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$\mathscr{G}_w$-CLOSED SETS IN WEAK STRUCTURE SPACES

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Abstract

In this paper, we introduce the notion of $\mathscr{G}_w$-closed sets in weak structure spaces. Also, we give some characterizations and applications of $\mathscr{G}_w$-closed sets. Finally, some characterizations of $\mathscr{R}_w$-regular and $\mathscr{N}_w$-normal spaces have been given.

1. Introduction

Császár [3] introduced a generalized structure called *generalized topology*. Also, Császár [2, 4] introduced and studied generalized operators. After then, Császár [1] introduced a new notion of structures called *weak structure*. Every generalized topology [3] and every minimal structure is a weak structure. In [1], Császár defined some structures and operators under

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more general conditions. Levine [6] introduced the concept of generalized closed sets. This notion has been studied extensively, in recent years, by many topologies. A subset A of a topological space (X, τ) is said to be *generalized closed* (briefly, *g-closed*) if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). In this paper, the notion of $\mathscr{G}_w$-closed sets in weak structure spaces and their properties are discussed. Also, we give some applications of $\mathscr{G}_w$-closed sets.

2. Preliminaries

Let X be a nonempty set and $w \subseteq \mathscr{P}$, where $\mathscr{P}$ is the power set of X. Then w is called a *weak structure* [1] on X if $\varnothing \in w$. A nonempty set X with a weak structure w, is denoted by the pair (X, w) and is called a *weak structure space* (X, w). The elements of w are called *w-open sets* and the complements of w-open sets are called *w-closed sets*. For a weak structure w on X, the intersection of all w-closed sets containing a subset A of X is denoted by $c_w(A)$ and the union of all w-open sets contained in A is denoted by $i_w(A)$. The following lemmas will be useful in the sequel.

Lemma 2.1 [1]**.** *Let w be a weak structure on X and $A, B \subseteq X$. Then $i_w i_w(A) = i_w(A)$, $c_w c_w(A) = c_w(A)$, $c_w(X - A) = X - i_w(A)$, $i_w(X - A) = X - c_w(A)$ and $A \subseteq B$ implies $i_w(A) \subseteq i_w(B)$ and $c_w(A) \subseteq c_w(B)$.*

Lemma 2.2 [1]**.** *Let w be a weak structure on X and $A \subseteq X$. Then $x \in i_w(A)$ if and only if there exists a w-open set U such that $x \in U \subseteq A$.*

Lemma 2.3 [1]**.** *Let w be a weak structure on X and $A \subseteq X$. Then $x \in c_w(A)$ if and only if $U \cap A \neq \varnothing$ whenever $x \in U \in w$.*

Lemma 2.4 [1]**.** *Let w be a weak structure on X and $A \subseteq X$. If $A \in w$, then $A = i_w(A)$ and if A is w-closed, then $A = c_w(A)$.*

3. $\mathscr{G}_w$-closed Sets

In this section, we introduce the notion of $\mathscr{G}_w$-closed sets in weak structure spaces and give characterizations of $\mathscr{G}_w$-closed sets.

Definition 3.1. Let (X, w) be a weak structure space. A subset A of X is said to be $\mathscr{G}_w$-*closed* if $c_w(A) \subseteq U$ whenever $A \subseteq U$ and $U \in w$. The complement of a $\mathscr{G}_w$-closed set is said to be a $\mathscr{G}_w$-*open set*.

Remark 3.2. For a weak structure space (X, w), every w-closed set is a $\mathscr{G}_w$-closed set. The converse is not true is shown by the following example.

Example 3.3. Let $X = \{a, b, c\}$ and $w = \{\varnothing, \{a\}\}$ be a weak structure on X. It is easy to check that $A = \{a, c\}$ is $\mathscr{G}_w$-closed but not w-closed.

The next two examples show that the union and the intersection of two $\mathscr{G}_w$-closed sets are not, in general, $\mathscr{G}_w$-closed.

Example 3.4. Let $X = \{a, b, c\}$ and $w = \{\varnothing, \{a, b\}, \{a, c\}, \{b, c\}\}$ be a weak structure on X. Then it can be easily checked that $A = \{a\}$ and $B = \{c\}$ are two $\mathscr{G}_w$-closed sets and $A \bigcup B = \{b, c\}$ is not a $\mathscr{G}_w$-closed set.

Example 3.5. Let $X = \{a, b, c\}$ and $w = \{\varnothing, \{a\}, \{c\}\}$ be a weak structure on X. Then it can be easily checked that $A = \{a, c\}$ and $B = \{a, b\}$ are two $\mathscr{G}_w$-closed sets and $A \bigcap B = \{a\}$ is not a $\mathscr{G}_w$-closed set.

Proposition 3.6. *Let (X, w) be a weak structure on X. If A is $\mathscr{G}_w$-closed, then $c_w(A) - A$ does not contain any nonempty w-closed set.*

Proof. Let F be a w-closed subset of X such that $F \subseteq c_w(A) - A$, where A is $\mathscr{G}_w$-closed. Since $X - F$ is w-open, $A \subseteq X - F$ and A is $\mathscr{G}_w$-closed, $c_w(A) \subseteq X - F$ and thus $F \subseteq X - c_w(A)$. Thus, $F \subseteq [X - c_w(A)] \bigcap c_w(A)$ $= \varnothing$ and hence $F = \varnothing$. $\square$

Corollary 3.7. *Let* (X, w) *be a weak structure space and A be a* $\mathscr{G}_w$-*closed subset of X. Then* $c_w(A) = A$ *if and only if* $c_w(A) - A$ *is w-closed.*

Proof. Let A be a w-closed set. If $c_w(A) = A$, then $c_w(A) - A = \varnothing$, and $c_w(A) - A$ is a w-closed set.

Conversely, let $c_w(A) - A$ be a w-closed set, where A is $\mathscr{G}_w$-closed. By Proposition 3.6, $c_w(A) - A$ does not contain any nonempty set. Since $c_w(A) - A$ is a w-closed subset of itself, $c_w(A) - A = \varnothing$ and hence $c_w(A) = A$. □

Theorem 3.8. *Let* (X, w) *be a weak structure space and* $A \subseteq X$. *Then A is* $\mathscr{G}_w$-*closed if and only if* $c_w(\{x\}) \cap A \neq \varnothing$ *for every* $x \in c_w(A)$.

Proof. Let A be a $\mathscr{G}_w$-closed set and suppose that there exists $x \in c_w(A)$ such that $c_w(\{x\}) \cap A = \varnothing$. Therefore, $A \subseteq X - c_w(\{x\})$, and so $c_w(A) \subseteq X - c_w(\{x\})$. Hence $x \notin c_w(A)$, which is a contradiction.

Conversely, suppose that the condition of the theorem holds and let U be any w-open set containing A. Let $x \in c_w(A)$. Then, by hypothesis $c_w(A) \cap A \neq \varnothing$, so there exists $y \in c_w(\{x\}) \cap A$ and so $y \in A \subseteq U$. Thus, $\{x\} \cap U \neq \varnothing$. Hence $x \in U$, which implies that $c_w(A) \subseteq U$. This shows that A is $\mathscr{G}_w$-closed. □

Proposition 3.9. *Let* (X, w) *be a weak structure space and* $A, B \subseteq X$. *If A is* $\mathscr{G}_w$-*closed such that* $A \subseteq B \subseteq c_w(A)$, *then B is* $\mathscr{G}_w$-*closed.*

Proof. Let $B \subseteq U \in w$. Since A is $\mathscr{G}_w$-closed and $A \subseteq U$, $c_w(A) \subseteq U$. Now, $B \subseteq c_w(A)$, $c_w(B) \subseteq c_w(A)$ and hence $c_w(B) \subseteq U$. □

Theorem 3.10. *Let* (X, w) *be a weak structure space and* $A \subseteq X$. *Then A is* $\mathscr{G}_w$-*open if and only if* $F \subseteq i_w(A)$ *whenever* $F \subseteq A$ *and F is w-closed.*

Proof. Let A be a $\mathscr{G}_w$-open set and $F \subseteq A$, where F is w-closed. Then $X - A$ is a $\mathscr{G}_w$-closed set contained in a w-open set $X - F$. Hence $c_w(X - A) \subseteq X - F$, i.e., $X - i_w(A) \subseteq X - F$. So $F \subseteq i_w(A)$.

Conversely, suppose that $F \subseteq i_w(A)$ for any w-closed set F whenever $F \subseteq A$. Let $X - A \subseteq U$, where $U \in w$. Then $X - U \subseteq A$ and $X - U$ is w-closed. By assumption, $X - U \subseteq i_w(A)$ and hence $c_w(X - A) = X - i_w(A) \subseteq U$. Therefore, $X - A$ is $\mathscr{G}_w$-closed and hence A is $\mathscr{G}_w$-open. □

Theorem 3.11. *Let (X, w) be a weak structure space. Then the following properties are equivalent*:

(1) *for every w-open set U of X,* $c_w(U) \subseteq U$;

(2) *every subset of X is* $\mathscr{G}_w$*-closed.*

Proof. (1) $\Rightarrow$ (2) Let A be any subset of X and $A \subseteq U \in w$. By (1), $c_w(U) \subseteq U$ and hence $c_w(A) \subseteq c_w(U) \subseteq U$. Hence, A is $\mathscr{G}_w$-closed.

(2) $\Rightarrow$ (1) Let $U \in w$. By (2), U is $\mathscr{G}_w$-closed and hence $c_w(U) \subseteq U$. □

Proposition 3.12. *Let (X, w) be a weak structure space and A be a subset of X. If A is w-open and* $\mathscr{G}_w$*-closed, then A is w-closed.*

Proof. This is obvious. □

Proposition 3.13. *Let (X, w) be a weak structure space and A be a subset of X. If A is* $\mathscr{G}_w$*-open, then* $U = X$ *whenever U is w-open and* $i_w(A) \cup (X - A) \subseteq U$.

Proof. Let A be a $\mathscr{G}_w$-open set and $U \in w$ such that $i_w(A) \cup (X - A) \subseteq U$. Then $X - U \subseteq [X - i_w(A)] \cap A$, i.e., $X - U \subseteq c_w(X - A) - (X - A)$. Since $X - A$ is $\mathscr{G}_w$-closed, by Proposition 3.6, $X - U = \varnothing$ and hence $X = U$. □

Proposition 3.14. *Let* (X, w) *be a weak structure space and A be a subset of X. If A is* $\mathscr{G}_w$*-open and* $i_w(A) \subseteq B \subseteq A$, *then B is* $\mathscr{G}_w$*-open.*

Proof. We have $X - A \subseteq X - B \subseteq X - i_w(A) = c_w(X - A)$. Since $X - A$ is $\mathscr{G}_w$-closed, it follows from Proposition 3.9 that $X - B$ is $\mathscr{G}_w$-closed and hence B is $\mathscr{G}_w$-open. □

4. Some Separation Axioms in Weak Structure Spaces

Definition 4.1. A weak structure w on a nonempty set X is said to have *property* $\mathscr{H}$ if $X \in w$ and the union of elements of w belongs to w.

Lemma 4.2. *Let X be a nonempty set and w be a weak structure on X satisfying property* $\mathscr{H}$. *For a subset A of X, the following properties hold*:

(1) $A \in w$ *if and only if* $i_w(A) = A$;

(2) *A is w-closed if and only if* $c_w(A) = A$.

Recall that a topological space (X, τ) is called a $T_{\frac{1}{2}}$-*space* [6] if for every g-closed set is closed or equivalently [5] if every singleton is open or closed. We introduce the following new definition:

Definition 4.3. A weak structure space (X, w) is called a $T^w_{\frac{1}{2}}$-*space* if for every $\mathscr{G}_w$-closed set is w-closed.

Theorem 4.4. *Let* (X, w) *be a weak structure space and w have property* $\mathscr{H}$. *Then the following properties are equivalent*:

(1) *X is a* $T^w_{\frac{1}{2}}$*-space*;

(2) *every singleton is w-closed or w-open.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\{x\}$ is not w-closed subset for some $x \in X$. Then $X - \{x\}$ is not w-open and hence X is the only w-open set containing $X - \{x\}$. Hence, $X - \{x\}$ is $\mathscr{G}_w$-closed. Since X is a $T^w_{\frac{1}{2}}$-space, $X - \{x\}$ is w-closed and hence $\{x\}$ is w-open.

(2) $\Rightarrow$ (1) Let A be a $\mathscr{G}_w$-closed subset of X and $x \in c_w(A)$. Suppose that $x \notin A$. (i) In case the singleton $\{x\}$ is w-closed, $A \subseteq X - \{x\}$. Since A is a $\mathscr{G}_w$-closed set and $X - \{x\}$ is a w-open, $c_w(A) \subseteq X - \{x\}$ and hence $\{x\} \subseteq X - c_w(A)$. Therefore, $\{x\} \subseteq c_w(A) \bigcap [X - c_w(A)] = \emptyset$. This is a contradiction. (ii) In case the singleton $\{x\}$ is w-open, since $x \in c_w(A)$, $\{x\} \bigcap A \neq \emptyset$ and $x \in A$. This is a contradiction. Therefore, $x \in A$ and hence $c_w(A) \subseteq A$. This shows that X is $T^w_{\frac{1}{2}}$-space. $\square$

Definition 4.5. A weak structure space (X, w) is said to be $\mathscr{R}_w$*-regular* if for each w-closed set F of X and each $x \notin F$, there exist disjoint w-open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 4.6. *Let (X, w) be a weak structure space and w have property $\mathscr{H}$. Then the following properties are equivalent*:

(1) *X is $\mathscr{R}_w$-regular*;

(2) *for each $x \in X$ and each $U \in w$ with $x \in U$, there exists $V \in w$ such that $x \in V \subseteq c_w(V) \subseteq U$*;

(3) *for each w-closed set F of X, $\bigcap\{c_w(V) : F \subseteq V \in w\} = F$*;

(4) *for each $A \subseteq X$ and each $U \in w$ with $A \bigcap U \neq \emptyset$, there exists $V \in w$ such that $A \bigcap V \neq \emptyset$ and $c_w(V) \subseteq U$*;

(5) *for each nonempty subset A of X and each w-closed subset F of X with* $A \bigcap F = \varnothing$, *there exist* $V, W \in w$ *such that* $A \bigcap V \neq \varnothing$, $F \subseteq W$ *and* $W \bigcap V = \varnothing$;

(6) *for each w-closed set F and* $x \notin F$, *there exist* $U \in w$ *and a* $\mathscr{G}_w$*-open set V such that* $x \in U$, $F \subseteq V$ *and* $U \bigcap V = \varnothing$;

(7) *for each* $A \subseteq X$ *and each w-closed set F with* $A \bigcap F = \varnothing$, *there exist* $U \in w$ *and a* $\mathscr{G}_w$*-open set V such that* $A \bigcap U \neq \varnothing$, $F \subseteq V$ *and* $U \bigcap V = \varnothing$.

Proof. (1) $\Rightarrow$ (2) Let $x \notin X - U$, where $U \in w$. Then there exist disjoint $G, V \in w$ such that $X - U \subseteq G$ and $x \in V$. Thus, $V \subseteq X - G$ and so $x \in V \subseteq c_w(V) \subseteq X - G \subseteq U$.

(2) $\Rightarrow$ (3) Let $X - F \in w$ with $x \in X - F$. Then, by (2), there exists $U \in w$ such that $x \in U \subseteq c_w(U) \subseteq X - F$. So $F \subseteq X - c_w(U) = V \in w$ and $U \bigcap V = \varnothing$. Then $x \in c_w(V)$. Thus, $F \supseteq \bigcap \{c_w(V) : F \subseteq V \in w\}$.

(3) $\Rightarrow$ (4) Let *A* be a subset of *X* such that $U \in w$ with $A \bigcap U \neq \varnothing$. Let $x \in A \bigcap U$. Then $x \notin X - U$. Hence, by (3), there exists $W \in w$ such that $X - U \subseteq W$ and $x \notin c_w(W)$. Put $V = X - c_w(W)$ which is a *w*-open set containing *x* and hence $A \bigcap V \neq \varnothing$. Now $V \subseteq X - W$ and so $c_w(V) \subseteq X - W \subseteq U$.

(4) $\Rightarrow$ (5) Let *A* be a nonempty subset of *X* and *F* be a *w*-closed subset *X* with $A \bigcap F = \varnothing$. Then $X - F \in w$ such that $A \bigcap (X - F) \neq \varnothing$ and hence by (4), there exists $V \in w$ such that $A \bigcap V \neq \varnothing$ and $c_w(V) \subseteq X - F$. If we put $W = X - c_w(V)$, then $F \subseteq W$ and $V \bigcap W = \varnothing$.

(5) $\Rightarrow$ (1) Let *F* be a *w*-closed set not containing *x*. Then $F \bigcap \{x\} = \varnothing$. Thus, by (5), there exist $V, W \in w$ such that $x \in V$, $F \subseteq W$ and $V \bigcap W = \varnothing$.

(1) $\Rightarrow$ (6) This is obvious.

(6) $\Rightarrow$ (7) Let A be a subset of X and F be a w-closed set with $A \cap F = \emptyset$. Then, for $x \in A$, $x \notin F$, and hence by (6), there exist $U \in w$ and a $\mathcal{G}_w$-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. So $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

(7) $\Rightarrow$ (1) Let $x \notin F$, where F is w-closed in X. Since $\{x\} \cap F = \emptyset$, by (7), there exist $U \in w$ and a $\mathcal{G}_w$-open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \emptyset$. Then $F \subseteq i_w(W) = V \in w$ and hence $U \cap V = \emptyset$. This shows that X is $\mathcal{N}_w$-normal. □

Definition 4.7. A weak structure space (X, w) is said to be $\mathcal{N}_w$*-normal* if for any two disjoint w-closed sets A and B, there exist two disjoint w-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 4.8. *Let* (X, w) *be a weak structure space and w have property* $\mathcal{H}$. *Then the following properties are equivalent*:

(1) *X is* $\mathcal{N}_w$*-normal*;

(2) *for any pair of disjoint w-closed sets A and B of X, there exist disjoint* $\mathcal{G}_w$*-open sets U and V of X such that* $A \subseteq U$ *and* $B \subseteq V$;

(3) *for each w-closed set A and each w-open set B containing A, there exists a* $\mathcal{G}_w$*-open set U such that* $A \subseteq U \subseteq c_w(U) \subseteq B$;

(4) *for each w-closed set A and each* $\mathcal{G}_w$*-open set B containing A, there exists a w-open set U such that* $A \subseteq U \subseteq c_w(U) \subseteq i_w(B)$;

(5) *for each w-closed set A and each* $\mathcal{G}_w$*-open set B containing A, there exists a* $\mathcal{G}_w$*-open set U such that* $A \subseteq U \subseteq c_w(U) \subseteq i_w(B)$;

(6) *for each* $\mathcal{G}_w$*-closed set A and each w-open set B containing A, there exists a w-open set U such that* $c_w(A) \subseteq U \subseteq c_w(U) \subseteq i_w(B)$;

(7) *for each* $\mathscr{G}_w$*-closed set A and each w-open set B containing A, there exists a* $\mathscr{G}_w$*-open set U such that* $c_w(A) \subseteq U \subseteq c_w(U) \subseteq i_w(B)$.

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) Let A be a w-closed set and B be a w-open set containing A. Then A and $X - B$ are two disjoint w-closed sets. Hence, by (2), there exist disjoint $\mathscr{G}_w$-open sets U and V of X such that $A \subseteq U$ and $X - B \subseteq V$. Since V is $\mathscr{G}_w$-open and $X - B$ is a w-closed set, by Theorem 3.10, $X - B \subseteq i_w(V)$. Therefore, $c_w(X - V) = X - i_w(V) \subseteq B$ and hence $A \subseteq U \subseteq c_w(U) \subseteq c_w(X - V) \subseteq B$.

(3) $\Rightarrow$ (1) Let A and B be two disjoint w-closed subsets of X. Then A is a w-closed set and $X - B$ is a w-open set containing A. Thus, by (3), there exists a $\mathscr{G}_w$-open set U such that $A \subseteq U \subseteq c_w(U) \subseteq X - B$. Thus, by Theorem 3.10, $A \subseteq i_w(U)$, $B \subseteq X - c_w(U)$, where $i_w(U)$ and $X - c_w(U)$ are two disjoint w-open sets.

(4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (2) are obvious.

(6) $\Rightarrow$ (7) and (7) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (5) Let A be a w-closed set and B be a $\mathscr{G}_w$-open set containing A. Since A is w-closed and B is $\mathscr{G}_w$-open, by Theorem 3.10, $A \subseteq i_w(B)$. Thus, by (3), there exists a $\mathscr{G}_w$-open set U such that $A \subseteq U \subseteq c_w(U) \subseteq i_w(B)$.

(5) $\Rightarrow$ (7) Let A be a $\mathscr{G}_w$-closed subset of X and B be a w-open set containing A. Then $c_w(A) \subseteq B$, where B is $\mathscr{G}_w$-open. Thus, there exists a $\mathscr{G}_w$-open set U such that $c_w(A) \subseteq G \subseteq c_w(G) \subseteq B$. Since G is $\mathscr{G}_w$-open and $c_w(A) \subseteq G$, by Theorem 3.10, $c_w(A) \subseteq i_w(G)$. Put $U = i_w(G)$. Then U is w-open and $c_w(A) \subseteq U \subseteq c_w(U) = c_w[i_w(G)] \subseteq c_w(G) \subseteq B$.

(6) $\Rightarrow$ (4) Let A be a w-closed set and B be a $\mathcal{G}_w$-open set containing A. Then, by Theorem 3.10, $c_w(A) = A \subseteq i_w(B)$ and $i_w(B)$ is w-open. Thus, by (6), there exists a w-open set U such that $c_w(A) = A \subseteq U \subseteq c_w(U) \subseteq i_w(B)$.

$\square$

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