_2\mathbf{K}_1 & \mathbf{0} & \mathbf{0} \\ -t\,{}^t\mathbf{K}_1\mathbf{K}_2 & a\mathbf{I}_m + (1-u)t^2\,{}^t\mathbf{K}_1\mathbf{K}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & t\mathbf{K}_1 & a\mathbf{I}_\varepsilon & \mathbf{0} \\ t\mathbf{K}_2 & \mathbf{0} & \mathbf{0} & a\mathbf{I}_\varepsilon \end{bmatrix} $$

$$= \begin{bmatrix} a\mathbf{I}_n + (1-s)t^2\mathbf{D}_{V_1} & -t\,{}^t\mathbf{K}_2\mathbf{K}_1 & \mathbf{0} & \mathbf{0} \\ -t\,{}^t\mathbf{K}_1\mathbf{K}_2 & a\mathbf{I}_m + (1-u)t^2\mathbf{D}_{V_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & t\mathbf{K}_1 & a\mathbf{I}_\varepsilon & \mathbf{0} \\ t\mathbf{K}_2 & \mathbf{0} & \mathbf{0} & a\mathbf{I}_\varepsilon \end{bmatrix},$$

where $a = 1-(1-u)(1-s)t^2$. By (2), we have

$$\det(\mathbf{PQ}) = (1-(1-u)(1-s)t^2)^{2\varepsilon}\det(\mathbf{I}_\nu - t\mathbf{A}(G)$$
$$+ t^2((1-s)(\mathbf{D}_{V_1} - (1-u)\mathbf{I}_n) \oplus (1-u)(\mathbf{D}_{V_2} - (1-s)\mathbf{I}_m))).$$

Furthermore, we have

$$\mathbf{QP} = \begin{bmatrix} a\mathbf{I}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & a\mathbf{I}_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & at\mathbf{K}_1 & a\mathbf{I}_\varepsilon + (1-u)t^2\mathbf{K}_1{}^t\mathbf{K}_1 & -t\mathbf{K}_1{}^t\mathbf{K}_1 \\ at\mathbf{K}_2 & \mathbf{0} & -t\mathbf{K}_2{}^t\mathbf{K}_2 & a\mathbf{I}_\varepsilon + (1-s)t^2\mathbf{K}_2{}^t\mathbf{K}_2 \end{bmatrix}$$

and so

$$\det(\mathbf{QP}) = (1-(1-u)(1-s)t^2)^\nu$$
$$\cdot \det\left(\begin{bmatrix} a\mathbf{I}_\varepsilon + (1-u)t^2\mathbf{K}_1{}^t\mathbf{K}_1 & -t\mathbf{K}_1{}^t\mathbf{K}_1 \\ -t\mathbf{K}_2{}^t\mathbf{K}_2 & a\mathbf{I}_\varepsilon + (1-s)t^2\mathbf{K}_2{}^t\mathbf{K}_2 \end{bmatrix}\right).$$

But,

$$\det\left(\begin{bmatrix} a\mathbf{I}_\varepsilon + (1-u)t^2\mathbf{K}_1{}^t\mathbf{K}_1 & -t\mathbf{K}_1{}^t\mathbf{K}_1 \\ -t\mathbf{K}_2{}^t\mathbf{K}_2 & a\mathbf{I}_\varepsilon + (1-s)t^2\mathbf{K}_2{}^t\mathbf{K}_2 \end{bmatrix}\right)$$
$$= \det\left(\begin{bmatrix} \mathbf{I}_\varepsilon & -t(\mathbf{K}_1{}^t\mathbf{K}_1 - (1-s)\mathbf{I}_\varepsilon) \\ -t(\mathbf{K}_2{}^t\mathbf{K}_2 - (1-u)\mathbf{I}_\varepsilon) & \mathbf{I}_\varepsilon \end{bmatrix}\right)$$
$$\cdot \det\left(\begin{bmatrix} \mathbf{I}_\varepsilon & -(1-s)t\mathbf{I}_\varepsilon \\ -(1-u)t\mathbf{I}_\varepsilon & \mathbf{I}_\varepsilon \end{bmatrix}\right).$$

By (1), we have

$$\det\left(\begin{bmatrix} \mathbf{I}_{\varepsilon} & -t(\mathbf{K}_1 {}^t\mathbf{K}_1 - (1-s)\mathbf{I}_{\varepsilon}) \\ -t(\mathbf{K}_2 {}^t\mathbf{K}_2 - (1-u)\mathbf{I}_{\varepsilon}) & \mathbf{I}_{\varepsilon} \end{bmatrix}\right)$$

$$= \det(\mathbf{I}_{2\varepsilon} - t(\mathbf{B}(G) - (1-s)\mathbf{J}_1 - (1-u)\mathbf{J}_2)).$$

Furthermore, we have

$$\det\left(\begin{bmatrix} \mathbf{I}_{\varepsilon} & -(1-s)t\mathbf{I}_{\varepsilon} \\ -(1-u)t\mathbf{I}_{\varepsilon} & \mathbf{I}_{\varepsilon} \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} \mathbf{I}_{\varepsilon} & -(1-s)t\mathbf{I}_{\varepsilon} \\ -(1-u)t\mathbf{I}_{\varepsilon} & \mathbf{I}_{\varepsilon} \end{bmatrix}\right)\det\left(\begin{bmatrix} \mathbf{I}_{\varepsilon} & \mathbf{0} \\ (1-u)t\mathbf{I}_{\varepsilon} & \mathbf{I}_{\varepsilon} \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} (1-(1-u)(1-s)t^2)\mathbf{I}_{\varepsilon} & -(1-s)t\mathbf{I}_{\varepsilon} \\ \mathbf{0} & \mathbf{I}_{\varepsilon} \end{bmatrix}\right)$$

$$= (1-(1-u)(1-s)t^2)^{\varepsilon}.$$

Since $\det(\mathbf{PQ}) = \det(\mathbf{QP})$, we have

$$(1-(1-u)(1-s)t^2)^{2\varepsilon}\det(\mathbf{I}_{\nu} - t\mathbf{A}(G)$$

$$+ t^2((1-s)(\mathbf{D}_{V_1} - (1-u)\mathbf{I}_n) \oplus (1-u)(\mathbf{D}_{V_2} - (1-s)\mathbf{I}_m)))$$

$$= (1-(1-u)(1-s)t^2)^{\varepsilon+\nu}\det(\mathbf{I}_{2\varepsilon} - t(\mathbf{B}(G) - (1-s)\mathbf{J}_1 - (1-u)\mathbf{J}_2)).$$

Therefore, it follows that

$$\det(\mathbf{I}_{2\varepsilon} - t(\mathbf{B}(G) - (1-s)\mathbf{J}_1 - (1-u)\mathbf{J}_2))$$

$$= (1-(1-u)(1-s)t^2)^{\varepsilon-\nu}\det(\mathbf{I}_{\nu} - t\mathbf{A}(G)$$

$$+ t^2((1-s)(\mathbf{D}_{V_1} - (1-u)\mathbf{I}_n) \oplus (1-u)(\mathbf{D}_{V_2} - (1-s)\mathbf{I}_m))).$$

Next, let

$$D(G) = \{f_1, \ldots, f_{\varepsilon}, f_{\varepsilon+1}, \ldots, f_{2\varepsilon}\}$$

such that $f_{\varepsilon+i} = f_i^{-1}$ $(1 \le i \le \varepsilon)$, and consider the lexicographic order on

$D(G) \times D(G)$ derived from a total order of $D(G)$: $f_1 < f_2 < \cdots < f_{2\varepsilon}$. If (f_i, f_j) is the rth pair under the above order, then we define the $2\varepsilon \times 2\varepsilon$ matrix $\mathbf{T}_r = ((\mathbf{T}_r)_{p,q})_{1 \le p, q \le 2\varepsilon}$ as follows:

$$(\mathbf{T}_r)_{p,q} = \begin{cases} t & \text{if } p = f_i,\ q = f_j,\ t(f_i) = o(f_j) \text{ and } f_j \neq f_i^{-1}, \\ ut & \text{if } p = f_i,\ q = f_j,\ t(f_i) \in V_1 \text{ and } f_j = f_i^{-1}, \\ st & \text{if } p = f_i,\ q = f_j,\ t(f_i) \in V_2 \text{ and } f_j = f_i^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{M} = \mathbf{T}_1 + \cdots + \mathbf{T}_k$, $k = 4\varepsilon^2$. Then we have

$$\mathbf{M} = t(\mathbf{B}(G) - (1-s)\mathbf{J}_1 - (1-u)\mathbf{J}_2).$$

Let L be the set of all Lyndon words in $D(G) \times D(G)$. Then we can also consider L as the set of all Lyndon words in $\{1, \ldots, k\}$: $(f_{i_1}, f_{j_1}) \cdots (f_{i_q}, f_{j_q})$ corresponds to $r_1 r_2 \cdots r_q$, where (f_{i_p}, f_{j_p}) $(1 \le p \le q)$ is the r_p th pair. Theorem 3 implies that

$$\det(\mathbf{I}_{2\varepsilon} - \mathbf{M}) = \prod_{w \in L} \det(\mathbf{I}_{2\varepsilon} - \mathbf{T}_w),$$

where

$$\mathbf{T}_w = \mathbf{T}_{i_1} \cdots \mathbf{T}_{i_p}$$

for $w = i_1 \cdots i_p$. Note that $\det(\mathbf{I}_{2\varepsilon} - \mathbf{T}_w)$ is the alternating sum of the diagonal minors of $\mathbf{T}_w$. Thus, we have

$$\det(\mathbf{I}_{2\varepsilon} - \mathbf{T}_w) = \begin{cases} 1 - u^{cbc_1(C)} s^{cbc_2(C)} t^{|C|} & \text{if } w \text{ is a prime cycle } C, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, it follows that

$$\begin{aligned} \zeta(G, u, s, t)^{-1} &= \prod_{[C]} (1 - u^{cbc_1(C)} s^{cbc_2(C)} t^{|C|}) \\ &= \det(\mathbf{I}_{2\varepsilon} - t(\mathbf{B}(G) - (1-s)\mathbf{J}_1 - (1-u)\mathbf{J}_2)), \end{aligned}$$

where $[C]$ runs over all equivalence classes of prime cycles of G. □

3. A Generalized Bartholdi Zeta Function of a Hypergraph

Let H be a hypergraph. In a path $P = (v_1, e_1, v_2, e_2, \ldots, e_n, v_{n+1})$, subsequences (e, v, e) and (v, e, v) are called a *vertex bump* and an *edge bump*, respectively. Furthermore, the *vertex cyclic bump count* $vcbc(C)$ and *edge cyclic bump count* $ecbc(C)$ of a cycle $C = (v_1, e_1, v_2, e_2, \ldots, e_n, v_1)$ are defined by

$$vcbc(C) = |\{i = 1, \ldots, n \,|\, e_i = e_{i+1}\}|$$

and

$$ecbc(C) = |\{i = 1, \ldots, n \,|\, v_i = v_{i+1}\}|,$$

respectively, where $v_{n+1} = v_1$ and $e_{n+1} = e_1$.

The *generalized Bartholdi zeta function* of a hypergraph H is defined by

$$\zeta(H, u, s, t) = \prod_{[C]} (1 - u^{vcbc(C)} s^{ecbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of H, and u, s, t are complex variables with $|u|, |s|, |t|$ sufficiently small.

If $u = s = 0$, then the Bartholdi zeta function of H is the Ihara-Selberg zeta function of H.

A determinant expression of the generalized Bartholdi zeta function of a hypergraph is given as follows:

Theorem 7. *Let H be a finite, connected hypergraph with n hypervertices and m hyperedges. Then*

$$\begin{aligned}
\zeta(H, u, s, t)^{-1} &= \zeta(B_H, u, s, \sqrt{t})^{-1} \\
&= \det(\mathbf{I}_{2\varepsilon} - \sqrt{t}(\mathbf{B}(B_H) - (1-s)\mathbf{J}_1 - (1-u)\mathbf{J}_2)) \\
&= (1-(1-u)(1-s)t)^{\varepsilon-\nu} \det(\mathbf{I}_\nu - \sqrt{t}\mathbf{A}(B_H) + t((1-s)(\mathbf{D}_{V(H)} \\
&\qquad - (1-u)\mathbf{I}_n) \oplus (1-u)(\mathbf{D}_{E(H)} - (1-s)\mathbf{I}_m))),
\end{aligned}$$

where $\nu = |V(B_H)|$ and $\varepsilon = |E(B_H)|$.

Proof. The argument is an analogue of Storm's method [11].

Let $V_1 = V(H)$ and $V_2 = E(H)$. At first, we show that there exists a one-to-one correspondence between equivalence classes of prime cycles of length l in H and those of prime cycles of length $2l$ in B_H, and $vcbc(C) = cbc_1(\tilde{C})$, $ecbc(C) = cbc_2(\tilde{C})$ for any prime cycle C in H and the corresponding cycle $\tilde{C}$ in B_H.

Let $C = (v_1, e_1, v_2, \ldots, v_l, e_l, v_1)$ be a prime cycle of length l in H. Then a cycle $\tilde{C} = (v_1, (v_1, e_1), e_1, \ldots, v_l, (v_l, e_l), e_l, (e_l, v_1), v_1)$ is a prime cycle of length $2l$ in B_H. Thus, there exists a one-to-one correspondence between equivalence classes of prime cycles of length l in H and those of prime cycles of length $2l$ in B_H.

Let C be a prime cycle in H and $\tilde{C}$ be a prime cycle corresponding to C in B_H. Then there exists a subsequence (v, e, v) (or (e, v, e)) in C if and only if there exists a subsequence $(v, (v, e), e, (e, v), v)$ (or $(e, (e, v), v, (v, e), e)$) in $\tilde{C}$. Thus, we have $vcbc(C) = cbc_1(\tilde{C})$ and $ecbc(C) = cbc_2(\tilde{C})$.

Therefore, it follows that

$$\zeta(H, u, s, t) = \prod_{[C]} (1 - u^{vcbc(C)} s^{ecbc(C)} t^{|C|})^{-1}$$

$$= \prod_{[\tilde{C}]} (1 - u^{cbc_1(\tilde{C})} s^{cbc_2(\tilde{C})} t^{|\tilde{C}|/2})^{-1} = \zeta(B_H, u\ s, \sqrt{t}),$$

where $[C]$ and $[\tilde{C}]$ run over all equivalence classes of prime cycles in H and B_H, respectively.

By Theorem 6, we have

$$\zeta(H, u, s, t)^{-1}$$

$$= \det(\mathbf{I}_{2\varepsilon} - \sqrt{t}(\mathbf{B}(B_H) - (1 - s)\mathbf{J}_1 - (1 - u)\mathbf{J}_2))$$

$$= (1-(1-u)(1-s)t)^{\varepsilon-\nu} \det(\mathbf{I}_{\nu} - \sqrt{t}\mathbf{A}(B_H) + t((1-s)(\mathbf{D}_{V(H)}$$

$$- (1-u)\mathbf{I}_n) \oplus (1-u)(\mathbf{D}_{E(H)} - (1-s)\mathbf{I}_m))),$$

where $\nu = |V(B_H)|$ and $\varepsilon = |E(B_H)|$. □

Corollary 1. *Let H be a finite, connected hypergraph. Then*

$$\zeta(H, u, s, t) = \zeta(H^*, u, s, t).$$

Proof. By the fact that $B_H = B_{H^*}$. □

If $u = 0$, then the following result holds.

Corollary 2. *Let H be a finite, connected hypergraph with n hypervertices and m hyperedges. Then*

$$\zeta(H, 0, s, t)^{-1} = \prod_{[C_1]} (1 - s^{ecbc(C_1)} t^{|C_1|})$$

$$= \det(\mathbf{I}_{2\varepsilon} - \sqrt{t}(\mathbf{B}(B_H) - (1-s)\mathbf{J}_1 - \mathbf{J}_2))$$

$$= (1-(1-s)t)^{\varepsilon-\nu} \det(\mathbf{I}_{\nu} - \sqrt{t}\mathbf{A}(B_H)$$

$$+ t((1-s)(\mathbf{D}_{V(H)} - \mathbf{I}_n) \oplus (\mathbf{D}_{E(H)} - (1-s)\mathbf{I}_m))),$$

where $\nu = |V(B_H)|$, $\varepsilon = |E(B_H)|$, *and* $[C_1]$ *runs over all equivalence classes of prime cycles without vertex bumps in H.*

If $s = 0$, then the following result holds.

Corollary 3. *Let H be a finite, connected hypergraph with n hypervertices and m hyperedges. Then*

$$\zeta(H, u, 0, t)^{-1} = \prod_{[C_2]} (1 - u^{vcbc(C_2)} t^{|C_2|})$$

$$= \det(\mathbf{I}_{2\varepsilon} - \sqrt{t}(\mathbf{B}(B_H) - \mathbf{J}_1 - (1-u)\mathbf{J}_2))$$

$$= (1-(1-u)t)^{\varepsilon-\nu}\det(\mathbf{I}_\nu - \sqrt{t}\mathbf{A}(B_H)$$

$$+ t(\mathbf{D}_{V(H)} - (1-u)\mathbf{I}_n) \oplus (1-u)(\mathbf{D}_{E(H)} - \mathbf{I}_m))),$$

where $\nu = |V(B_H)|$, $\varepsilon = |E(B_H)|$, *and* $[C_2]$ *runs over all equivalence classes of prime cycles without edge bumps in H*.

In the case of $s = u$, we also obtain Theorem 5.

Next, in the case that $H = G$ is a graph, we show that $\zeta(G, u, 0, t) = \zeta(B_G, u, 0, \sqrt{t})$ is equal to the original Bartholdi zeta function $\zeta(G, u, t)$ of G.

Corollary 4. *Let* $H = G$ *be a finite, connected graph with n vertices and m edges*. *Then*

$$\zeta(G, u, 0, t) = \zeta(G, u, t).$$

Proof. Let $H = G$ be a connected graph, $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. Furthermore, let B_G be the bipartite graph with ν vertices and ε edges corresponding to G. Then we have $\varepsilon = 2m$, $\nu = m + n$, and

$$\mathbf{D}_{V(G)} = \mathbf{D}_G, \quad \mathbf{D}_{E(G)} = 2\mathbf{I}_m.$$

By Corollary 3, we have

$$\zeta(G, u, 0, t)^{-1} = (1-(1-u)t)^{2m-(m+n)}\det(\mathbf{I}_\nu - \sqrt{t}\mathbf{A}(B_G)$$

$$+ t((\mathbf{D}_G - (1-u)\mathbf{I}_n) \oplus (1-u)\mathbf{I}_m)).$$

Let $\mathbf{H} = (h_{ve})_{v\in V(G);\, e\in E(G)}$ be the incidence matrix of G:

$$h_{ve} = \begin{cases} 1 & \text{if } v \text{ and } e \text{ are incident,} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\mathbf{A}(B_G) = \begin{bmatrix} \mathbf{0} & \mathbf{H} \\ {}^t\mathbf{H} & \mathbf{0} \end{bmatrix}.$$

Thus,

$$\det(\mathbf{I}_\nu - \sqrt{t}\mathbf{A}(B_G) + t((\mathbf{D}_G - (1-u)\mathbf{I}_n) \oplus (1-u)\mathbf{I}_m))$$

$$= \det\left(\begin{bmatrix} \mathbf{I}_n + t(\mathbf{D}_G - (1-u)\mathbf{I}_n) & -\sqrt{t}\mathbf{H} \\ -\sqrt{t}\,{}^t\mathbf{H} & (1+(1-u)t)\mathbf{I}_m \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} \mathbf{I}_n + t(\mathbf{D}_G - (1-u)\mathbf{I}_n) - t/(1+(1-u)t)\mathbf{H}{}^t\mathbf{H} & -\sqrt{t}\mathbf{H} \\ \mathbf{0} & (1+(1-u)t)\mathbf{I}_m \end{bmatrix}\right).$$

Since

$$\mathbf{H}{}^t\mathbf{H} = \mathbf{A}(G) + \mathbf{D}_G,$$

we have

$$\det(\mathbf{I}_\nu - \sqrt{t}\mathbf{A}(B_G) + t((\mathbf{D}_G - (1-u)\mathbf{I}_n) \oplus (1-u)\mathbf{I}_m))$$

$$= (1+(1-u)t)^{m-n}\det((1-(1-u)^2t^2)\mathbf{I}_n - t\mathbf{A}(G) + (1-u)t^2\mathbf{D}_G)$$

$$= (1+(1-u)t)^{m-n}\det(\mathbf{I}_n - t\mathbf{A}(G) + (1-u)t^2(\mathbf{D}_G - (1-u)\mathbf{I}_n)).$$

Therefore, it follows that

$$\zeta(G, u, 0, t)^{-1}$$

$$= (1-(1-u)^2t^2)^{m-n}\det(\mathbf{I}_n - t\mathbf{A}(G) + (1-u)t^2((\mathbf{D}_G - (1-u)\mathbf{I}_n))$$

$$= \zeta(G, u, t)^{-1}.$$ □

In the case of $s = u = 0$, Theorem 7 implies Storm Theorem.

4. Two New Determinant Expressions of the Generalized Bartholdi Zeta Function of a Hypergraph

Let $H = (V(H), E(H))$ be a hypergraph, $V(H) = \{v_1, ..., v_n\}$ and $E(H) = \{e_1, ..., e_m\}$. Let B_H have ν vertices and ε edges, where $\nu = n + m$. Then we have

$$D(B_H) = \{(v, e), (e, v) | v \in V(H), e \in E(H)\}.$$

Let $f_1, \ldots, f_\varepsilon$ be arcs in B_H such that $o(f_i) \in V(H)$ for each $i = 1, \ldots, \varepsilon$. Then two $\varepsilon \times \varepsilon$ matrices $\mathbf{X} = (X_{ij})$ and $\mathbf{Y} = (Y_{ij})$ are defined as follows:

$$X_{ij} = \begin{cases} 1 & \text{if there exists an arc } f_k^{-1} \text{ such that } (f_i, f_k^{-1}, f_j) \text{ is a reduced path,} \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_{ij} = \begin{cases} 1 & \text{if there exists an arc } f_k \text{ such that } (f_i^{-1}, f_k, f_j^{-1}) \text{ is a reduced path,} \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let

$$\mathbf{B}(B_H) - \mathbf{J}_1 - \mathbf{J}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{F} \\ \mathbf{G} & \mathbf{0} \end{bmatrix}.$$

Theorem 8. *Let H be a finite, connected hypergraph. Set* $\varepsilon = |E(B_H)|$. *Then*

$$\zeta(H, u, s, t)^{-1} = \det(\mathbf{I}_\varepsilon - t(\mathbf{X} + u\mathbf{F} + s\mathbf{G} + us\mathbf{I}_\varepsilon))$$
$$= \det(\mathbf{I}_\varepsilon - t(\mathbf{Y} + u\mathbf{F} + s\mathbf{G} + us\mathbf{I}_\varepsilon)).$$

Proof. Let $H = (V(H), E(H))$ be a hypergraph, $V(H) = \{v_1, \ldots, v_n\}$ and $E(H) = \{e_1, \ldots, e_m\}$. Let B_H have ν vertices and ε edges. By Theorem 7, we have

$$\zeta(H, u, s, t)^{-1} = \det(\mathbf{I}_{2\varepsilon} - \sqrt{t}(\mathbf{B}(B_H) - (1-s)\mathbf{J}_1 - (1-u)\mathbf{J}_2)).$$

Arrange arcs of B_H as follows: $f_1, \ldots, f_\varepsilon, f_1^{-1}, \ldots, f_\varepsilon^{-1}$. We consider three matrices $\mathbf{B}(B_H)$, $\mathbf{J}_1$ and $\mathbf{J}_2$ under this order. Then we have

$$\mathbf{B}(B_H) - (1-s)\mathbf{J}_1 - (1-u)\mathbf{J}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{F} + s\mathbf{I}_\varepsilon \\ \mathbf{G} + u\mathbf{I}_\varepsilon & \mathbf{0} \end{bmatrix}.$$

It is clear that both $\mathbf{F}$ and $\mathbf{G}$ are symmetric, but $\mathbf{F} \neq {}^t\mathbf{G}$. Furthermore,

$$\mathbf{FG} = \mathbf{X} \quad \text{and} \quad \mathbf{GF} = \mathbf{Y}. \tag{6}$$

Thus, we have

$$\det(\mathbf{I}_{2\varepsilon} - \sqrt{t}(\mathbf{B}(B_H) - (1-s)\mathbf{J}_1 - (1-u)\mathbf{J}_2))$$

$$= \det\left(\begin{bmatrix} \mathbf{I}_\varepsilon & -\sqrt{t}(\mathbf{F} + s\mathbf{I}_\varepsilon) \\ -\sqrt{t}(\mathbf{G} + u\mathbf{I}_\varepsilon) & \mathbf{I}_\varepsilon \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} \mathbf{I}_\varepsilon - t(\mathbf{F} + s\mathbf{I}_\varepsilon)(\mathbf{G} + u\mathbf{I}_\varepsilon) & -\sqrt{t}(\mathbf{F} + s\mathbf{I}_\varepsilon) \\ \mathbf{0} & \mathbf{I}_\varepsilon \end{bmatrix}\right)$$

$$= \det(\mathbf{I}_\varepsilon - t(\mathbf{FG} + u\mathbf{F} + s\mathbf{G} + us\mathbf{I}_\varepsilon)) = \det(\mathbf{I}_\varepsilon - t(\mathbf{X} + u\mathbf{F} + s\mathbf{G} + us\mathbf{I}_\varepsilon))$$

$$= \det(\mathbf{I}_\varepsilon - t(\mathbf{GF} + u\mathbf{F} + s\mathbf{G} + us\mathbf{I}_\varepsilon)) = \det(\mathbf{I}_\varepsilon - t(\mathbf{Y} + u\mathbf{F} + s\mathbf{G} + us\mathbf{I}_\varepsilon)).$$

Therefore, the result follows. □

For the bipartite graph B_H corresponding to a hypergraph H with n hypervertices and m hyperedges, let $V_1 = V(H)$ and $V_2 = E(H)$. Then, the *broad halved graph* $B_H^{(i)}$ of B_H is defined to be the graph with vertex set V_i and arc set $\{P : \text{path} \,|\, |P| = 2;\, o(P), t(P) \in V_i\}$ for $i = 1, 2$. Furthermore, let $\{c_1, \ldots, c_m\}$ be a set of m colors such that $c(e_i) = c_i$ for $i = 1, \ldots, m$. We color each arc of $B_H^{(1)}$ as follows:

$$c(P) = c(e) \text{ for } P = (v, e, w) \in D(B_H^{(1)}).$$

Then the *line digraph* $\vec{L}(B_H^{(i)})$ of $B_H^{(i)}$ $(i = 1, 2)$ is defined as follows: $V(\vec{L}(B_H^{(i)})) = D(B_H^{(i)})$, and $(P, Q) \in A(\vec{L}(B_H^{(i)}))$ if and only if $t(P) = o(Q)$ in B_H.

Let B_H have ν vertices and ε edges, and

$$D(B_H) = \{f_1, \dots, f_\varepsilon, f_1^{-1}, \dots, f_\varepsilon^{-1}\}$$

such that $o(f_i) \in V(H)$ for each $i = 1, \dots, \varepsilon$. Let $\mathcal{R}$ (or $\mathcal{S}$) be the set of reduced paths P in B_H with length two such that $o(P), t(P) \in V(H)$ (or $o(P), t(P) \in E(H)$). Set $r = |\mathcal{R}|$ and $s = |\mathcal{S}|$. Furthermore, let $\mathcal{R}'$ (or $\mathcal{S}'$) be the set of paths P in B_H with length two such that $o(P), t(P) \in V(H)$ (or $\in E(H)$). Next, let $f_k = (v_{i_k}, e_{j_k})$, $P_k = (v_{i_k}, e_{j_k}, v_{i_k})$ and $Q_k = (e_{j_k}, v_{i_k}, e_{j_k})$ for each $k = 1, \dots, \varepsilon$. Then we have

$$\mathcal{R}' = \mathcal{R} \cup \{P_1, \dots, P_\varepsilon\} \text{ and } \mathcal{S}' = \mathcal{S} \cup \{Q_1, \dots, Q_\varepsilon\}.$$

Furthermore, we have $\mathcal{R}' = D((B_H^{(1)}))$, $\mathcal{S}' = D((B_H^{(2)}))$, $|\mathcal{R}'| = r + \varepsilon$ and $|\mathcal{S}'| = s + \varepsilon$.

Now, we introduce an $(r + \varepsilon) \times (r + \varepsilon)$ matrix $\mathbf{T}' = (T''_{PP'})_{P, P' \in \mathcal{R}'}$ for the line digraph $\vec{L}(B_H^{(1)})$ of the halved graph $B_H^{(1)}$ is defined as follows:

$$T''_{PP'} = \begin{cases} us & \text{if } t(P) = o(P'),\ P = P' = P_i \text{ for some } i = 1, \dots, \varepsilon, \\ us & \text{if } t(P) = o(P'),\ P = P_i,\ P' \in \mathcal{R} \text{ and } c(P) = c(P'), \\ s & \text{if } t(P) = o(P'),\ P = P_i,\ P' = P_j \text{ and } c(P) \neq c(P'), \\ s & \text{if } t(P) = o(P'),\ P = P_i,\ P' \in \mathcal{R} \text{ and } c(P) \neq c(P'), \\ u & \text{if } t(P) = o(P'),\ P \in \mathcal{R},\ P' = P_i \text{ and } c(P) = c(P'), \\ u & \text{if } t(P) = o(P'),\ P \in \mathcal{R},\ P' \in \mathcal{R} \text{ and } c(P) = c(P'), \\ 1 & \text{if } t(P) = o(P'),\ P \in \mathcal{R},\ P' = P_i \text{ and } c(P) \neq c(P'), \\ 1 & \text{if } t(P) = o(P'),\ P, P' \in \mathcal{R} \text{ and } c(P) \neq c(P'), \\ 0 & \text{otherwise.} \end{cases}$$

We present another new determinant expression for the Bartholdi zeta function of a hypergraph.

Theorem 9. *Let H be a finite, connected hypergraph. Set* $\varepsilon = |E(B_H)|$ *and* $r = |\mathcal{R}|$. *Then*

$$\zeta(H, u, s, t)^{-1} = \det(\mathbf{I}_{r+\varepsilon} - t\mathbf{T}'').$$

Proof. Let $H = (V(H), E(H))$ be a hypergraph, $V(H) = \{v_1, \ldots, v_n\}$ and $E(H) = \{e_1, \ldots, e_m\}$ such that $o(f_i) \in V(H)(1 \le i \le \varepsilon)$. Let B_H have ν vertices and ε edges, and $D(B_H) = \{f_1, \ldots, f_\varepsilon, f_1^{-1}, \ldots, f_\varepsilon^{-1}\}$. Furthermore, let $\mathcal{R}$ (or $\mathcal{S}$) be the set of reduced paths P in B_H with length two such that $o(P), t(P) \in V(H)$(or $o(P), t(P) \in E(H)$). Set $r = |\mathcal{R}|$ and $s = |\mathcal{S}|$. For a path $P = (x, y, z)$ of length two in B_H, let

$$oe(P) = (x, y), \quad te(P) = (y, z),$$

where $(x, y, z) = (v, e, w)$ or $(x, y, z) = (e, v, f)(v, w \in V(H); e, f \in E(H))$.

Now, we introduce two $r \times \varepsilon$ matrices $\mathbf{K} = (K_{Pf_j^{-1}})_{P \in R; 1 \le j \le \varepsilon}$ and $\mathbf{L} = (L_{Pf_j})_{P \in R; 1 \le j \le \varepsilon}$ are defined as follows:

$$K_{Pf_j^{-1}} = \begin{cases} 1 & \text{if } te(P) = f_j^{-1}, \\ 0 & \text{otherwise,} \end{cases} \quad L_{Pf_j} = \begin{cases} 1 & \text{if } oe(P) = f_j, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, two $s \times \varepsilon$ matrices $\mathbf{M} = (M_{Qf_j^{-1}})_{Q \in S; 1 \le j \le \varepsilon}$ and $\mathbf{N} = (N_{Qf_j})_{Q \in S; 1 \le j \le \varepsilon}$ are defined as follows:

$$M_{Qf_j^{-1}} = \begin{cases} 1 & \text{if } oe(Q) = f_j^{-1}, \\ 0 & \text{otherwise,} \end{cases} \quad N_{Qf_j} = \begin{cases} 1 & \text{if } te(Q) = f_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$^t\mathbf{LK} = \mathbf{F} \quad \text{and} \quad ^t\mathbf{MN} = \mathbf{G}. \tag{7}$$

Arrange elements of $\mathcal{R}'$ and $\mathcal{S}'$ are as follows:

$$P_1, \ldots, P_\varepsilon, \mathcal{R}; \quad Q_1, \ldots, Q_\varepsilon, \mathcal{S}.$$

Then we introduce two $(r+\varepsilon)\times\varepsilon$ matrices $\mathbf{K}' = (K'_{Pf_j^{-1}})_{P\in\mathcal{R}';1\le j\le\varepsilon}$ and $\mathbf{L}' = (L'_{Pf_j})_{P\in\mathcal{R}';1\le j\le\varepsilon}$ are defined as follows:

$$K'_{Pf_j^{-1}} = \begin{cases} 1 & \text{if } te(P) = f_j^{-1} \text{ and } te(P) \neq te(P^{-1}), \\ s & \text{if } te(P) = te(P^{-1}) = f_j^{-1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$L'_{Pf_j} = \begin{cases} 1 & \text{if } oe(P) = f_j, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, two $(s+\varepsilon)\times\varepsilon$ matrices $\mathbf{M}' = (M'_{Qf_j^{-1}})_{Q\in\mathcal{S}';1\le j\le\varepsilon}$ and $\mathbf{N}' = (N'_{Qf_j})_{Q\in\mathcal{S}';1\le j\le\varepsilon}$ are defined as follows:

$$M'_{Qf_j^{-1}} = \begin{cases} 1 & \text{if } oe(Q) = f_j^{-1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$N'_{Qf_j} = \begin{cases} 1 & \text{if } te(Q) = f_j \text{ and } te(Q) \neq te(Q^{-1}), \\ u & \text{if } te(Q) = te(Q^{-1}) = f_j, \\ 0 & \text{otherwise.} \end{cases}$$

But, we have

$$\mathbf{K}' = \begin{bmatrix} s\mathbf{I}_\varepsilon \\ \mathbf{K} \end{bmatrix}, \mathbf{L}' = \begin{bmatrix} \mathbf{I}_\varepsilon \\ \mathbf{L} \end{bmatrix}, \mathbf{M}' = \begin{bmatrix} \mathbf{I}_\varepsilon \\ \mathbf{M} \end{bmatrix} \text{ and } \mathbf{N}' = \begin{bmatrix} u\mathbf{I}_\varepsilon \\ \mathbf{N} \end{bmatrix}.$$

Thus, we have

$$\mathbf{K}'\,{}^t\mathbf{M}'\mathbf{N}'\,{}^t\mathbf{L}' = \begin{bmatrix} us\mathbf{I}_\varepsilon + s\,{}^t\mathbf{M}\mathbf{N} & us\,{}^t\mathbf{L} + s\,{}^t\mathbf{M}\mathbf{N}\,{}^t\mathbf{L} \\ u\mathbf{K} + \mathbf{K}\,{}^t\mathbf{M}\mathbf{N} & u\mathbf{K}\,{}^t\mathbf{L} + \mathbf{K}\,{}^t\mathbf{M}\mathbf{N}\,{}^t\mathbf{L} \end{bmatrix}.$$

A nonzero element of $us\mathbf{I}_\varepsilon$, $s\,{}^t\mathbf{M}\mathbf{N}$ $us\,{}^t\mathbf{L}$, $s\,{}^t\mathbf{M}\mathbf{N}\,{}^t\mathbf{L}$, $u\mathbf{K}$, $\mathbf{K}\,{}^t\mathbf{M}\mathbf{N}$, $u\mathbf{K}\,{}^t\mathbf{L}$ and $\mathbf{K}\,{}^t\mathbf{M}\mathbf{N}\,{}^t\mathbf{L}$ corresponds to a sequence of eight paths of length two, respectively:

$$P_i \to Q_i \to P_i; \quad P_i \to Q \to P_j(c(P_i) \neq c(P_j));$$

$$P_i \to Q_i \to R(c(P_i) = c(R)); \quad P_i \to Q \to R(c(P_i) \neq c(R));$$

$$P \to Q_i \to P_i(c(P) = c(P_i)); \quad P \to Q \to P_i(c(P) \neq c(P_i));$$

$$P \to Q_i \to R(c(P) = c(R)); \quad P \to Q \to R(c(P) \neq c(R)),$$

where $P, R \in \mathcal{R}$, $Q \in \mathcal{S}$, $i = 1, \ldots, \varepsilon$, and the notation $P \to Q$ implies that $te(P) = oe(Q)$ in B_H. Therefore, it follows that

$$\mathbf{K}'\,{}^t\mathbf{M}'\mathbf{N}'\,{}^t\mathbf{L}' = \mathbf{T}''. \tag{8}$$

By (6) and (7), we have

$${}^t\mathbf{L}'\mathbf{K}'\,{}^t\mathbf{M}'\mathbf{N}' = us\mathbf{I}_\varepsilon + u\,{}^t\mathbf{L}\mathbf{K} + s\,{}^t\mathbf{M}\mathbf{N} + {}^t\mathbf{L}\mathbf{K}\,{}^t\mathbf{M}\mathbf{N} = us\mathbf{I}_\varepsilon + u\mathbf{F} + s\mathbf{G} + \mathbf{X}. \tag{9}$$

But, it is known that, for an $m \times n$ matrix **A** and an $n \times m$ matrix **B**,

$$\det(\mathbf{I}_m + \mathbf{AB}) = \det(\mathbf{I}_n + \mathbf{BA}). \tag{10}$$

By (8) and (9), it follows that

$$\det(\mathbf{I}_{r+\varepsilon} - t\mathbf{T}'') = \det(\mathbf{I}_\varepsilon - t(\mathbf{X} + u\mathbf{F} + s\mathbf{G} + us\mathbf{I}_\varepsilon)). \qquad \square$$

If $u = s = 0$, then Theorem 9 implies the first formula of Theorem 4.

Corollary 5. *Let H be a finite, connected hypergraph such that every hypervertex is in at least two hyperedges. Set $r = |\mathcal{R}|$. Then*

$$\zeta_H(t)^{-1} = \det(\mathbf{I}_r - t\mathbf{T}).$$

Proof. Set $\varepsilon = |E(B_H)|$ and $u = s = 0$. By Theorem 9 and the definition of $\mathbf{T}''$, we have

$$\zeta_H(t)^{-1} = \det(\mathbf{I}_{r+\varepsilon} - t\mathbf{T}'') = \det\left(\begin{bmatrix} \mathbf{I}_\varepsilon & \mathbf{0} \\ -t\mathbf{K}\,{}^t\mathbf{M}\mathbf{N} & \mathbf{I}_r - t\mathbf{T} \end{bmatrix}\right) = \det(\mathbf{I}_r - t\mathbf{T}).$$

$\square$

5. Bartholdi Zeta Functions of (d, r)-regular Hypergraphs

At first, we state a decomposition formula for the generalized Bartholdi zeta function of a semiregular bipartite graph. Hashimoto [5] presented a determinant expression for the Ihara zeta function of a semiregular bipartite graph. We generalize Hashimoto's result on the Ihara zeta function to the generalized Bartholdi zeta function.

A graph G is called *bipartite*, denoted by $G = (V_1, V_2)$ if there exists a partition $V(G) = V_1 \cup V_2$ of $V(G)$ such that the vertices in V_i are mutually nonadjacent for $i = 1, 2.$ A bipartite graph $G = (V_1, V_2)$ is called $(q_1 + 1, q_1 + 2)$-*semiregular* if $\deg_G v = q_i + 1$ for each $v \in V_i$ $(i = 1, 2)$. Then $G^{[1]}$ is $(q_1 + 1)q_2$-regular, and $G^{[2]}$ is $(q_2 + 1)q_1$-regular.

A determinant expression for the generalized Bartholdi zeta function of a semiregular bipartite graph is given as follows. For a graph G, let $Spec(G)$ be the set of all eigenvalues of the adjacency matrix of G.

Theorem 10. *Let* $G = (V_1, V_2)$ *be a connected* $(q_1 + 1, q_2 + 1)$-*semiregular bipartite graph with* ν *vertices and* ε *edges. Set* $|V_1| = n$ *and* $|V_2| = m$ $(n \le m)$. *Then*

$$\begin{aligned}
\zeta(G, u, s, t)^{-t} &= (1 - (1-u)(1-s)t^2)^{\varepsilon - \nu}(1 + (1-u)(q_2 + s)t^2)^{m-n} \\
&\quad \times \prod_{j=1}^{n}(1 - (\lambda_j^2 - (1-s)(q_1 + u) - (1-u)(q_2 + s))t^2 \\
&\quad + (1-u)(1-s)(q_1 + u)(q_2 + s)t^4) \\
&= (1 - (1-u)(1-s)t^2)^{\varepsilon - \nu} \\
&\quad \cdot (1 + (1-u)(q_2 + s)t^2)^{m-n} \det(\mathbf{I}_n - (\mathbf{A}^{[1]} \\
&\quad - (q_2 - 1 - (q_1 - 1)s - (q_2 - 1)u - 2us)\mathbf{I}_n)t^2 \\
&\quad + (1-u)(1-s)(q_1 + u)(q_2 + s)t^4\mathbf{I}_n)
\end{aligned}$$

$$= (1-(1-u)(1-s)t^2)^{\varepsilon-\nu}(1+(1-s)(q_1+u)t^2)^{n-m}\det(\mathbf{I}_m$$

$$-(\mathbf{A}^{[2]}-(q_1-1-(q_1-1)s-(q_2-1)u-2us)\mathbf{I}_m)t^2$$

$$+(1-u)(1-s)(q_1+u)(q_2+s)t^4\mathbf{I}_m),$$

where $Spec(G) = \{\pm\lambda_1, \dots, \pm\lambda_n, 0, \dots, 0\}$ *and* $\mathbf{A}^{[i]} = \mathbf{A}(G^{[i]})(i = 1, 2)$.

Proof. The argument is an analogue of Hashimoto's method [5].

By Theorem 6, we have

$$\zeta(G, u, s, t)^{-1} = (1-(1-u)(1-s)t^2)^{\varepsilon-\nu}\det(\mathbf{I}_\nu - t\mathbf{A}(G)$$

$$+t^2((1-s)(q_1+u)\mathbf{I}_n \oplus (1-u)(q_2+s)\mathbf{I}_m)).$$

Let $V_1 = \{v_1, \dots, v_n\}$ and $V_2 = \{w_1, \dots, w_m\}$. Arrange vertices of G are as follows: $v_1, \dots, v_n$; $w_1, \dots, w_m$. We consider the matrix $\mathbf{A} = \mathbf{A}(G)$ under this order. Then, let

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{E} \\ {}^t\mathbf{E} & \mathbf{0} \end{bmatrix},$$

where ${}^t\mathbf{E}$ is the transpose of $\mathbf{E}$.

Since $\mathbf{A}$ is symmetric, there exists an orthogonal matrix $\mathbf{W} \in O(m)$ such that

$$\mathbf{EW} = [\mathbf{R} \quad \mathbf{0}] = \begin{bmatrix} \mu_1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ \star & & \mu_n & 0 & \cdots & 0 \end{bmatrix}.$$

Now, let

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{bmatrix}.$$

Then we have

$$ {}^t\mathbf{PAP} = \begin{bmatrix} \mathbf{0} & \mathbf{R} & \mathbf{0} \\ {}^t\mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. $$

Furthermore, we have

$$ {}^t\mathbf{P}((1-s)(q_1+u)\mathbf{I}_n \oplus (1-u)(q_2+s)\mathbf{I}_m)\mathbf{P} $$

$$ = (1-s)(q_1+u)\mathbf{I}_n \oplus (1-u)(q_2+s)\mathbf{I}_m. $$

Thus,

$$ \zeta(G, u, s, t)^{-1} $$

$$ = (1-(1-u)(1-s)t^2)^{\varepsilon-\nu}(1+(1-u)(q_2+s)t^2)^{m-n} \det\left(\begin{bmatrix} a\mathbf{I}_n & -t\mathbf{R} \\ -t\,{}^t\mathbf{R} & b\mathbf{I}_n \end{bmatrix} \right) $$

$$ = (1-(1-u)(1-s)t^2)^{\varepsilon-\nu}(1+(1-u)(q_2+s)t^2)^{m-n} $$

$$ \cdot \det\left(\begin{bmatrix} a\mathbf{I}_n & \mathbf{0} \\ -t\,{}^t\mathbf{R} & b\mathbf{I}_n - a^{-1}t^2\,{}^t\mathbf{RR} \end{bmatrix} \right) $$

$$ = (1-(1-u)(1-s)t^2)^{\varepsilon-\nu}(1+(1-u)(q_2+s)t^2)^{m-n} \det(ab\mathbf{I}_n - t^2\,{}^t\mathbf{RR}), $$

where $a = 1+(1-s)(q_1+u)t^2$ and $b = 1+(1-u)(q_2+s)t^2$.

Since $\mathbf{A}$ is symmetric, ${}^t\mathbf{RR}$ is symmetric and positive semi-definite, i.e., the eigenvalues of ${}^t\mathbf{RR}$ are of form:

$$ \lambda_1^2, \ldots, \lambda_n^2 (\lambda_1, \ldots, \lambda_n \geq 0). $$

Therefore, it follows that

$$ \zeta(G, u, s, t)^{-1} $$

$$ = (1-(1-u)(1-s)t^2)^{\varepsilon-\nu}(1+(1-u)(q_2+s)t^2)^{m-n} \prod_{j=1}^{n} (ab - \lambda_j^2 t^2). $$

But, we have

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^{m-n} \det(\lambda^2 \mathbf{I} - {}^t\mathbf{R}\mathbf{R})$$

and so

$$Spec(\mathbf{A}) = \{\pm\lambda_1, \ldots, \pm\lambda_n, 0, \ldots, 0\}.$$

Thus, there exists an orthogonal matrix S such that

$$ {}^t\mathbf{S}\mathbf{A}^2\mathbf{S} = \begin{bmatrix} \lambda_1^2 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & \lambda_n^2 & & & & & & \\ & & & \lambda_1^2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_n^2 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & 0 \end{bmatrix}, \ \mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{0} & * \end{bmatrix},$$

where $\mathbf{S}_1$ is an $n \times n$ matrix. Furthermore, we have

$$\mathbf{A}^2 = \mathbf{A}_2 + \mathbf{D}_G,$$

where $\mathbf{A}_2 = ((\mathbf{A}_2)_{uv})_{u, v \in V(G)}$ is given as follows:

$$(\mathbf{A}_2)_{uv} = \text{ the number of reduced } (u, v)\text{-paths with length 2.}$$

By the definition of the graphs $G^{[i]}$ $(i = 1, 2)$,

$$\mathbf{A}^2 = \begin{bmatrix} \mathbf{A}^{[1]} + (q_1 + 1)\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{[2]} + (q_2 + 1)\mathbf{I}_m \end{bmatrix},$$

where $\mathbf{A}^{[i]} = \mathbf{A}(G^{[i]})(i = 1, 2)$. Thus,

$${}^t\mathbf{S}\mathbf{A}^2\mathbf{S} = \begin{bmatrix} \mathbf{S}_1^{-1}\mathbf{A}^{[1]}\mathbf{S}_1 + (q_1 + 1)\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & * \end{bmatrix}.$$

Therefore, it follows that

$$\mathbf{S}_1^{-1}\mathbf{A}^{[1]}\mathbf{S}_1 = \begin{bmatrix} \lambda_1^2 - (q_1+1) & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 - (q_1+1) \end{bmatrix}.$$

Hence

$$\det(ab\mathbf{I}_n - (\mathbf{A}^{[1]} + (q_1+1)\mathbf{I}_n)t^2) = \prod_{j=1}^{n}(ab - \lambda_j^2 t^2).$$

Thus, the second equation follows.

Similarly to the proof of the second equation, the third equation is obtained. □

A hypergraph H is a (d, r)-*regular* if every hypervertex is incident to d hyperedges, and every hyperedge contains r hypervertices. If H is a (d, r)-regular hypergraph, then the associated bipartite graph B_H is (d, r)-semiregular. Let $V_1 = V(H)$, $V_2 = E(H)$ and $d \geq r$. Set $n = |V_1|$ and $m = |V_2|$. Then we have $\mathbf{A}^{[1]} = \mathbf{A}(H)$ and $\mathbf{A}^{[2]} = \mathbf{A}(H^*)$. By Theorems 7 and 10, we obtain the following result. Let $Spec(\mathbf{B})$ be the set of all eigenvalues of the square matrix $\mathbf{B}$.

Theorem 11. *Let H be a finite, connected (d, r)-regular hypergraph with $d \geq r$. Set $n = |V(H)|$ and $m = |E(H)|$. Then*

$$\begin{aligned} &\zeta(H, u, s, t)^{-1} \\ &= (1-(1-u)(1-s)t)^{\varepsilon-\nu}(1+(1-u)(r-1+s)t)^{m-n} \\ &\quad \times \prod_{j=1}^{n}(1-(\lambda_j^2 - (1-s)(d-1+u) - (1-u)(r-1+s))t \\ &\quad + (1-u)(1-s)(d-1+u)(r-1+s)t^2) \end{aligned}$$

$$= (1-(1-u)(1-s)t)^{\varepsilon-\nu}(1+(1-u)(r-1+s)t)^{m-n}$$

$$\times \det(\mathbf{I}_n - (\mathbf{A}(H) - (r-2-(d-2)s-(r-2)u-2us)\mathbf{I}_n)t$$

$$+ (1-u)(1-s)(d-1+u)(r-1+s)t^2\mathbf{I}_n)$$

$$= (1-(1-u)(1-s)t)^{\varepsilon-\nu}(1+(1-s)(d-1+u)t)^{n-m}$$

$$\times \det(\mathbf{I}_m - (\mathbf{A}(H^*) - (d-2-(d-2)s-(r-2)u-2us)\mathbf{I}_m)t$$

$$+ (1-u)(1-s)(d-1+u)(r-1+s)t^2\mathbf{I}_m),$$

where $\varepsilon = nd = mr,\ \nu = n + m$ *and* $Spec(\mathbf{A}(B_H)) = \{\pm\lambda_1, \ldots, \pm\lambda_n, 0, \ldots, 0\}$.

In the case of $s = u = 0,$ we obtain Theorem 16 in [11].

Corollary 6 (Storm)**.** *Let H be a finite, connected (d, r)-regular hypergraph with $d \geq r$. Set $n = |V(H)|,\ m = |E(H)|$ and $q = (d-1)(r-1)$. Then*

$$\zeta_H(t)^{-1} = (1-t)^{\varepsilon-\nu}(1+(r-1)t)^{m-n}\det(\mathbf{I}_n - (\mathbf{A}(H) - r + 2)t + qt^2)$$

$$= (1-t)^{\varepsilon-\nu}(1+(d-1)t)^{n-m}\det(\mathbf{I}_m - (\mathbf{A}(H^*) - d + 2)t + qt^2),$$

where $\varepsilon = nd = mr$ *and* $\nu = n + m$*.*

6. Example

Let $G = (V_1, V_2)$ be the bipartite graph with $V_1 = \{v_1, v_2, v_3\},\ V_2 = \{v_4, v_5, v_6\}$ and

$$E(G) = \{v_1v_4, v_1v_5, v_1v_6, v_2v_4, v_2v_6, v_3v_5, v_3v_6\}.$$

Then we have $n = m = 3,\ \varepsilon = 7,\ \nu = 6$ and

$$\mathbf{A}(G) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{V_1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$\mathbf{D}_{V_2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

By Theorem 6, we have

$$\begin{aligned}
&\zeta(G, u, s, t)^{-1} \\
&= (1-(1-s)(1-u)t^2)^{\varepsilon-\nu} \times \det(\mathbf{I}_\nu - t\mathbf{A}(G) \\
&\quad + t^2((1-s)(\mathbf{D}_{V_1} - (1-u)\mathbf{I}_n) \oplus (1-u)(\mathbf{D}_{V_2} - (1-s)\mathbf{I}_m))) \\
&= (1-(1-s)(1-u)t^2) \\
&\quad \times \det\left(\begin{bmatrix} 1+at^2 & 0 & 0 & -t & -t & -t \\ 0 & 1+bt^2 & 0 & -t & 0 & -t \\ 0 & 0 & 1+bt^2 & 0 & -t & -t \\ -t & -t & 0 & 1+ct^2 & 0 & 0 \\ -t & 0 & -t & 0 & 1+ct^2 & 0 \\ -t & -t & -t & 0 & 0 & 1+dt^2 \end{bmatrix} \right),
\end{aligned}$$

where $a = (1-s)(2+u)$, $b = (1-s)(1+u)$, $c = (1-u)(1+s)$ and $d = (1-s)(2+u)$. Thus, we obtain

$$\begin{aligned}
\zeta(G, u, s, t)^{-1} &= (1-(1-s)(1-u)t^2)(1+(1-2us)t^2 + (1-u^2)(1-s^2)t^4) \\
&\quad \times \{1-(s+u+4us)t^2 + (-4-u-3u^2 + (-1+u+3u^2)s \\
&\quad + (-3+3u+6u^2)s^2)t^4 - (1-u)(1-s)(1+2u+u^2
\end{aligned}$$

$$+ (2 + 12u + 7u^2)s + (1 + 7u + 4u^2)s^2)t^6$$

$$+ (1 - s)^2(1 - u)^2(1 + u)(2 + u)(1 + s)(2 + s)t^8\}.$$

Now, let H be the hypergraph with $V(H) = \{v_1, v_2, v_3\}$ and $E(H) = \{e_1, e_2, e_3\}$, where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_1, v_3\}$ and $e_3 = \{v_1, v_2, v_3\}$. Then the above bipartite graph G is the bipartite graph B_H associated with H, where $V_1 = V(H)$ and $V_2 = E(H)$. By Theorem 7, we have

$$\begin{aligned}\zeta(H, u, s, t)^{-1} &= \zeta(G, u, s, \sqrt{t})^{-1} \\ &= (1 - (1 - s)(1 - u)t)(1 + (1 - 2us)t + (1 - u^2)(1 - s^2)t^2) \\ &\quad \times \{1 - (s + u + 4us)t + (-4 - u - 3u^2 + (-1 + u + 3u^2)s \\ &\quad + (-3 + 3u + 6u^2)s^2)t^2 - (1 - u)(1 - s)(1 + 2u + u^2 \\ &\quad + (2 + 12u + 7u^2)s + (1 + 7u + 4u^2)s^2)t^3 \\ &\quad + (1 - s)^2(1 - u)^2(1 + u)(2 + u)(1 + s)(2 + s)t^4\}.\end{aligned}$$

If $u = 0$, then

$$\begin{aligned}\zeta(H, 0, s, t)^{-1} &= (1 - (1 - s)t)(1 + t + (1 - s^2)t^2) \\ &\quad \times (1 - st + (-4 - s - 3s^2)t^2 - (1 - s)(1 + s)^2t^3 \\ &\quad + 2(1 - s)^2(1 + s)(2 + s)t^4).\end{aligned}$$

In the case of $s = 0$, we have

$$\begin{aligned}\zeta(H, u, 0, t)^{-1} &= (1 - (1 - u)t)(1 + t + (1 - u^2)t^2) \\ &\quad \times (1 - ut + (-4 - u - 3u^2)t^2 - (1 - u)(1 + u)^2t^3 \\ &\quad + 2(1 - u)^2(1 + u)(2 + u)t^4).\end{aligned}$$

Furthermore, let $s = u.$ Then

$$\zeta(H, u, u, t)^{-1} = \zeta(H, u, t)^{-1} = (1-(1-u)^2 t)(1+(1-2u^2)t+(1-u^2)^2 t^2)$$

$$\times (1-2u(1+2u)t + (-4-2u-5u^2-6u^3+6u^4)t^2$$

$$-(1-u)^2(1+4u+14u^2+14u^3+4u^4)t^3$$

$$+(1-u)^4(1+u)^2(2+u)^2 t^4).$$

If $s = u = 0,$ then we have

$$\zeta(H, 0, 0, t)^{-1} = \zeta(H, t)^{-1} = (1-t)(1+t+t^2)(1-4t^2-t^3+4t^4).$$

Next, let $f_1 = (v_1, e_1),$ $f_2 = (v_1, e_2),$ $f_3 = (v_1, e_3),$ $f_4 = (v_2, e_1),$ $f_5 = (v_2, e_3),$ $f_6 = (v_3, e_2)$ and $f_7 = (v_3, e_3).$ Then we have

$$D(B_H) = \{f_1, \ldots, f_7, f_1^{-1}, \ldots, f_7^{-1}\}.$$

Three matrices **X**, **F** and **G** are given as follows:

$$\mathbf{X} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{G} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then it is certain that $\mathbf{FG} = \mathbf{X}$.

Furthermore,

$$\mathbf{X} + u\mathbf{F} + s\mathbf{G} + us\mathbf{I}_7 = \begin{bmatrix} us & s & s & u & 1 & 0 & 0 \\ s & us & s & 0 & 0 & u & 1 \\ s & s & us & 1 & u & 1 & u \\ u & 1 & 1 & us & s & 0 & 0 \\ 1 & 1 & u & s & us & 1 & u \\ 1 & u & 1 & 0 & 0 & us & s \\ 1 & 1 & u & 1 & u & s & us \end{bmatrix},$$

and so, we have

$$\det(\mathbf{I}_7 - t(\mathbf{X} + u\mathbf{F} + s\mathbf{G} + us\mathbf{I}_7)) = \zeta(H, u, s, t)^{-1}.$$

Finally, we consider arcs of $B_H^{(1)}$. Let

$R_1 = (v_1, e_1, v_2)$, $R_2 = (v_1, e_2, v_3)$, $R_3 = (v_1, e_3, v_2)$,

$R_4 = (v_1, e_3, v_3)$, $R_5 = R_1^{-1}$, $R_6 = R_3^{-1}$, $R_7 = (v_2, e_3\ v_3)$,

$R_8 = R_2^{-1}$, $R_9 = R_4^{-1}$, $R_{10} = R_7^{-1}$,

and $P_i = (f_i, f_i^{-1})(1 \le i \le 7)$. Arrange elements of $\mathcal{R}' = D(B_H^{(1)})$ are as follows: $P_1, \ldots, P_7$, $R_1, \ldots, R_{10}$. We consider the matrix $\mathbf{T}''$ under this order, and then, we have

$$\mathbf{T}'' = \left[\begin{array}{ccccccccc}
us & s & s & 0 & 0 & 0 & 0 & us & s \\
s & us & s & 0 & 0 & 0 & 0 & s & us \\
s & s & us & 0 & 0 & 0 & 0 & s & s \\
0 & 0 & 0 & us & s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s & us & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & us & s & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s & us & 0 & 0 \\
0 & 0 & 0 & u & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & u & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & u & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & u & 0 & 0 \\
u & 1 & 1 & 0 & 0 & 0 & 0 & u & 1 \\
1 & 1 & u & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & u & 0 & 0 \\
1 & u & 1 & 0 & 0 & 0 & 0 & 1 & u \\
1 & 1 & u & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & u & 0 & 0 & 0 & 0
\end{array}\right.$$

$$\left.\begin{matrix}
s & s & 0 & 0 & 0 & 0 & 0 & 0 \\
s & s & 0 & 0 & 0 & 0 & 0 & 0 \\
us & us & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & us & s & s & 0 & 0 & 0 \\
0 & 0 & s & us & us & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & us & s & s \\
0 & 0 & 0 & 0 & 0 & s & us & us \\
0 & 0 & u & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & u & 1 & 1 \\
0 & 0 & 1 & u & u & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & u & u \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
u & u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & u & u \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
u & u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & u & u & 0 & 0 & 0
\end{matrix}\right].$$

By Theorem 9, we have

$$\det(\mathbf{I}_{17} - t\mathbf{T}'') = \zeta(H, u, s, t)^{-1}.$$

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