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TRANSVERSALLY SYMMETRIC FOLIATION ON THE FOLIATE KÄHLER MANIFOLD

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Abstract

We study the properties of transversally symmetric foliation on the foliate Kähler manifold and prove that if a foliate Kähler manifold is locally symmetric, then $\mathcal{F}$ is transversally symmetric.

1. Introduction

On a foliated Riemannian manifold M, the transversal geometry can be considered as a generalization of ordinary manifolds which carry the trivial foliation by points. In this sense, the study of the transversal geometry of M is very important and has been actively developed by many authors [2, 5, 6, 7]. One interesting problem in the transversal geometry is to study the relations of objects between the quotient space $M/\mathcal{F}$ defined by a foliation $\mathcal{F}$ and the ambient manifold M. In fact, many

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objects on M are inherited to the transversal geometry. For example [2, 3], every Killing (resp. conformal and Jacobi) fields on the ambient manifold are the transversal Killing (resp. transversal conformal and transversal Jacobi) fields. But it is not true for symmetric property. In fact, the author [3] proved that if the normal bundle Q of $\mathcal{F}$ is integrable on the Riemannian symmetric space, then $\mathcal{F}$ is transversally symmetric.

In this paper, we prove that if a foliate Kähler manifold is locally symmetric, then $\mathcal{F}$ is transversally symmetric.

2. Preliminaries

Let (M, g_M, J) be a real $2n$-dimensional Kähler manifold with metric g_M and almost complex structure J. And let L be a real $2p$-dimensional complex analytic distribution on M. Then L is said to define a *complex analytic foliation* $\mathcal{F}$ on M if it is also integrable, i.e., an involutive distribution. In this case, $(M, g_M, J, \mathcal{F})$ is said to be a *foliate Kähler manifold* and the maximal connected integrable manifolds of L are the leaves of the foliation $\mathcal{F}$.

Let $(M, g_M, J, \mathcal{F})$ be a (real) $2n$-dimensional foliate Kähler manifold with a bundle-like metric g_M, an almost complex structure J and an analytic foliation $\mathcal{F}$ defined by the integrable analytic distribution L. Let us consider now the usual real structure of TM. Then the complex distribution L is also a real subbundle of TM, hence it is defined as the image of a real projector $\pi_L : TM \to TM$ with the supplementary projector $\pi_Q = 1 - \pi_L$. It is convenient to take for π_L the orthogonal projection with respect to g_M. This is equivalent with the condition

$$g_M(\pi_L X, \pi_Q Y) = 0$$

for any vector fields X, Y on M. For a distinguished chart $\mathcal{U} \subset M$ the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a Riemannian submersion $f : \mathcal{U} \to \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of model space N and this makes it possible to use the tensors A and T introduced by O'Neill [4]:

$$\begin{cases} A_E F = \pi_Q \nabla^M_{\pi_Q E} \pi_L F + \pi_L \nabla^M_{\pi_Q E} \pi_Q F, \\ T_E F = \pi_Q \nabla^M_{\pi_L E} \pi_L F + \pi_L \nabla^M_{\pi_L} \pi_Q F, \end{cases} \tag{2.1}$$

for any vector fields X, Y on M, where ∇^M denotes the Levi Civita connection on M. Then we have the following properties for A and T [4]:

$$\begin{cases} T_U X = T_U V = 0,\ T_X Y = \pi_Q \nabla^M_X Y,\ T_X U = \pi_L \nabla^M_X U,\ T_X Y = T_Y X, \\ T_X \text{ is alternating; in particular, } g_M(T_X Y, U) = -g_M(T_X U, Y), \end{cases} \tag{2.2}$$

$$\begin{cases} A_X U = A_X Y = 0,\ A_U X = \pi_Q \nabla^M_U X,\ A_U V = \pi_L \nabla^M_U V,\ A_U V = -A_V U, \\ A_U \text{ is alternating; in particular, } g_M(A_U V, X) = -g_M(A_U X, V), \end{cases} \tag{2.3}$$

for any $X, Y \in \Gamma L$ and $U, V \in \Gamma Q$. The Riemannian foliation is said to be *totally geodesic* if the leaves are totally geodesic submanifolds, that is, if $T = 0$. Moreover, the normal bundle $L^{\perp}$ is integrable if $A = 0$ (in this case the integral submanifolds of $L^{\perp}$ are totally geodesic). Also, let $\{E_i,\ i = 1, \ldots, 2p\}$ be a (local) orthonormal basis for L. Then

$$H = \sum_{i=1}^{2p} T_{E_i} E_i \tag{2.4}$$

is called the *mean curvature vector field* of $\mathcal{F}$. $\mathcal{F}$ is said to be a *harmonic foliation* if $H = 0$, that is, if the leaves are minimal submanifolds.

Let g_L be the metric on $\mathcal{F}$ induced by g_M and g_Q the holonomy invariant metric, i.e., $\theta(X) g_Q = 0$ for any $X \in \Gamma L$, where $\theta(X)$ is a Lie derivative with respect to X. Let us define the metric connection ∇^L on L by

$$\nabla^L_X Y = \pi_L \nabla^M_X Y \tag{2.5}$$

for any $Y \in \Gamma L$ and $X \in TM$ and define the canonical connection ∇ on the normal bundle $Q = TM/L$ of $\mathcal{F}$ as

$$\begin{cases} \nabla_X s = \pi_Q([X, Y_s]) & \text{for } X \in \Gamma L, \\ \nabla_X s = \pi_Q(\nabla^M_X Y_s) & \text{for } X \in \Gamma L^\perp, \end{cases} \tag{2.6}$$

where $s \in \Gamma Q$, and $Y_s \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $L^\perp \cong Q$. Then ∇ is metric and torsion free with respect to g_Q. Denote R, R^L and R^∇ by the curvature tensors with respect to ∇^M, ∇^L and ∇, respectively. For the convenience we introduce some notations:

$$R_{XYZW} = g_M(R(X, Y)Z, W),$$

$$\nabla^M_U R_{XYZW} = g_M((\nabla^M_U R)(X, Y)Z, W) \tag{2.7}$$

for any vector fields U, X, Y, Z and W on M. Now we recall some relations between these curvature tensors [1, 6]:

$$R_{XYZZ'} = R^L_{XYZZ'} - g_M(T_X Z, T_Y Z') + g_M(T_Y Z, T_X Z') \tag{2.8}$$

$$R_{UXVY} = g_M((\nabla^M_U T)_X Y, V) - g_M(T_X U, T_Y V) + g_M((\nabla^M_X A)_U V, Y)$$
$$+ g_M(A_U X, A_V Y) \tag{2.9}$$

$$R_{UVWX} = g_M((\nabla^M_W A)_U V, X) + g_M(A_U V, T_X W) - g_M(A_V W, T_X U)$$
$$- g_M(A_W U, T_X V) \tag{2.10}$$

$$R_{UVWW'} = R^\nabla_{UVWW'} - 2g_M(A_U V, A_W W') + g_M(A_V W, A_U W')$$
$$- g_M(A_U W, A_V W') \tag{2.11}$$

for any $X, Y, Z, Z' \in \Gamma L$ and $U, V, W, W' \in \Gamma Q$. From (2.8), (2.9), (2.10) and (2.11), we have the following theorem.

Theorem 2.1 [1, 4]. *For any* $X, Y \in \Gamma L$ *and* $U, V \in \Gamma Q$ *with* $|X| = |U| = 1$, $|X \wedge Y| = 1$ *and* $|U \wedge V| = 1$, *we have*

(1) $K(X, Y) = K^L(X, Y) + |T_X Y|^2 - g_M(T_X X, T_Y Y)$,

(2) $K(X, U) = g_M((\nabla^M_U T)_X X, U) - |T_X U|^2 + |A_U X|^2$,

(3) $K(U, V) = K^{\nabla}(U, V) - 3|A_U V|^2$,

where K, K^L, K^{∇} *are sectional curvatures with respect to* ∇^M, ∇^L, ∇.

3. Transversally Symmetric Foliation on a Kähler Manifold

Let $(M, g_M, J, \mathcal{F})$ be a foliate Kähler manifold with a bundle-like metric g_M, an almost complex structure J and with a complex analytic foliation $\mathcal{F}$. Since the complex structure of L is induced by the complex structure J of M, it is clear that the leaves of L are complex analytic submanifolds of M. In other words, the leaves are Kähler manifolds with respect to the induced structure J. Hence we have

$$\pi_L J = J\pi_L \text{ and } \pi_Q J = J\pi_Q. \tag{3.1}$$

This implies that

$$T_X JY = JT_X Y \text{ and } A_U JV = JA_U V \tag{3.2}$$

for any vector fields X, Y, U and V on M. Also we have the following proposition (see [8]).

Proposition 3.1 [8]. $(M, g_M, J, \mathcal{F})$ *is a foliate Kähler manifold if and only if*

$$J^2 = -1, \quad g_M(JX, JY) = g_M(X, Y), \quad \nabla_X^M J = 0,$$

$$\pi_L^2 = \pi_L, \quad \pi_L J = J\pi_L, \quad g_M(\pi_L X, \pi_Q Y) = 0,$$

$$\pi_Q([JX, \pi Y] - J[X, \pi_L Y]) = 0, \quad \pi_Q[\pi_L X, \pi_L Y] = 0$$

for any vector fields X, Y on M.

Here, the first relations express that g_M is a Kähler metric and the last condition expresses the integrability of L.

Proposition 3.2. *Let* $(M, g_M, J, \mathcal{F})$ *be a foliate Kähler manifold with bundle-like metric* g_M. *Then every leaves is always minimal.*

Proof. Let $\{E_i, JE_i\}_{i=1, \dots, p}$ be an orthonormal basis for L. Then we have from (3.2),

$$H = \sum_{i=1}^{p} \{T_{E_i}E_i + T_{JE_i}JE_i\} = \sum_{i=1}^{p} \{T_{E_i}E_i - T_{E_i}E_i\} = 0.$$

Proposition 3.3. *Let $(M, g_M, J, \mathcal{F})$ be a foliate Kähler manifold of constant holomorphic sectional curvature c. Then $\mathcal{F}$ is totally geodesic if and only if every leaves of $\mathcal{F}$ has constant holomorphic sectional curvature c.*

Proof. From Theorem 2.1 and (3.2), we have

$$K(X, JX) = K^L(X, JX) + 2| T_X X|^2$$

for any $X \in \Gamma L$. This implies that our result holds.

From (3.1), we know that for any $U \in \Gamma Q$, $\pi_L JU = J\pi_L U = 0$. This implies that the complex structure J maps Q onto Q. Moreover, the holonomy invariant metric g_Q satisfies that $g_Q(JU, JV) = g_Q(U, V)$. Trivially, $\nabla J = 0$. Hence we know that $\mathcal{F}$ is Kähler foliation on M (cf. [5]).

A Riemannian foliation $\mathcal{F}$ is *transversally symmetric* if its transversal geometry is locally modeled on a locally symmetric space. Then we have the following theorem.

Theorem 3.4 [7]. *The Kähler foliation $\mathcal{F}$ on M is transversally symmetric if and only if*

$$\nabla_U R^{\nabla}_{UJUUJU} = 0$$

for any $U \in \Gamma Q$.

It is well-known that a Kähler manifold is locally symmetric if and only if $\nabla^M_X R_{XJXXJX} = 0$ for any vector field X on M.

Theorem 3.5. *Let $(M, g_M, J, \mathcal{F})$ be a locally symmetric foliate Kähler manifold. Then $\mathcal{F}$ is a transversally symmetric.*

Proof. From (2.3) and (2.11), we have that for any $U, V \in \Gamma Q$,

$$R_{UVUV} = R^{\nabla}_{UVUV} - 3g_M(A_UV, A_UV). \tag{3.3}$$

We assume that $U \in \Gamma Q$ satisfies $\nabla^M_U U = 0$ locally. Hence we have

$$\nabla^M_U R_{UVUV} = UR_{UVUV} - 2R_{UVU\nabla^M_U V}. \tag{3.4}$$

By similar argument, we have

$$\nabla_U R^{\nabla}_{UVUV} = UR^{\nabla}_{UVUV} - 2R^{\nabla}_{UVU\nabla_U V}. \tag{3.5}$$

From (3.4) and (3.5), we have

$$\begin{aligned}\nabla^M_U R_{UVUV} - \nabla_U R^{\nabla}_{UVUV} &= -3Ug_M(A_UV, A_UV) - 2R_{UVUA_UV} \\ &\quad - 2(R_{UVU\nabla_U V} - R^{\nabla}_{UVU\nabla_U V}).\end{aligned} \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}Ug_M(A_UV, A_UV) &= 2g_M((\nabla^M_U A)_UV, A_UV) + 2g_M(A_U(\nabla_U V), A_UV) \\ &\quad + 2g_M(A_U(A_UV), A_UV).\end{aligned}$$

Since $A_U : L \to Q$ and $A_U : Q \to L$, the last term of the right hand side on the above equation is zero. Hence

$$\begin{aligned}&Ug_M(A_UV, A_UV) \\ &= 2g_M((\nabla^M_U A)_UV, A_UV) + 2g_M(A_U(\nabla_U V), A_UV).\end{aligned} \tag{3.7}$$

From (2.11), since $A_UU = 0$, we have

$$R_{UVU\nabla_U V} - R^{\nabla}_{UVU\nabla_U V} = -3g_M(A_UV, A_U(\nabla_U V)). \tag{3.8}$$

From (3.7) and (3.8), we have

$$\begin{aligned}&-3Ug_M(A_UV, A_UV) - 2(R_{UVU\nabla_U V} - R^{\nabla}_{UVU\nabla_U V}) \\ &= -6g_M((\nabla^M_U A)_UV, A_UV).\end{aligned} \tag{3.9}$$

From (3.6) and (3.9), we have

$$\nabla^M_U R_{UVUV} - \nabla_U R^{\nabla}_{UVUV} = -2R_{UA_UVUV} - 6g_M((\nabla^M_U A)_UV, A_UV). \tag{3.10}$$

From (2.3) and (3.2), we have $A_U JU = JA_U U = 0$. Then we have from (3.10),

$$\nabla_U^M R_{UJUUJU} = \nabla_U R_{UJUUJU}^{\nabla}.$$

From this formula, the proof is completed.

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