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CONSISTENT ESTIMATION OF THE MEAN FUNCTION OF A COMPOUND CYCLIC POISSON PROCESS

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Abstract

An estimator of the mean function of a compound cyclic Poisson process is constructed and investigated. We do not assume any particular parametric form for the intensity function except that it is periodic. Moreover, we consider the case when there is only a single realization of the Poisson process is observed in a bounded interval. The proposed estimator is proved to be consistent when the size of the interval indefinitely expands.

1. Introduction

Let $\{N(t), t \geq 0\}$ be a Poisson process with (unknown) locally

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integrable intensity function λ. We consider the case when the intensity function λ is a periodic function with (known) period $\tau > 0$. We do not assume any (parametric) form of λ except that it is periodic, that is, the equality

$$\lambda(s + k\tau) = \lambda(s) \tag{1.1}$$

holds for all $s \geq 0$ and $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers. This condition of intensity function is also considered in [5].

Let $\{Y(t),\ t \geq 0\}$ be a process with

$$Y(t) = \sum_{i=1}^{N(t)} X_i, \tag{1.2}$$

where $\{X_i,\ i \geq 1\}$ is a sequence of independent and identically distributed random variables with mean μ and variance σ^2, which is also independent of the process $\{N(t),\ t \geq 0\}$. The process $\{Y(t),\ t \geq 0\}$ is said to be a *compound cyclic Poisson process*. The model presented in (1.2) is a generalization of the (well known) compound Poisson process, which assume that $\{N(t),\ t \geq 0\}$ is a homogeneous Poisson process.

There are many applications of the compound Poisson model. Some examples are as follows. Applications of the compound Poisson model in insurance and financial problems can be found in [1], while its applications in physics can be seen in [2]. We refer to [6], [9] and [10] for some applications of the compound Poisson model in other areas.

Suppose that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of the cyclic Poisson process $\{N(t),\ t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Furthermore, suppose that for each data point in the observed realization $N(\omega) \cap [0, n]$, say ith data point, $i = 1, 2, \ldots, N([0, n])$, its corresponding random variable X_i is also observed. Our goals in this paper are to construct an estimator for the mean function of the process $\{Y(t),\ t \geq 0\}$ using the observed realization and to prove its consistency.

The mean function (expected value) of $Y(t)$, denoted by $\psi(t)$, is given by

$$\psi(t) = E[N(t)]E[X_1] = \Lambda(t)\mu$$

with $\Lambda(t) = \int_0^t \lambda(s)ds$. Let $t_r = t - \left\lfloor \frac{t}{\tau} \right\rfloor \tau$, where for any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, and let also $k_{t,\tau} = \left\lfloor \frac{t}{\tau} \right\rfloor$. Then, for any given real number $t \geq 0$, we can write $t = k_{t,\tau}\tau + t_r$, with $0 \leq t_r < \tau$. Let $\theta = \frac{1}{\tau}\int_0^\tau \lambda(s)ds$, that is, the global intensity of the cyclic Poisson process $\{N(t),\ t \geq 0\}$. We assume that

$$\theta > 0. \tag{1.3}$$

Then, for any given $t \geq 0$, we have

$$\Lambda(t) = k_{t,\tau}\tau\theta + \Lambda(t_r)$$

which implies

$$\psi(t) = (k_{t,\tau}\tau\theta + \Lambda(t_r))\mu.$$

2. The Estimator and Main Results

The estimator of the mean function $\psi(t)$ using the available data set at hand is given by

$$\hat{\psi}_n(t) = (k_{t,\tau}\tau\hat{\theta}_n + \hat{\Lambda}_n(t_r))\hat{\mu}_n, \tag{2.1}$$

where

$$\hat{\theta}_n = \frac{N([0,\ n])}{n},$$

$$\hat{\Lambda}_n(t_r) = \frac{\tau}{n} \sum_{k=0}^{\infty} N([k\tau,\ k\tau + t_r] \cap [0,\ n])$$

and

$$\hat{\mu}_n = \frac{1}{N([0,\ n])} \sum_{i=1}^{N([0, n])} X_i,$$

with the understanding that $\hat{\mu}_n = 0$ when $N([0,\ n]) = 0$. Thus, $\hat{\psi}_n(t) = 0$ when $N([0,\ n]) = 0$. We refer to [4] and [7] for some related work in estimation of θ and $\Lambda(t_r)$ for different purposes.

Our main results are presented in the following two theorems.

Theorem 1 (Weak consistency)**.** *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable. If, in addition,* $Y(t)$ *satisfies condition of* (1.2), *then*

$$\hat{\psi}_n(t) \xrightarrow{P} \psi(t)$$

as $n \to \infty$. *Hence* $\hat{\psi}_n(t)$ *is a weak consistent estimator of* $\psi(t)$.

Theorem 2 (Strong consistency)**.** *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable. If, in addition,* $Y(t)$ *satisfies condition of* (1.2), *then*

$$\hat{\psi}_n(t) \xrightarrow{a.s.} \psi(t)$$

as $n \to \infty$. *Hence* $\hat{\psi}_n(t)$ *is a strong consistent estimator of* $\psi(t)$.

3. Some Technical Lemmas

In this section, we present some results which are needed in the proofs of our theorems. Proofs of Lemmas 2 and 3 can also be seen in [8].

Lemma 1. *Let N be a Poisson random variable with* $E[N] > 0.$ *Then for any* $\varepsilon > 0,$ *we have*

$$\mathbf{P}\left(\frac{|N - E[N]|}{(E[N])^{\frac{1}{2}}} > \varepsilon\right) \leq 2\exp\left\{-\frac{\varepsilon^2}{2 + \varepsilon(E[N])^{-\frac{1}{2}}}\right\}.$$

Proof. We refer to [11, p. 222].

Lemma 2. *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable. Then*

$$E[\hat{\theta}_n] = E\left[\frac{N([0, n])}{n}\right] = \frac{1}{n}\int_0^n \lambda(s)ds = \theta + \mathcal{O}\left(\frac{1}{n}\right)$$

as $n \to \infty.$

Proof. Let $k_{n,\tau} = \left\lfloor \frac{n}{\tau} \right\rfloor.$ Then we can write

$$E[\hat{\theta}_n] = \frac{k_{n,\tau}\tau}{n}\frac{1}{k_{n,\tau}\tau}\int_0^{k_{n,\tau}\tau}\lambda(s)ds + \frac{1}{n}\int_{k_{n,\tau}\tau}^{n}\lambda(s)ds. \tag{3.1}$$

First note that

$$\frac{1}{k_{n,\tau}\tau}\int_0^{k_{n,\tau}\tau}\lambda(s)ds = \theta$$

because λ is periodic with period τ. Since $(n - k_{n,\tau}\tau) < \tau$ for all n, we have that

$$\frac{k_{n,\tau}\tau}{n} = \frac{n - (n - k_{n,\tau}\tau)}{n} = 1 + \mathcal{O}\left(\frac{1}{n}\right)$$

as $n \to \infty.$ Because λ is locally integrable and $n - k_{n,\tau}\tau = \mathcal{O}(1)$ as $n \to \infty,$ we also know that

$$\int_{k_{n,\tau}\tau}^{n}\lambda(s)ds = \mathcal{O}(1) \quad \text{as } n \to \infty.$$

Hence, the first term on the r.h.s. of (3.1) is $\theta + \mathcal{O}(n^{-1})$, while its second term is of order $\mathcal{O}(n^{-1})$ as $n \to \infty$. This completes the proof of Lemma 2.

Throughout this paper, for any random variables X_n and X on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, we write $X_n \xrightarrow{c} X$ to denote that X_n converges completely to X as $n \to \infty$. We say that X_n converges completely to X if $\sum_{n=1}^{\infty} \mathbf{P}(|X_n - X| > \varepsilon) < \infty$, for every $\varepsilon > 0$.

Lemma 3. *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable. Then*

$$\hat{\theta}_n \xrightarrow{P} \theta \tag{3.2}$$

and

$$\hat{\theta}_n \xrightarrow{c} \theta \tag{3.3}$$

as $n \to \infty$.

Proof. Since (3.3) implies (3.2), we only need to prove (3.3). To verify (3.3), we must show that, for each $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}(|\hat{\theta}_n - \theta| > \varepsilon) < \infty. \tag{3.4}$$

Now we see that

$$\mathbf{P}(|\hat{\theta}_n - \theta| > \varepsilon) \leq \mathbf{P}(|\hat{\theta}_n - E[\hat{\theta}_n]| + |E[\hat{\theta}_n] - \theta| > \varepsilon).$$

By Lemma 2, for sufficiently large *n*, we have $|E[\hat{\theta}_n] - \theta| < \frac{\varepsilon}{2}$. Then, for sufficiently large *n*, we have

$$\mathbf{P}(|\hat{\theta}_n - \theta| > \varepsilon) \leq \mathbf{P}\left(|\hat{\theta}_n - E[\hat{\theta}_n]| > \frac{\varepsilon}{2}\right). \tag{3.5}$$

Then, to prove (3.4), it suffices to check that the probability on the r.h.s. of (3.5) is summable. By an application of Lemma 1, the probability on the r.h.s. of (3.5) can be bounded above as follows:

$$\mathbf{P}\left(|\,\hat{\theta}_n - E[\hat{\theta}_n]\,| > \frac{\varepsilon}{2}\right)$$

$$= \mathbf{P}\left(\frac{|\,N([0, n]) - E[N([0, n])]\,|}{n} > \frac{\varepsilon}{2}\right)$$

$$= \mathbf{P}\left(\frac{|\,N([0, n]) - E[N([0, n])]\,|}{(E[N([0, n])])^{\frac{1}{2}}} > \frac{\varepsilon n}{2(E[N([0, n])])^{\frac{1}{2}}}\right)$$

$$\leq 2\exp\left\{-\frac{\varepsilon^2 n^2}{4E[N([0, n])](2 + \varepsilon n 2^{-1}(E[N([0, n])])^{-1})}\right\}$$

$$= 2\exp\left\{-\frac{\varepsilon^2 n}{8E[\hat{\theta}_n] + 2\varepsilon}\right\}. \tag{3.6}$$

By Lemma 2, we have $E[\hat{\theta}_n] = \theta + o(1)$ as $n \to \infty$. Then, for sufficiently large n, the r.h.s. of (3.6) does not exceed $2\exp\{-\varepsilon^2 n(16\theta + 2\varepsilon)^{-1}\}$ which is summable. This completes the proof of Lemma 3.

Lemma 4. *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable. Then*

$$E[\hat{\Lambda}_n(t_r)] = \Lambda(t_r) + \mathcal{O}\left(\frac{1}{n}\right) \tag{3.7}$$

as $n \to \infty$.

Proof. The mean of $\hat{\Lambda}_n(t_r)$ can be computed as follows:

$$E[\hat{\Lambda}_n(t_r)] = \frac{\tau}{n}\sum_{k=0}^{\infty} E[N([k\tau, k\tau + t_r] \cap [0, n])]$$

$$= \frac{\tau}{n} \sum_{k=0}^{\infty} \int_{k\tau}^{k\tau + t_r} \lambda(s) I(s \in [0, n]) ds$$

$$= \frac{\tau}{n} \sum_{k=0}^{\infty} \int_{0}^{t_r} \lambda(s + k\tau) I(s + k\tau \in [0, n]) ds$$

$$= \frac{\tau}{n} \sum_{k=0}^{\infty} \int_{0}^{t_r} \lambda(s) I(s + k\tau \in [0, n]) ds$$

$$= \frac{\tau}{n} \int_{0}^{t_r} \lambda(s) \sum_{k=0}^{\infty} I(s + k\tau \in [0, n]) ds$$

$$= \frac{\tau}{n} \int_{0}^{t_r} \lambda(s) \left(\frac{n}{\tau} + \mathcal{O}(1) \right) ds$$

$$= \int_{0}^{t_r} \lambda(s) ds + \mathcal{O}\left(\frac{1}{n} \right)$$

$$= \Lambda(t_r) + \mathcal{O}\left(\frac{1}{n} \right).$$

Then we have (3.7). This completes the proof of Lemma 4.

Lemma 5. *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable*. *Then*

$$\hat{\Lambda}_n(t_r) \xrightarrow{P} \Lambda(t_r) \tag{3.8}$$

and

$$\hat{\Lambda}_n(t_r) \xrightarrow{c} \Lambda(t_r) \tag{3.9}$$

as $n \to \infty$.

Proof. Since (3.9) implies (3.8), we only need to check (3.9). To prove (3.9), we must verify that, for each $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}(|\hat{\Lambda}_n(t_r) - \Lambda(t_r)| > \varepsilon) < \infty. \tag{3.10}$$

We observe that

$$\mathbf{P}(|\hat{\Lambda}_n(t_r) - \Lambda(t_r)| > \varepsilon)$$

$$\leq \mathbf{P}(|\hat{\Lambda}_n(t_r) - E[\hat{\Lambda}_n(t_r)]| + |E[\hat{\Lambda}_n(t_r)] - \Lambda(t_r)| > \varepsilon).$$

By Lemma 4, for sufficiently large n, we have $|E[\hat{\Lambda}_n(t_r)] - \Lambda(t_r)| < \frac{\varepsilon}{2}$. Then, for sufficiently large n, we have

$$\mathbf{P}(|\hat{\Lambda}_n(t_r) - \Lambda(t_r)| > \varepsilon) \leq \mathbf{P}\left(|\hat{\Lambda}_n(t_r) - E[\hat{\Lambda}_n(t_r)]| > \frac{\varepsilon}{2}\right). \tag{3.11}$$

Then, to show (3.10), it suffices to check that the probability on the r.h.s. of (3.11) is summable. Let $M = \sum_{k=0}^{\infty} N([k\tau, k\tau + t_r] \cap [0, n])$. Applying Lemma 1, the probability on the r.h.s. of (3.11) can be bounded above as follows:

$$\mathbf{P}\left(|\hat{\Lambda}_n(t_r) - E[\hat{\Lambda}_n(t_r)]| > \frac{\varepsilon}{2}\right)$$

$$= \mathbf{P}\left(\frac{|M - E[M]|}{n\tau^{-1}} > \frac{\varepsilon}{2}\right)$$

$$= \mathbf{P}\left(\frac{|M - E[M]|}{(E[M])^{\frac{1}{2}}} > \frac{\varepsilon n\tau^{-1}}{2(E[M])^{\frac{1}{2}}}\right)$$

$$\leq 2\exp\left\{-\frac{\varepsilon^2 n^2 \tau^{-2}}{4E[M](2 + \varepsilon n\tau^{-1}2^{-1}(E[M])^{-1})}\right\}$$

$$= 2\exp\left\{-\frac{\varepsilon^2 n\tau^{-1}}{8E[\hat{\Lambda}_n(t_r)] + 2\varepsilon}\right\}. \tag{3.12}$$

By Lemma 4, we have $E[\hat{\Lambda}_n(t_r)] = \Lambda(t_r) + o(1)$ as $n \to \infty$. Then, for sufficiently large n, the r.h.s. of (3.12) does not exceed $2\exp\{-\varepsilon^2 n\tau^{-1}(16\Lambda(t_r) + 2\varepsilon)^{-1}\}$ which is summable. This completes the proof of Lemma 5.

Lemma 6. *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable*. *If*, *in addition*, (1.3) *holds*, *then with probability* 1,

$$N([0, n]) \to \infty \tag{3.13}$$

as $n \to \infty$.

Proof. First note that, Lemma 2 and condition (1.3) lead to

$$E[N([0, n])] = \int_0^n \lambda(s)ds = \theta n + \mathcal{O}(1) \to \infty$$

as $n \to \infty$. Then, by the Borel-Cantelli Lemma, we have (3.13). This completes the proof of Lemma 6.

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. By (2.1), to prove Theorem 1, it suffices to check

$$\hat{\theta}_n \xrightarrow{P} \theta, \tag{4.1}$$

$$\hat{\Lambda}_n(t_r) \xrightarrow{P} \Lambda(t_r) \tag{4.2}$$

and

$$\hat{\mu}_n \xrightarrow{P} \mu \tag{4.3}$$

as $n \to \infty$. By Lemma 3, we have (4.1), by Lemma 5, we have (4.2), and by Lemma 6 and the weak law of large numbers, we have (4.3). This completes the proof of Theorem 1.

Proof of Theorem 2. By (2.1), to prove Theorem 2, it suffices to check

$$\hat{\theta}_n \xrightarrow{a.s.} \theta, \tag{4.4}$$

$$\hat{\Lambda}_n(t_r) \xrightarrow{a.s.} \Lambda(t_r) \tag{4.5}$$

and

$$\hat{\mu}_n \xrightarrow{a.s.} \mu \tag{4.6}$$

as $n \to \infty$ (cf. [3]). By Lemma 3 and the Borel-Cantelli Lemma, we have (4.4). Similarly, Lemma 5 and the Borel-Cantelli Lemma also lead to (4.5). Finally, by Lemma 6 and the strong law of large numbers, we obtain (4.6). This completes the proof of Theorem 2.

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