Far East Journal of Mathematical Sciences (FJMS)
Volume 77, Number 1, 2013, Pages 85-104
Published Online: June 2013
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

FIXED POINT THEOREMS AND CONVERGENCE THEOREMS FOR GENERALIZED NONEXPANSIVE MAPPINGS

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Abstract

We prove the unique common fixed point theorems for a pair of R-weakly commuting mappings satisfying the generalized contractive conditions in metric spaces. Moreover, we also prove the weak convergence theorems for the sequences of Mann iterations and modified Ishikawa iterations in uniformly convex Banach spaces satisfying the Opial's condition where T is generalized S-nonexpansive type and S satisfies condition (C).

1. Introduction and Preliminaries

The common fixed point theorems for mappings satisfying certain contractive conditions in metric spaces have been continually studied for decade (see [1-3, 6-9, 14] and references contained therein). In 1986, Jungck

2010 Mathematics Subject Classification: 47H10, 47H09.

Keywords and phrases: common fixed points, R-weakly commuting mappings, condition (C), nonexpansive type mappings.

Submitted by K. K. Azad

Received February 7, 2013

[6] introduced the concept of compatible mappings and proved that weakly commuting mappings are compatible mappings. After that, Pant [14] introduced R-weakly commuting mappings and assured the existence of common fixed points where the result requires the continuity of at least one of the mappings.

Definition 1.1. Let E be a nonempty subset of a metric space (X, d) and $T, S : E \to E$. Then a pair of mappings (T, S) is said to be R-*weakly commuting* if there exists $R > 0$ such that

$$d(STx, TSx) \le Rd(Sx, Tx), \text{ for all } x \in E.$$

Recently, Mishra et al. [12] assured the existence of the unique common fixed point theorem as the following:

Theorem 1.2 [12]**.** *Let* (X, d) *be a metric space and E be a nonempty subset of X. Suppose that* $T, S : E \to E$ *are mappings such that*

$$d(Tx, Ty) \le k\left(\max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\}\right), \quad (1)$$

for all $x, y \in E$, *where* $k \in (0, 1)$. *Assume that* $T(E) \subseteq S(E)$ *and the pair* (T, S) *is R-weakly commuting on E. Then*

(a) *if* $T(E)$ *is complete*, *then T and S have a unique common fixed point in* $T(E)$;

(b) *if* $S(E)$ *is complete*, *then T and S have a unique common fixed point in* $S(E)$.

Definition 1.3. Let E be a nonempty subset of a Banach space X. Then a mapping $T : E \to E$ is said to be *nonexpansive* if

$$\| Tx - Ty \| \le \| x - y \|,$$

for all $x, y \in E$. Let $F(T) := \{x \in E : Tx = x\}$ be denoted as the set of all fixed points of T.

Definition 1.4. Let E be a nonempty subset of a Banach space X and $T, S : E \to E$. Then a mapping T is said to be *S-nonexpansive* if

$$\| Tx - Ty \| \leq \| Sx - Sy \|,$$

for all $x, y \in E$.

Mishra et al. [12] introduced more general class of S-nonexpansiveness.

Definition 1.5. Let E be a nonempty subset of a Banach space X and $T, S : E \to E$. Then a mapping T is *generalized S-nonexpansive type* if

$$\| Tx - Ty \| \leq M(x, y),$$

for all $x, y \in E$, where

$$M(x, y)$$
$$= \max\left\{ \| Sx - Sy \|, \frac{\| Sx - Tx \| + \| Sy - Sy \|}{2}, \frac{\| Sx - Ty \| + \| Sy - Tx \|}{2} \right\}.$$

Definition 1.6. A Banach space X is said to satisfy Opial's condition if whenever a sequence $\{x_n\}$ in X converges weakly to x, then

$$\liminf_{n\to\infty} \| x_n - x \| < \liminf_{n\to\infty} \| x_n - y \|,$$

for all $y \in X$ with $y \neq x$ (see [13]).

Let X be a Banach space and E be a nonempty convex subset of X. Assume that $T : E \to E$, $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta_n\} \subseteq [0, 1)$. The sequence $\{x_n\}$ of Mann iterations [11] defined, for an arbitrary $x_0 \in E$, by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \in \mathbb{N} \cup \{0\}, \tag{2}$$

and the sequence $\{x_n\}$ of Ishikawa iterations [5] defined, for an arbitrary $x_0 \in E$, by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n \in \mathbb{N} \cup \{0\}. \tag{3}$$

In (3), if $\beta_n = 0$, for all n, then (3) is reduced to (2).

Recently, Mishra et al. [12] obtained some common fixed point theorems for mappings T and S where T is generalized S-nonexpansive type under the certain assumptions. Furthermore, they also proved the weak convergence of a sequence $\{x_n\}$ of Mann iterations to a common fixed point of the mentioned mappings.

Theorem 1.7 [12]**.** *Let E be a bounded closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition. Let $T, S : E \to E$ be mappings where T is generalized S-nonexpansive type and S is nonexpansive. Suppose that $F(T) \cap F(S)$ is nonempty and $\{\alpha_n\}$ is a real sequence in $(0, 1)$. Then, for an arbitrary $x_0 \in E$, the sequence $\{x_n\}$ of Mann iterations converges weakly to a common fixed point of T and S.*

Definition 1.8. Let X be a Banach space. Then a subset E of X is said to be a *retract* of X if there exists a continuous mapping $P : X \to E$ such that $Px = x$, for all $x \in E$.

Let X be a uniformly convex Banach space and E be a closed convex subset of X with a nonexpansive retraction P. Assume that $T : E \to X$, $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta_n\} \subseteq [0, 1)$. The sequence $\{x_n\}$ of modified Ishikawa iterations defined, for $x_0 \in E$, by

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n Ty_n),$$

$$y_n = P((1-\beta_n)x_n + \beta_n Tx_n), \quad n \in \mathbb{N} \cup \{0\}. \tag{4}$$

If T is a self mapping on E, then (4) is reduced to (3).

Suzuki [16] introduced condition (C) as the following:

Definition 1.9. Let E be a nonempty subset of a Banach space X. Then a mapping $T : E \to E$ is said to satisfy condition (C) if

$$\frac{1}{2}\| x - Tx \| \le \| x - y \| \text{ implies } \| Tx - Ty \| \le \| x - y \|,$$

for all $x, y \in E$.

Dhompongsa and Kaewcharoen [4] gave the example of the mapping satisfying condition (C).

Example 1.10 [4]**.** Define a mapping T on $\left[0,\ 3\frac{1}{2}\right]$ by

$$Tx = \begin{cases} 0 & \text{if } x \in [0,\ 3], \\ 4x - 12 & \text{if } x \in \left[3,\ 3\frac{1}{4}\right], \\ -4x + 14 & \text{if } x \in \left[3\frac{1}{4},\ 3\frac{1}{2}\right]. \end{cases}$$

Therefore, T is a continuous mapping satisfying condition (C) and T is not nonexpansive.

Lemma 1.11 [16]**.** *Let E be a subset of a Banach space X and $T : E \to E$. Assume that T satisfies condition* (C)*. Then*

$$\| x - Ty \| \le 3 \| Tx - x \| + \| x - y \|,$$

for all $x,\ y \in E$.

Theorem 1.12 [16]**.** *Let T be a self mapping on a weakly compact convex subset E of a uniformly convex Banach space X. If T satisfies condition* (C)*, then T has a fixed point.*

In this paper, we assure the existence of the unique common fixed point theorem for mappings satisfying generalized contractive contraction where the mentioned mappings are R-weakly commuting in metric spaces. Moreover, we also prove the weak convergence of Mann iterations and modified Ishikawa iterations to a common fixed point for the class of mappings that is general than nonexpansiveness.

2. Fixed Point Theorems

Let Φ be the set of all mappings ϕ such that $\phi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing mapping satisfying $\sum_{n=0}^{\infty} \phi^n(t) < \infty$, for all $t \in (0, +\infty)$. If

$\phi \in \Phi$, then ϕ is called a Φ-*map*. Furthermore, if ϕ is a Φ-map, then

(i) $\phi(t) < t$, for all $t \in (0, +\infty)$,

(ii) $\phi(0) = 0$.

For more detail, see [15]. From now on, unless otherwise stated, ϕ is meant the Φ-map.

Theorem 2.1. *Let (X, d) be a metric space and E be a nonempty subset of X. Suppose that $T, S : E \to E$ are mappings such that*

$$d(Tx, Ty)$$

$$\leq \phi\left(\max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\}\right), \quad (5)$$

for all $x, y \in E$. Assume that $T(E) \subseteq S(E)$ and the pair (T, S) is R-weakly commuting on E. Then

(a) *if $T(E)$ is complete, then T and S have a unique common fixed point in $T(E)$;*

(b) *if $S(E)$ is complete, then T and S have a unique common fixed point in $S(E)$.*

Proof. Let x_0 be an arbitrary point in E. Since $T(E) \subseteq S(E)$, there exists $x_1 \in E$ such that $Sx_1 = Tx_0$. Let a be a positive real number such that

$$\phi(d(Sx_0, Sx_1)) \leq \phi(a).$$

Again, since $T(E) \subseteq S(E)$, there exists $x_2 \in E$ such that $Sx_2 = Tx_1$. By (5), we have

$$d(Sx_1, Sx_2)$$

$$= d(Tx_0, Tx_1)$$

$$\leq \phi\left(\max\left\{d(Sx_0, Sx_1), d(Sx_0, Tx_0), d(Sx_1, Tx_1), \frac{d(Sx_0, Tx_1) + d(Sx_1, Tx_0)}{2}\right\}\right)$$

$$= \phi\left(\max\left\{d(Sx_0, Sx_1), d(Sx_0, Sx_1), d(Sx_1, Sx_2), \frac{d(Sx_0, Sx_2) + d(Sx_1, Sx_1)}{2}\right\}\right)$$

$$\le \phi\left(\max\left\{d(Sx_0, Sx_1), d(Sx_1, Sx_2), \frac{d(Sx_0, Sx_1) + d(Sx_1, Sx_2)}{2}\right\}\right)$$

$$\le \phi(\max\{d(Sx_0, Sx_1), d(Sx_1, Sx_2)\}).$$

By continuing this process, we can construct the sequence $\{Sx_n\}$ in E such that

$$Sx_n = Tx_{n-1}$$

and

$$d(Sx_n, Sx_{n+1}) \le \phi(\max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\}), \text{ for all } n \in \mathbb{N}.$$

If $Sx_n = Sx_{n+1}$, for some $n \in \mathbb{N}$, then $Sx_n = Tx_n$. Suppose that $Sx_n \neq Sx_{n+1}$, for all $n \in \mathbb{N}$. If $\max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\} = d(Sx_n, Sx_{n+1})$, then

$$d(Sx_n, Sx_{n+1}) \le \phi(d(Sx_n, Sx_{n+1})) < d(Sx_n, Sx_{n+1}),$$

which leads to a contradiction. This implies that

$$d(Sx_n, Sx_{n+1}) \le \phi(d(Sx_{n-1}, Sx_n)), \text{ for all } n \in \mathbb{N}.$$

This yields

$$d(Sx_n, Sx_{n+1}) \le \phi(d(Sx_{n-1}, Sx_n)) \le \cdots \le \phi^n(a), \text{ for all } n \in \mathbb{N}.$$

Therefore,

$$\sum_{n=0}^{\infty} d(Sx_n, Sx_{n+1}) \le \sum_{n=0}^{\infty} \phi^n(a) < \infty.$$

This implies that $\{Sx_n\}$ is a Cauchy sequence in E. Suppose that $T(E)$ is complete. It follows that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z, \text{ for some } z \in T(E).$$

Since $T(E) \subseteq S(E)$, thus there exists $u \in E$ such that $Su = z$. By applying (5), we obtain that

$$d(Tx_n, Tu)$$

$$\leq \phi\left(\max\left\{d(Sx_n, Su), d(Sx_n, Tx_n), d(Su, Tu), \frac{d(Sx_n, Tu) + d(Su, Tx_n)}{2}\right\}\right).$$

If

$$\max\left\{d(Sx_n, Su), d(Sx_n, Tx_n), d(Su, Tu), \frac{d(Sx_n, Tu) + d(Su, Tx_n)}{2}\right\}$$

$$= d(Sx_n, Su),$$

then

$$d(Tx_n, Tu) \leq \phi(d(Sx_n, Su)) < d(Sx_n, Su).$$

By taking the limit as $n \to \infty$, we have $d(Su, Tu) = 0$ and this yields $Su = Tu$.

If

$$\max\left\{d(Sx_n, Su), d(Sx_n, Tx_n), d(Su, Tu), \frac{d(Sx_n, Tu) + d(Su, Tx_n)}{2}\right\}$$

$$= d(Sx_n, Tx_n),$$

then

$$d(Tx_n, Tu) \leq \phi(d(Sx_n, Tx_n)) < d(Sx_n, Tx_n).$$

By taking the limit as $n \to \infty$, we have $d(Su, Tu) = 0$ and this yields $Su = Tu$.

If

$$\max\left\{d(Sx_n, Su), d(Sx_n, Tx_n), d(Su, Tu), \frac{d(Sx_n, Tu) + d(Su, Tx_n)}{2}\right\}$$

$$= d(Su, Tu),$$

then

$$d(Tx_n, Tu) \leq \phi(d(Su, Tu)).$$

By taking the limit as $n \to \infty$, we have $d(Su, Tu) \leq \phi(d(Su, Tu))$. If $Su \neq Tu$, then

$$d(Su, Tu) \leq \phi(d(Su, Tu)) < d(Su, Tu),$$

which leads to a contradiction. It follows that $Su = Tu$.

If

$$\max\left\{d(Sx_n, Su), d(Sx_n, Tx_n), d(Su, Tu), \frac{d(Sx_n, Tu) + d(Su, Tx_n)}{2}\right\}$$

$$= \frac{d(Sx_n, Tu) + d(Su, Tx_n)}{2},$$

then

$$d(Tx_n, Tu) \leq \phi\left(\frac{d(Sx_n, Tu) + d(Su, Tx_n)}{2}\right) < \frac{d(Sx_n, Tu) + d(Su, Tx_n)}{2}.$$

By taking the limit as $n \to \infty$, we have $d(Su, Tu) \leq \frac{d(Su, Tu)}{2}$. This implies that $Su = Tu$. Since the pair (T, S) is R-weakly commuting on E, we obtain that

$$d(STu, TSu) \leq Rd(Su, Tu) = 0.$$

Therefore, $Sz = Tz$. By applying (5), we obtain that

$$d(Tx_n, Tz)$$

$$\leq \phi\left(\max\left\{d(Sx_n, Sz), d(Sx_n, Tx_n), d(Sz, Tz), \frac{d(Sx_n, Tz) + d(Sz, Tx_n)}{2}\right\}\right).$$

Using the mentioned argument as above, we can conclude that $d(z, Tz) = 0$ and then $z = Tz = Sz$. Suppose that w is any common fixed point of T and S. By applying (5), we have

$$
\begin{aligned}
d(z, w) &= d(Tz, Tw) \\
&\le \phi\left(\max\left\{d(Sz, Sw), d(Sz, Tz), d(Sw, Tw), \frac{d(Sz, Tw) + d(Sw, Tz)}{2}\right\}\right) \\
&\le \phi\left(\max\left\{d(z, w), d(z, z), d(w, w), \frac{d(z, w) + d(w, z)}{2}\right\}\right) \\
&= \phi(d(z, w)).
\end{aligned}
$$

This yields $d(z, w) = 0$ and then $z = w$. Hence z is a unique common fixed point of T and S in $T(E)$. If $S(E)$ is complete, then

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z, \text{ for some } z \in S(E).$$

Thus there exists $u \in E$ such that $Su = z$. By using the analogous proof as before, we can conclude that z is a unique common fixed point of T and S in $S(E)$. □

By applying Theorem 2.1, we obtain the following corollaries.

Corollary 2.2 [12, Theorem 10]**.** *Let (X, d) be a metric space and E be a nonempty subset of X. Suppose that $T, S : E \to E$ are mappings such that*

$$
\begin{aligned}
&d(Tx, Ty) \\
&\le k \max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\}, \qquad (6)
\end{aligned}
$$

for all $x, y \in E$, where $k \in (0, 1)$. Assume that $T(E) \subseteq S(E)$ and the pair (T, S) is R-weakly commuting on E. Then

(a) *if $T(E)$ is complete, then T and S have a unique common fixed point in $T(E)$;*

(b) *if $S(E)$ is complete, then T and S have a unique common fixed point in $S(E)$.*

Proof. Define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = kt$. Therefore, ϕ is a nondecreasing mapping and $\sum_{n=0}^{\infty} \phi^n(t) < \infty,$ for all $t \in (0, +\infty)$. It follows that the contractive condition (5) in Theorem 2.1 is now satisfied. This completes the proof. □

Corollary 2.3. *Let (X, d) be a metric space and E be a nonempty subset of X. Suppose that $T, S : E \to E$ are mappings such that*

$$d(Tx, Ty)$$

$$\leq \phi\left(\max\left\{d(Sx, Sy), \frac{d(Sx, Tx) + d(Sy, Ty)}{2}, \frac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\}\right), \quad (7)$$

for all $x, y \in E$. Assume that $T(E) \subseteq S(E)$ and the pair (T, S) is R-weakly commuting on E. Then

(a) *if $T(E)$ is complete, then T and S have a unique common fixed point in $T(E)$;*

(b) *if $S(E)$ is complete, then T and S have a unique common fixed point in $S(E)$.*

Proof. Since the contractive condition (7) implies the contractive condition (5), we obtain that all assumptions in Theorem 2.1 are now satisfied. Therefore, the proof is complete. □

Corollary 2.4 [12, Lemma 12]**.** *Let (X, d) be a metric space and E be a nonempty subset of X. Suppose that $T, S : E \to E$ are mappings such that*

$$d(Tx, Ty)$$

$$\leq k \max\left\{d(Sx, Sy), \frac{d(Sx, Tx) + d(Sy, Ty)}{2}, \frac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\}, \quad (8)$$

for all $x, y \in E$, where $k \in (0, 1)$. Assume that $T(E) \subseteq S(E)$ and the pair (T, S) is R-weakly commuting on E. Then

(a) *if* $T(E)$ *is complete*, *then T and S have a unique common fixed point in* $T(E)$;

(b) *if* $S(E)$ *is complete*, *then T and S have a unique common fixed point in* $S(E)$.

3. Convergence Theorems

We now prove the weak convergence of a sequence of Mann iterations and modified Ishikawa iterations to a common fixed point of mappings using the techniques appeared in [10, 12].

Theorem 3.1. *Let E be a bounded closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition*. *Let* $T, S : E \to E$ *be mappings where T is generalized S-nonexpansive type and S satisfies condition* (*C*).

Suppose that $F(T) \cap F(S)$ is nonempty and $\{\alpha_n\}$ is a real sequence in $(0, 1)$. Then, for an arbitrary $x_0 \in E$, the sequence $\{x_n\}$ of Mann iterations converges weakly to a common fixed point of *T* and *S*.

Proof. Let $z \in F(T) \cap F(S)$. For each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} \| x_{n+1} - z \| &= \| (1 - \alpha_n) x_n + \alpha_n T x_n - z \| \\ &= \| (1 - \alpha_n)(x_n - z) + \alpha_n (T x_n - z) \|. \end{aligned}$$

This implies that

$$\| x_{n+1} - z \| \leq (1 - \alpha_n) \| (x_n - z) \| + \alpha_n \| T x_n - z \|. \tag{9}$$

Since *T* is generalized *S*-nonexpansive type, we obtain that

$$\| T x_n - z \| \leq M(x_n, z),$$

where

$$M(x_n, z)$$

$$= \max \left\{ \| S x_n - S z \|, \frac{\| S x_n - T x_n \| + \| S z - T z \|}{2}, \frac{\| S x_n - T z \| + \| S z - T x_n \|}{2} \right\}.$$

We separate the proof into the following cases:

Case 1. If $M(x_n, z) = \| Sx_n - Sz \|$, then

$$\| Tx_n - z \| \leq \| Sx_n - Sz \|.$$

Since S satisfies condition (C) and Lemma 1.11, we have

$$\begin{aligned} \| Sx_n - Sz \| &= \| Sx_n - z \| \\ &\leq 3\| Sz - z \| + \| x_n - z \| \\ &= \| x_n - z \|. \end{aligned}$$

It follows that

$$\| Tx_n - z \| \leq \| x_n - z \|.$$

Case 2. If $M(x_n, z) = \dfrac{\| Sx_n - Tx_n \| + \| Sz - Tz \|}{2}$, then

$$\begin{aligned} \| Tx_n - z \| &\leq \frac{\| Sx_n - Tx_n \| + \| Sz - Tz \|}{2} \\ &= \frac{\| Sx_n - Tx_n \|}{2} \\ &\leq \frac{\| Sx_n - z \| + \| z - Tx_n \|}{2}. \end{aligned}$$

This yields

$$\| Tx_n - z \| \leq \| Sx_n - z \|.$$

Since S satisfies condition (C) and Lemma 1.11, we have

$$\| Tx_n - z \| \leq \| x_n - z \|.$$

Case 3. If $M(x_n, z) = \dfrac{\| Sx_n - Tz \| + \| Sz - Tx_n \|}{2}$, then

$$\begin{aligned} \| Tx_n - z \| &\leq \frac{\| Sx_n - Tz \| + \| Sz - Tx_n \|}{2} \\ &= \frac{\| Sx_n - z \| + \| z - Tx_n \|}{2}. \end{aligned}$$

This yields

$$\| Tx_n - z \| \leq \| Sx_n - z \|.$$

Since S satisfies condition (C) and Lemma 1.11, we have

$$\| Tx_n - z \| \leq \| x_n - z \|.$$

From the above three cases, we conclude that

$$\| Tx_n - z \| \leq \| x_n - z \|, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{10}$$

Using (9) and (10), we obtain that

$$\| x_{n+1} - z \| \leq (1 - \alpha_n) \| (x_n - z) \| + \alpha_n \| x_n - z \| = \| x_n - z \|,$$

for all $n \in \mathbb{N} \cup \{0\}$.

Therefore, $\{\| x_n - z \|\}$ is a nonincreasing sequence and then $\lim_{n\to\infty} \| x_n - z \|$ exists. We will prove that $\{x_n\}$ converges weakly to a common fixed point of T and S. Suppose that $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are the subsequences of $\{x_n\}$ which converge to z and $\bar{z}$, respectively. Assume that $z \neq \bar{z}$. Since X satisfies the Opial's condition and $\lim_{n\to\infty} \| x_n - z \|$ exists, for all $z \in F(T) \cap F(S)$, we obtain that

$$\begin{aligned}
\lim_{n\to\infty} \| x_n - z \| &= \lim_{k\to\infty} \| x_{n_k} - z \| \\
&< \lim_{k\to\infty} \| x_{n_k} - \bar{z} \| \\
&= \lim_{n\to\infty} \| x_n - \bar{z} \| \\
&= \lim_{k\to\infty} \| x_{m_k} - \bar{z} \| \\
&< \lim_{k\to\infty} \| x_{m_k} - z \| \\
&= \lim_{n\to\infty} \| x_n - z \|,
\end{aligned}$$

which leads to a contradiction. Therefore, we can conclude that $z = \bar{z}$. □

By using Theorem 3.1, we immediately obtain the following corollary.

Corollary 3.2 [12, Theorem 17]**.** *Let E be a bounded closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition. Let* $T, S: E \to E$ *be mappings where T is generalized S-nonexpansive type and S is nonexpansive. Suppose that* $F(T) \cap F(S)$ *is nonempty and* $\{\alpha_n\}$ *is a real sequence in* $(0, 1)$. *Then, for an arbitrary* $x_0 \in E$, *the sequence* $\{x_n\}$ *of Mann iterations converges weakly to a common fixed point of T and S.*

Theorem 3.3. *Let E be a bounded closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition. Assume that* $T, S: E \to X$ *are mappings where T is generalized S-nonexpansive type and S satisfies condition* (*C*). *Suppose that* $F(T) \cap F(S)$ *is nonempty,* $\{\alpha_n\} \subseteq (0, 1)$ *and* $\{\beta_n\} \subseteq [0, 1)$. *Then, for an arbitrary* $x_0 \in E$, *the sequence* $\{x_n\}$ *of modified Ishikawa iterations with a nonexpansive retraction P converges weakly to a common fixed point of T and S.*

Proof. Let $z \in F(T) \cap F(S)$. Since P is a nonexpansive retraction, for each $n \in \mathbb{N} \cup \{0\}$, we obtain that

$$
\begin{aligned}
&\| x_{n+1} - z \| \\
&= \| P((1-\alpha_n)x_n + \alpha_n Ty_n) - z \| \\
&= \| P((1-\alpha_n)x_n + \alpha_n Ty_n) - Pz \| \\
&\le \| (1-\alpha_n)x_n + \alpha_n TP((1-\beta_n)x_n + \beta_n Tx_n) - \alpha_n z + \alpha_n z - z \| \\
&\le (1-\alpha_n)\| x_n - z \| + \alpha_n \| TP((1-\beta_n)x_n + \beta_n Tx_n) - Tz \|.
\end{aligned}
$$

This implies that

$$
\| x_{n+1} - z \| \le (1-\alpha_n)\| x_n - z \| + \alpha_n \| TP((1-\beta_n)x_n + \beta_n Tx_n) - Tz \|. \tag{11}
$$

For each $n \in \mathbb{N} \cup \{0\}$, let $w_n = P((1-\beta_n)x_n + \beta_n Tx_n)$. Since T is generalized S-nonexpansive type, we obtain that

$$\| Tw_n - Tz \| \leq M(w_n, z),$$

where

$$M(w_n, z)$$

$$= \max\left\{\| Sw_n - Sz \|, \frac{\| Sw_n - Tw_n\| + \| Sz - Tz\|}{2}, \frac{\| Sw_n - Tz \| + \| Sz - Tw_n\|}{2}\right\}.$$

We separate the proof into the following cases:

Case 1. If $M(w_n, z) = \| Sw_n - Sz \|,$ then

$$\| Tw_n - Tz \| \leq \| Sw_n - Sz \|.$$

Since S satisfies condition (C) and Lemma 1.11, we have

$$\begin{aligned} \| Sw_n - Sz \| &= \| Sw_n - z \| \\ &\leq 3\| Sz - z \| + \| w_n - z \| \\ &= \| w_n - z \|. \end{aligned}$$

It follows that

$$\| Tx_n - Tz \| \leq \| w_n - z \|.$$

Case 2. If $M(w_n, z) = \dfrac{\| Sw_n - Tw_n \| + \| Sz - Tz \|}{2},$ then

$$\begin{aligned} \| Tw_n - Tz \| &\leq \frac{\| Sw_n - Tw_n \| + \| Sz - Tz \|}{2} \\ &= \frac{\| Sw_n - Tw_n \|}{2} \\ &\leq \frac{\| Sw_n - z \| + \| z - Tw_n \|}{2}. \end{aligned}$$

This yields

$$\| Tw_n - Tz \| \leq \| Sw_n - Sz \|.$$

Since S satisfies condition (C) and Lemma 1.11, we have

$$\| Tw_n - Tz \| \leq \| w_n - z \|.$$

Case 3. If $M(w_n, z) = \frac{\| Sw_n - Tz \| + \| Sz - Tw_n \|}{2}$, then

$$\| Tw_n - Tz \| \leq \frac{\| Sw_n - Tz \| + \| Sz - Tw_n \|}{2}$$

$$= \frac{\| Sw_n - z \| + \| z - Tw_n \|}{2}.$$

This yields

$$\| Tw_n - Tz \| \leq \| Sw_n - z \|.$$

Since S satisfies condition (C) and Lemma 1.11, we have

$$\| Tw_n - Tz \| \leq \| w_n - z \|.$$

From the above three cases, we conclude that

$$\| Tw_n - Tz \| \leq \| w_n - z \|, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{12}$$

This implies that

$$\begin{aligned}
\| TP((1-\beta_n)x_n + \beta_n Tx_n) - Tz \| &= \| Tw_n - z \| \\
&\leq \| w_n - z \| \\
&= \| P((1-\beta_n)x_n + \beta_n Tx_n) - z \| \\
&= \| P((1-\beta_n)x_n + \beta_n Tx_n) - Pz \| \\
&\leq \| (1-\beta_n)x_n + \beta_n Tx_n - z \|.
\end{aligned}$$

By (11), we have

$$\| x_{n+1} - z \|$$

$$\leq (1-\alpha_n)\| x_n - z \| + \alpha_n(\| (1-\beta_n)x_n + \beta_n Tx_n - z + \beta_n Tz - \beta_n Tz \|)$$

$$= (1-\alpha_n)\| x_n - z \| + \alpha_n(\| (1-\beta_n)x_n + \beta_n Tx_n - z + \beta_n z - \beta_n Tz \|)$$

$$\leq (1-\alpha_n)\| x_n - z \| + \alpha_n(1-\beta_n)\| x_n - z \| + \alpha_n\beta_n\| Tx_n - Tz \|.$$

Since T is generalized S-nonexpansive type and S satisfies condition (C), we can prove that

$$\| Tx_n - Tz \| \leq \| x_n - z \|,$$

by using the analogous argument as before. For each $n \in \mathbb{N} \cup \{0\}$, we obtain that

$$\| x_{n+1} - z \| \leq (1-\alpha_n)\| x_n - z \| + \alpha_n(1-\beta_n)\| x_n - z \| + \alpha_n\beta_n\| x_n - z \|$$

$$= \| x_n - z \|.$$

Therefore, $\{\| x_n - z \|\}$ is a nonincreasing sequence and then $\lim_{n\to\infty}\| x_n - z \|$ exists. By the analogous argument in the proof of Theorem 3.1, we can conclude that $\{x_n\}$ converges weakly to a common fixed point of S and T. □

Since S-nonexpansive type mappings and condition (C) are weaker than S-nonexpansiveness and nonexpansiveness, respectively, we immediately obtain the following result.

Corollary 3.4 [10, Theorem 2.1]**.** *Let E be a bounded closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition. Assume that $T, S : E \to X$, where T is S-nonexpansive and S is nonexpansive. Suppose that $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta_n\} \subseteq [0, 1)$. Then, for an arbitrary $x_0 \in E$, the sequence $\{x_n\}$ of modified Ishikawa iterations with a nonexpansive retraction P converges weakly to a common fixed point of T and S.*

Acknowledgement

This research is supported by Naresuan University, Phitsanulok, Thailand.

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