JP Jour. Geometry & Topology **4**(1) (2004), 81-96

A COMPACT-OPEN TOPOLOGY FOR COLLECTIONS OF SET-VALUED FUNCTIONS AND SOME OF ITS PROPERTIES

TERRENCE A. EDWARDS
Department of Mathematics, *Dillard University*
New Orleans, *LA* 70122, *U. S. A.*
e-mail: tajedwards@netscape.net

JAMES E. JOSEPH
Department of Mathematics, *Howard University*
Washington, *D. C.* 20059, *U. S. A.*
e-mail: jjoseph@howard.edu

MYUNG H. KWACK
Department of Mathematics, *Howard University*
Washington, *D. C.* 20059, *U. S. A.*
e-mail: mkwack@howard.edu

www.pphmj.com

Abstract

It has been observed by T. Edwards that the compact-open topology on the family of continuous functions between topological spaces is the weak topology induced on the family by a certain collection of continuous compact-valued functions. In this paper, several of Edwards' results are generalized to the Φ-open topology of A. Wilansky, and it is shown that Edwards' discovery leads to a compact-open topology for families of set-

2000 Mathematics Subject Classification: Primary 32A10, 32C10, 32H20, 32A17; Secondary 54C20, 54C35, 54D35, 54C05.

Key words and phrases: set-valued functions, compact-open topology, function spaces, finite topology, upper semicontinuous, lower semicontinuous, continuous set-valued functions, Ascoli theorem.

Received September 18, 2003

© 2004 Pushpa Publishing House

valued functions between topological spaces. The collection of set-valued functions endowed with this compact-open topology gives rise to an imbedding theorem for the space of set-valued functions, which in turn, in conjunction with results found in old papers of E. Michael, D. Wulbert and R. Smithson, leads to characterizations of compact subsets of such families from Hausdorff spaces and arbitrary spaces to Hausdorff spaces, and sufficient conditions for members of such families to be first countable and second countable.

In addition, several theorems on the first countability of the space of compact-valued functions with one of the compact-open topologies of Smithson are proved. One such theorem is the following generalization of a result of R. Arens: If X is a Hausdorff hemicompact space and Y is a Hausdorff locally compact second countable space, then the collection of upper semicontinuous compact-valued functions α from X to Y satisfying $\alpha(\alpha^{-1}(F)) \subset F$ for each $F \subset Y$ is first countable.

All spaces will be topological spaces and no separation axioms will be assumed unless explicitly stated. For a nonempty set X, let Φ be a nonempty collection of nonempty subsets of X, and for a space Y, let Θ be a nonempty collection of functions from X to Y. For each $K \in \Phi$ and W open in Y, let $[K, W] = \{g \in \Theta : g(K) \subset W\}$. The collection of all such $[K, W]$ generates, as subbase, a topology on Θ which is called the Φ-*open topology* by Wilansky [10]. If X is a space and Φ is the collection of nonempty compact (closed) subsets of X, the Φ-open topology is the familiar compact-open topology (is called the *closed-open topology*). The family of nonempty compact (closed) subsets of X will be denoted by $\mathcal{K}X$ $(\mathcal{C}X)$, and the topology on $\mathcal{K}X$ $(\mathcal{C}X)$ will be the finite topology [6]. If X is a space, let $\mathcal{O}(X)$ be the collection of open subsets of X and, for $A \subset X$, let $\sum(A) = \{V \in \mathcal{O}(X) : A \subset V\}$ (simply $\sum(x)$ if $A = \{x\}$). For sets X, Y, a *set-valued function from* X *to* Y is a function on X with nonempty subsets of Y as values. If φ is a set-valued function from X to Y and $A \subset X$ $(B \subset Y)$, $\bigcup_{x \in A} \varphi(x)$ $(\{x \in X : \varphi(x) \cap B \neq \varnothing\})$ is denoted by $\varphi(A)$ $(\varphi^{-1}(B))$ and called the *image* of A (*inverse image* of B) under φ. If X, Y are spaces, a set-valued function $\varphi : X \to Y$ is *upper* (*lower*) *semicontinuous u.s.c.* (*l.s.c.*) *at* $x \in X$ if for each $W \in \sum(\varphi(x))$ $(W \in \mathcal{O}(Y)$

satisfying $x \in \varphi^{-1}(W)$), some $V \in \Sigma(x)$ satisfies $\varphi(V) \subset W$ ($V \subset \varphi^{-1}(W)$). It is known and readily verified that φ is u.s.c (l.s.c.) at each $x \in X$ if and only if $\varphi^{-1}(W)$ is closed (open) in X for each closed (open) subset W in Y. In this case φ is said to be *upper* (*lower*) *semicontinuous* (*u.s.c.*) (*l.s.c.*); a set-valued function which is u.s.c. and l.s.c. is called *continuous*. If Θ is a nonempty collection of set-valued functions from X to Y and $\varnothing \neq K \subset X$, then $H_K(\varphi) = \varphi(K)$ defines a set-valued function from Θ to Y. Let Y^X represent the set of functions from X to Y, let $\mathbf{SV}(X, Y)$ be the family of all set-valued functions from X into Y, and let $\mathbf{uSV}(X, Y)$ be the collection of upper semicontinuous members of $\mathbf{SV}(X, Y)$. Let $\mathbf{KV}(X, Y)$ be the collection of members of $\mathbf{uSV}(X, Y)$ with compact values. It is known that members of $\mathbf{KV}(X, Y)$ preserve compact subsets [8]. The discovery that the compact-open topology on a collection of continuous functions between topological spaces is the weak topology for a family of set-valued functions motivates an extension of the compact-open topology to collections of set-valued functions. This compact-open topology on a nonempty $\Theta \subset \mathbf{SV}(X, Y)$ will be denoted by $\mathcal{T}_{co}$. Smithson [7] has introduced a topology on a nonempty $\Theta \subset \mathbf{SV}(X, Y)$, which coincides with the compact-open topology in the case of single-valued functions. This topology, generated by $\{\alpha \in \Theta : \alpha(F) \subset W,\ F$ compact in X, W open in $Y\}$, as subbase, will be denoted by $\mathcal{C}_2$. Some of the main results established in this paper for these topologies are listed below. In the sequel, if X is a Hausdorff space, $\varnothing \neq \Gamma \subset \mathcal{K}X$, and $K \in \mathcal{K}X$, $\{F \in \Gamma : K \subset F\}$ will be denoted by Γ_K. Without loss, assume Γ_K is closed under finite intersections.

Definition 1 is from Smithson [9]. In this definition, a cover, Ω of a set A is called a *proper cover* if $N \cap A \neq \varnothing$ is satisfied for each $N \in \Omega$.

Definition 1. A topological space X is called *compactly second countable* if for each compact subset M of X, there is a countable collection of open sets Δ so that for each open finite proper cover Γ of M,

www.pphmj.com

there is a finite proper cover Λ of M such that $\Lambda \subset \Delta$, $\bigcup_{\Lambda} H \subset \bigcup_{\Gamma} J$ and, for each $J \in \Gamma$, some $H \in \Lambda$ satisfies $H \subset J$.

(1°) If X and Y are spaces, then $\mathbf{KV}(X, Y)$ with the compact-open topology is imbedded in the product space $(\mathcal{K}\, Y)^{\mathcal{K} X}$.

(2°) Let X, Y be spaces with Y Hausdorff and suppose $\mathbf{KV}(X, Y)$ has the compact-open topology. Then $\Theta \subset \mathbf{KV}(X, Y)$ is compact if and only if

(1) Θ is a closed subset of $\mathbf{KV}(X, Y)$ and

(2) $\bigcup_{\mu \in \Theta} \mu(K)$ is a compact subset of Y for each $K \in \mathcal{K}X$.

(3°) Let X, Y be spaces with X Hausdorff, Y Fréchet, and let $\Gamma \subset \mathcal{K}X$ such that $\bigcap_{\Gamma_K} F = K$. Then $\mathbf{KV}(X, Y)$ endowed with the compact-open topology is imbedded in the product space $(\mathcal{K}Y)^{\Gamma}$.

(4°) Let X, Y be spaces with X Hausdorff, Y Fréchet, and let $\Gamma \subset \mathcal{K}X$ be countable such that $\bigcap_{\Gamma_K} F = K$. Then $\mathbf{KV}(X, Y)$ endowed with the compact-open topology is second countable if and only if Y is second countable.

(5°) Let X, Y be spaces with X Hausdorff, Y Fréchet, and let $\Gamma \subset \mathcal{K}X$ be countable such that $\bigcap_{\Gamma_K} F = K$. Then $\mathbf{KV}(X, Y)$ endowed with the compact-open topology is metrizable if and only if Y is metrizable.

(6°) Let X be a Hausdorff space and let $\Gamma \subset \mathcal{K}X$ be countable such that $\bigcap_{\Gamma_K} F = K$. If Y is Hausdorff and compactly second countable, then $\mathbf{KV}(X, Y)$ endowed with the compact-open topology is first countable.

(7°) If X is a Hausdorff hemicompact space, and Y is a Hausdorff locally compact second countable space, then $\{\alpha \in \mathbf{KV}(X, Y) : \alpha(\alpha^{-1}(F)) \subset F$ for each $F \subset Y\}$ endowed with Smithson's topology $\mathcal{C}_2$ is first countable.

www.pphmj.com

Results. First, some of Edwards' results on the compact-open topology are generalized to the Φ-open topology of Wilansky [10].

Theorem 1. *If Φ is a nonempty collection of nonempty subsets of X and Y is a space, the Φ-open topology on a nonempty $\Theta \subset Y^X$ is the smallest topology on Θ for which $H_K : \Theta \to Y$ is u.s.c. on Θ for each $K \in \Phi$.*

Proof. Let $\mathcal{T}$ represent the Φ-open topology on Θ. If $K \in \Phi$ and $W \in \mathcal{O}(Y)$, then for $[K, W] \in \mathcal{T}$, $H_K([K, W]) \subset W$ so H_K is u.s.c. Assume now that $\mathcal{U}$ is a topology on Θ for which each H_K is u.s.c. Let $[K, W]$ be a subbasic element of $\mathcal{T}$ and $f \in [K, W]$. Then $H_K(f) = f(K) \subset W$. Since $H_K : (\Theta, \mathcal{U}) \to Y$ is u.s.c., choose $V_f \in \mathcal{U}$ with $f \in V_f$ and $H_K(V_f) \subset W$. Each $h \in V_f$ satisfies $H_K(h) = h(K) \subset W$ and therefore $h \in [K, W]$. Thus $[K, W] = \cup_{[K,W]} V_f$ and hence $[K, W] \in \mathcal{U}$. This establishes that $\mathcal{T}$ is the smallest topology on Θ for which each H_K is u.s.c.

Corollary 1. *If Φ is a nonempty collection of nonempty subsets of X and Y is a space, the Φ-open topology on a nonempty $\Theta \subset Y^X$ is generated by $\Lambda = \{[K, W] : K \in \Phi, W \in \mathcal{O}(Y)\}$ as subbase.*

Proof. Let $\mathcal{T}$ be the topology generated by Λ as subbase. Show that $\mathcal{T}$ is the Φ-open topology on Θ. Let $\mathcal{S} \subset \mathcal{T}$ be a topology on Θ such that H_K is u.s.c. for each $K \in \Phi$, let $[K, W] \in \Lambda$, and let $\varphi \in [K, W]$. Then $H_K(\varphi) \subset W$. Choose $V_\varphi \in \mathcal{S}$ such that $\varphi \in V_\varphi$ and $H_K(V_\varphi) \subset W$. Then $[K, W] = \cup_{[K,W]} V_\varphi \in \mathcal{S}$. Hence $\mathcal{T}$ is the Φ-open topology.

Theorem 2. *If Φ is a collection of nonempty subsets of a set X containing the family of singletons and Y is a space, the Φ-open topology on a nonempty $\Theta \subset Y^X$ is the smallest topology on Θ for which $H_K : \Theta \to Y$ is continuous for each $K \in \Phi$.*

Proof. In view of Theorem 1 it need be shown that H_K is l.s.c. and the proof will be complete. If $K \in \Phi$ and $W \in \mathcal{O}(Y)$, then

www.pphmj.com

$$H_K^{-1}(W) = \Theta - \bigcap_{x \in K} H_{\{x\}}^{-1}(Y - W).$$

Hence $H_K^{-1}(W)$ is open since $H_{\{x\}}$ is u.s.c. from Theorem 1. Thus H_K is l.s.c.

Corollary 2. *The compact-open (closed-open) topology on a nonempty $\Theta \subset Y^X$ for spaces X, Y is the smallest topology on Θ for which $H_K : \Theta \to Y$ is u.s.c. for each $K \in \mathcal{K}X(\mathcal{C}X)$.*

Corollary 3 [3]**.** *For spaces X, Y the compact-open topology on a nonempty $\Theta \subset Y^X$ is the smallest topology on Θ for which $H_K : \Theta \to Y$ is continuous for each $K \in \mathcal{K}X$.*

Corollary 4. *For spaces X, Y with X a Fréchet space, the closed-open topology on a nonempty $\Theta \subset Y^X$ is the smallest topology on Θ for which $H_K : \Theta \to Y$ is continuous for each $K \in \mathcal{C}X$.*

Let X and Y be non-empty sets, let $H : X \to Y$ be a set-valued function and let 2^Y be the collection of non-empty subsets of Y. Define the function $H^* : X \to 2^Y$ by $H^*(x) = H(x)$ for each x in X. The function H^* is called the *function induced by the set-valued function H*. For spaces X, Y, it is known that a set-valued function $H : X \to Y$ is continuous if and only if $H^* : X \to 2^Y$ is a continuous function when 2^Y has the finite topology. Hence the following theorem and corollaries are produced.

www.pphmj.com

Theorem 3. *If X, Y are spaces and Φ is a collection of subsets of X containing the family of singletons, then a nonempty $\Theta \subset Y^X$ with the Φ-open topology is imbedded in $(2^Y)^\Phi$ when 2^Y has the finite topology.*

Proof. The family of continuous functions $\{H_K^* : \Theta \to 2^Y : K \in \Phi\}$ separates points. So the conclusion follows from a well-known imbedding theorem (see 6.6.2 in [10]).

Let $\mathcal{C}(X, Y)$ be the collection of continuous functions from the space X to the space Y.

Corollary 5 [3]**.** *If X and Y are spaces, then a nonempty $\Theta \subset \mathcal{C}(X, Y)$ with the compact-open topology is imbedded in $(\mathcal{K}Y)^{\mathcal{K}X}$.*

Theorem 2 motivates a definition of a Φ-open (compact-open) topology on a family of set-valued functions.

Definition 2. Let Φ be a nonempty collection of nonempty subsets of X, and let Y be a space. The Φ-*open topology on a nonempty* $\Theta \subset \mathbf{SV}(X, Y)$ is the smallest topology on Θ for which the set-valued function $H_K : \Theta \to Y$ defined by $H_K(\mu) = \mu(K)$ is continuous for each $K \in \Phi$. The notation $\mathcal{T}_\Phi$ is used for this topology. If X is a space and $\Phi = \mathcal{K}X$, and $\mathcal{T}_\Phi = \mathcal{C}_2$ the topology is denoted by $\mathcal{T}_{co}$ and called the *compact-open topology*.

Useful characterizations of $\mathcal{T}_\Phi$ in terms of a subbasis and convergence of nets with respect to $\mathcal{T}_\Phi$ are given in Proposition 1, Corollary 6, and Corollary 7. For $\Theta \subset \mathbf{SV}(X, Y)$, a nonempty subset K of X and a subset A of Y, let $M_1(K, A) = \{\mu \in \Theta : \mu(K) \subset A\}$ and let $M_2(K, A) = \{\mu \in \Theta : K \subset \mu^{-1}(A)\}$.

Proposition 1. *If Φ is a collection of nonempty subsets of X which contains the family of singletons, Y is a space, and $\Theta \subset \mathbf{SV}(X, Y)$ is nonempty, then $\mathcal{T}_\Phi$ is the topology generated by all subsets $M_1(K, W)$, $M_2(\{x\}, W)$, where $x \in X$, $K \in \Phi$, $W \in \mathcal{O}(Y)$, as subbase.*

Proof. Let $\mathcal{U}$ be the topology generated by the family of subsets described in the statement of the proposition as subbase. For $x \in X$ and $W \in \mathcal{O}(Y)$, $H_{\{x\}}^{-1}(W) = M_2(\{x\}, W)$ and $M_2(\{x\}, W) \in \mathcal{T}_\Phi$. Since

$$H_K^{-1}(Y - W) = \Theta - M_1(K, W)$$

and, since $H_K^{-1}(Y - W)$ is a $\mathcal{T}_\Phi$-closed subset of Θ, $M_1(K, W) \in \mathcal{T}_\Phi$. Hence $\mathcal{U} \subset \mathcal{T}_\Phi$. From the equations

$$H_K^{-1}(W) = \bigcup_{x \in K} M_2(\{x\}, W), \quad H_K^{-1}(Y - W) = \Theta - M_1(K, W)$$

for each $K \in \Phi$ and $W \in \mathcal{O}(Y)$, it follows that H_K is a continuous set-valued function for each $K \in \Phi$ if Θ is endowed with the topology $\mathcal{U}$. Hence $\mathcal{T}_\Phi \subset \mathcal{U}$.

The proofs of Corollaries 6 and 7 are omitted.

Corollary 6. *Let Φ be a collection of nonempty subsets of X which contains the family of singletons and Y be a space. If $\mathbf{SV}(X, Y)$ has the topology $\mathcal{T}_\Phi$, then a net μ_n in $\mathbf{SV}(X, Y)$ converges to $\mu \in \mathbf{SV}(X, Y)$ if and only if* (1) *$\mu_n(K) \subset W$ is ultimately satisfied whenever $K \in \Phi$ and $W \in \mathcal{O}(Y)$ satisfy $\mu(K) \subset W$ and* (2) *$\mu_n(x) \cap W \neq \varnothing$ is ultimately satisfied whenever $x \in X$, $W \in \mathcal{O}(Y)$ satisfy $\mu(x) \cap W \neq \varnothing$.*

Corollary 7. *Let Φ be a collection of nonempty subsets of X which contains the family of singletons and Y be a space. If $\mathbf{SV}(X, Y)$ has the topology $\mathcal{T}_\Phi$, then a net μ_n in $\mathbf{SV}(X, Y)$ converges to $\mu \in \mathbf{SV}(X, Y)$ if and only if $\mu_n(K) \to \mu(K)$ in 2^Y for each $K \in \Phi$.*

Proposition 2 will be utilized in the sequel. A proof is given for the sake of completeness as the result is doubtless (at least in the case of functions) part of the folklore of topology (see [2], page 252, exercise 8, for the statement in the case of a continuous function and a decreasing sequence of nonempty compact subsets).

Proposition 2. *Let X, Y be spaces with X Hausdorff, Y Fréchet, and let $\varphi \in \mathbf{uSV}(X, Y)$. Then*

$$\varphi\left(\bigcap_{\Gamma} F\right) = \bigcap_{\Gamma} \varphi(F)$$

for any filterbase Γ of compact subsets on X.

Proof. Let $p \in \bigcap_\Gamma \varphi(F)$ and define $\leq$ on Γ by $F_1 \leq F_2$ if $F_2 \subset F_1$; for each $F \in \Gamma$ choose $x_F \in F$ such that $p \in \varphi(x_F)$. Fix $F_0 \in \Gamma$. Then the net $(x_F, \leq)$, which we simply call x_F, is ultimately in F_0. Some subnet of x_F, called again x_F, converges to $z \in F_0$. For any $F \in \Gamma$, such a subnet is ultimately in F and hence has a subnet converging to

www.pphmj.com

some point in F. Since X is Hausdorff, it follows that $z \in F$. Since φ is u.s.c. it follows that for $W \in \sum(\varphi(z))$, $\varphi(x_F) \subset W$ ultimately and, hence, $p \in \varphi(z)$ since Y is Fréchet.

Theorem 4. *Let X, Y be spaces with X Hausdorff, Y Fréchet, and let $\Gamma \subset \mathcal{K}X$ such that $\bigcap_{\Gamma_K} F = K$. For each nonempty $\Theta \subset \mathbf{uSV}(X, Y)$, $\{H_F^* : \Theta \to 2^Y : F \in \Gamma\}$ separates points.*

Proof. Suppose $\varphi, \vartheta \in \Theta$ and $\varphi(x) \neq \vartheta(x)$. Then, from Proposition 2,

$$\bigcap_{\Gamma_{\{x\}}} \varphi(F) = \varphi\left(\bigcap_{\Gamma_{\{x\}}} F\right) = \varphi(x) \neq \vartheta(x) = \vartheta\left(\bigcap_{\Gamma_{\{x\}}} F\right) = \bigcap_{\Gamma_{\{x\}}} \vartheta(F).$$

Choose $F \in \Gamma_{\{x\}}$ satisfying $\varphi(F) \neq \vartheta(F)$. For this F, $H_F^*(\vartheta) \neq H_F^*(\varphi)$.

Remark 1. The assumption of the existence of a collection $\Gamma \subset \mathcal{K}X$ such that $\bigcap_{\Gamma_K} F = K$ might appear to be somewhat artificial. However, in a second countable Hausdorff locally compact space, a countable base consisting of relatively compact open sets is such a Γ. Of course, $\mathcal{K}X$ is also such a Γ.

Remark 2. If X, Y are arbitrary spaces, and $\Gamma \subset \mathcal{K}X$ contains the collection of singletons, and $\Theta \subset \mathbf{KV}(X, Y)$, it is easy to see that $\{H_K^* : \Theta \to \mathcal{K}Y : K \in \Gamma\}$ separates points.

Remark 3. If X, Y are arbitrary spaces, and $\Gamma \subset \mathcal{K}X$ such that $\bigcap_{\Gamma_K} F = K$, the topology generated on $\Theta \subset \mathbf{SV}(X, Y)$ by $\{M_1(F, W) : F \in \Gamma, W \in \mathcal{O}(Y)\}$ as subbase will be denoted by $\mathcal{T}_\Gamma$.

Theorem 5. *Let X, Y be spaces with X Hausdorff, Y Fréchet, and let $\Gamma \subset \mathcal{K}X$ such that $\bigcap_{\Gamma_K} F = K$. Then $\mathcal{T}_\Gamma = \mathcal{C}_2$ on $\Theta \subset \mathbf{KV}(X, Y)$.*

Proof. Clearly, $\mathcal{T}_\Gamma \subset \mathcal{C}_2$. Let W be open in Y, let $\varphi \in \Theta$ and let $K \in \mathcal{K}X$ such that $\varphi(K) \subset W$. There is an $F \in \Gamma_K$ satisfying $\varphi(F) \subset W$. Otherwise, $\{\varphi(F) - W : F \in \Gamma_K\}$ is a filterbase of compact subsets on the closed set $Y - W$ and $\bigcap_{\Gamma_K} \varphi(F) - W \neq \varnothing$. It then follows from

www.pphmj.com

Proposition 2 that $\varphi(K) - W \neq \varnothing$, a contradiction. Choose such an F and let $\vartheta \in M_1(F, W)$. Then $\vartheta(K) \subset \vartheta(F) \subset W$. Hence $\varphi \in M_1(F, W) \subset M_1(K, W)$, and $\mathcal{C}_2 \subset \mathcal{T}_\Gamma$.

Corollary 8. *Let X, Y be spaces with X Hausdorff, Y Fréchet, and let $\Gamma \subset \mathcal{K}X$ such that $\bigcap_{\Gamma_K} F = K$. Then $\mathcal{T}_\Gamma = \mathcal{T}_{co}$ on $\Theta \subset \mathbf{KV}(X, Y)$.*

Proof. This follows from the definition of $\mathcal{T}_{co}$ and Theorem 5.

Theorem 6. *Let X, Y be spaces with X Hausdorff, Y Fréchet, and let $\Gamma \subset \mathcal{K}X$ such that $\bigcap_{\Gamma_K} F = K$. Then $\Theta \subset \mathbf{KV}(X, Y)$ with the compact-open topology is imbedded in $(\mathcal{K}Y)^\Gamma$.*

Proof. From Theorem 5, $\mathcal{T}_\Gamma$ is the smallest topology on Θ for which $H_F : \Theta \to Y$ is continuous for each $F \in \Gamma$, and $\mathcal{T}_\Gamma = \mathcal{T}_{co}$ from Corollary 8. Moreover, H_F is point-compact, so $\{H_F^* : \Theta \to \mathcal{K}Y : F \in \Gamma\}$ is a family of continuous functions which separates points by Theorem 4.

Corollary 9. *Let X, Y be spaces with X Hausdorff, Y Fréchet, let $\mathbf{KV}(X, Y)$ have the compact-open topology. Then $\mathbf{KV}(X, Y)$ is imbedded in the product space $(\mathcal{K}Y)^{\mathcal{K}X}$.*

Remark 4. The conditions imposed on X, Y in Corollary 9 are unnecessary since $\mathcal{K}X$ contains the collection of singletons, and hence $\{H_K^* : \mathbf{KV}(X, Y) \to \mathcal{K}Y : K \in \mathcal{K}X\}$ separates points.

The next theorem characterizes compact subsets of $\mathbf{KV}(X, Y)$.

Theorem 7. *Let X, Y be Hausdorff spaces, and let $\Gamma \subset \mathcal{K}X$ such that $\bigcap_{\Gamma_K} F = K$. Then $\Theta \subset \mathbf{KV}(X, Y)$ with the compact-open topology is compact if and only if*

(1) Θ *is a closed subset of* $\mathbf{KV}(X, Y)$ *and*

(2) $\bigcup_{\mu \in \Theta} \mu(K)$ *is a compact subset of Y for each* $K \in \Gamma$.

Proof. First, the necessity of conditions (1) and (2) is established. As

www.pphmj.com

for (1), Θ is a closed subset of $\mathbf{KV}(X, Y)$ since $\mathcal{K}Y$ is Hausdorff [6], Θ is compact, and $\mathbf{KV}(X, Y)$ is imbedded in the Hausdorff space $(\mathcal{K}Y)^{\Gamma}$. Now, let $K \in \Gamma$. Since H_K is u.s.c. and point-compact it follows that $H_K(\Theta) = \bigcup_{\mu \in \Theta} \mu(K)$ is a compact subset of Y and (2) is established. Assume now that conditions (1) and (2) hold. From Theorem 4.9.6 in [6], for each $K \in \Gamma$, $\mathcal{K}[H_K(\Theta)]$ is compact. From above, there is a homeomorphism (into)

$$\mathcal{H} : \Theta \to \mathcal{P} = \prod_{\Gamma} \mathcal{K}[H_K(\Theta)].$$

The topological space $\mathcal{P}$ is compact by Tychonoff's theorem. Since Θ is a closed subset of $\mathbf{KV}(X, Y)$, $\mathcal{H}(\Theta)$ is a closed subset of $(\mathcal{K}Y)^{\Gamma}$ and hence of $\mathcal{P}$. Hence $\mathcal{H}(\Theta)$ and Θ are compact.

The Hausdorff condition on X in Theorem 7 may be removed if $\Gamma = \mathcal{K}X$.

Theorem 8. *Let X, Y be spaces with Y Hausdorff. Then $\Theta \subset \mathbf{KV}(X, Y)$ with the compact-open topology is compact if and only if*

(1) Θ *is a closed subset of* $\mathbf{KV}(X, Y)$ *and*

(2) $\bigcup_{\mu \in \Theta} \mu(K)$ *is a compact subset of Y for each $K \in \mathcal{K}X$.*

Proof. See Remark 4.

Corollary 10. *Let X, Y be spaces with Y Hausdorff. Then $\mathbf{KV}(X, Y)$ with the compact-open topology is compact if and only if $\bigcup_{\mu \in \Theta} \mu(K)$ is a compact subset of Y for each $K \in \mathcal{K}X$.*

In what follows, spaces X, Y are produced for which the space $\mathbf{KV}(X, Y)$ with the compact-open topology satisfies the properties of second countability, metrizability, and first countability by imbedding the space in a product space. A different approach is utilized to prove Theorems 12 and 13, a second countability theorem and a first countability theorem for $\mathbf{KV}(X, Y)$ equipped with Smithson's topology

www.pphmj.com

$\mathcal{C}_2$. The search for function spaces with these properties is motivated by a paper of McCoy [5].

Theorem 9. *Let* X, Y *be spaces with* X *Hausdorff,* Y *Fréchet, and let* $\Gamma \subset \mathcal{K}X$ *be countable such that* $\bigcap_{\Gamma_K} F = K$. *Then* $\mathbf{KV}(X, Y)$ *endowed with the compact-open topology is second countable if and only if* Y *is second countable.*

Proof. Suppose Y is second countable. Then $\mathcal{K}Y$ is second countable [6]. From Theorem 6, $\mathbf{KV}(X, Y)$ is imbedded in $(\mathcal{K}Y)^{\Gamma}$, a countable product of second countable spaces. The proof of the other required implication is obvious since Y is homeomorphic to the subspace of constant functions from X to Y.

Theorem 10. *Let* X, Y *be spaces with* X *Hausdorff,* Y *Fréchet, and let* $\Gamma \subset \mathcal{K}X$ *be countable such that* $\bigcap_{\Gamma_K} F = K$. *Then* $\mathbf{KV}(X, Y)$ *endowed with the compact-open topology is metrizable if and only if* Y *is metrizable.*

Proof. Replace "second countable" with "metrizable" in the proof of Theorem 9.

Definition 3. A subset P of a topological space X has a *countable exterior base* if there is a countable family of open subsets Ω of X such that $P = \bigcap_{\Omega} A$ and for each $V \in \Sigma(P)$ some $A \in \Omega$ satisfies $A \subset V$.

It has been shown that the hyperspace of compact subsets of a Hausdorff space X is first countable if and only if X satisfies either of the following conditions:

(1) [9] The space X is compactly second countable.

(2) [11] The space X is separable and each compact subset has a countable exterior base.

Theorem 11. *Let* X *be a Hausdorff space and let* $\Gamma \subset \mathcal{K}X$ *be countable such that* $\bigcap_{\Gamma_K} F = K$. *If* Y *is Hausdorff and compactly second countable, then* $\mathbf{KV}(X, Y)$ *endowed with the compact-open topology is first countable.*

www.pphmj.com

Proof. From [9], $\mathcal{K}Y$ is first countable. From Theorem 6, $\mathbf{KV}(X, Y)$ is imbedded in $(\mathcal{K}Y)^{\Gamma}$, a countable product of first countable spaces.

Theorem 12. *Let X be a Hausdorff space and let $\Gamma \subset \mathcal{K}X$ be countable such that $\bigcap_{\Gamma_K} F = K$. If the Hausdorff space Y is second countable, then $\mathbf{KV}(X, Y)$ with the topology $\mathcal{C}_2$ is second countable.*

Proof. Let $\mathcal{B}$ be a countable base of open subsets of Y which is closed under finite unions. Let $\mathcal{D}$ be the collection of finite intersections of elements of $\{M_1(F, H) : F \in \Gamma, H \in \mathcal{B}\}$. Then $\mathcal{D}$ is countable. Let W be open in Y and let $K \in \mathcal{K}X$ such that $W \in \Sigma(\varphi(K))$. Since $\varphi(K)$ is compact, choose $V \in \mathcal{B} \cap \Sigma(\varphi(K))$ such that $V \subset W$. As in the proof of Theorem 5, choose $F \in \Gamma_K$ such that $\varphi(F) \subset V$. For such F, V, if $\varrho \in M_1(F, W)$, then $\varrho(K) \subset \varrho(F) \subset V \subset W$. Hence $\mathcal{D}$ is a base for the topology $\mathcal{C}_2$ on $\mathbf{KV}(X, Y)$.

Theorem 13. *Let X be a Hausdorff space and let $\Gamma \subset \mathcal{K}X$ be countable such that $\bigcap_{\Gamma_K} F = K$. If each nonempty compact subset of the Hausdorff space Y has a countable exterior base, then $\mathbf{KV}(X, Y)$ with the topology $\mathcal{C}_2$ is first countable.*

Proof. Let $\varphi \in \mathbf{KV}(X, Y)$ and for each $F \in \Gamma$, let $\{W_{F,k} : k = 1, 2, \ldots\}$ be a countable exterior base at $\varphi(F)$. Let $\mathcal{B}$ be the collection of finite intersections of elements of $\{M_1(F, W_{F,k}) : F \in \Gamma, k = 1, 2, \ldots\}$. Then $\mathcal{B}$ is countable. It will now be shown that $\mathcal{B}$ is a base at φ. Let W be open in Y and let $K \in \mathcal{K}X$ such that $\varphi(K) \subset W$. As in the proof of Theorem 5, choose $F \in \Gamma_K$ such that $\varphi(F) \subset W$, and choose an integer m such that $\varphi(F) \subset W_{F,m} \subset W$. If $\varrho \in M_1(F, W_{F,m})$, then $\varrho(K) \subset \varrho(F) \subset W_{F,m} \subset W$.

Theorem 14. *Let X be a locally compact space and let $K \subset X$ be a compact G_δ. Then K has a countable exterior base.*

Proof. Under the hypothesis K is the intersection of a sequence of

www.pphmj.com

compact neighborhoods K_n. If some $W \in \Sigma(K)$ and every K_n satisfies $K_n - W \neq \emptyset$, then $K - W \neq \emptyset$.

Theorem 15. *Let X be a countably compact regular space and let* $K \subset X$ *be a compact* G_δ. *Then K has a countable exterior base.*

Proof. Under the hypothesis K is the intersection of a sequence of closed neighborhoods K_n. If some $W \in \Sigma(K)$ and every K_n satisfies $K_n - W \neq \emptyset$, then $K - W \neq \emptyset$.

The following corollaries are consequences of the foregoing theorems.

Corollary 11. *Let X be a Hausdorff space and let* $\Gamma \subset \mathcal{K}X$ *be countable such that* $\bigcap_{\Gamma_K} F = K$. *If Y is a countably compact regular space such that each* $K \in \mathcal{K}Y$ *is a* G_δ, *then* $(\mathbf{KV}(X, Y), \mathcal{C}_2)$ *is first countable.*

Corollary 12. *Let X be a Hausdorff space and let* $\Gamma \subset \mathcal{K}X$ *be countable such that* $\bigcap_{\Gamma_K} F = K$. *If Y* is *locally compact and Hausdorff and each* $K \in \mathcal{K}Y$ *is a* G_δ, *then* $(\mathbf{KV}(X, Y), \mathcal{C}_2)$ *is first countable.*

Corollary 13. *If X is a locally compact second countable Hausdorff space and each* $K \in \mathcal{K}Y$ *has a countable exterior base, then* $\mathbf{KV}(X, Y)$ *with the topology* $\mathcal{C}_2$ *is first countable.*

Arens [1] introduced the class of hemicompact spaces in 1946 and proved that the space of real-valued continuous functions defined on such a space is first countable when equipped with the compact-open topology. A generalization of this result is established in Theorem 16.

Theorem 16. *Let X be Hausdorff, hemicompact and Y be Hausdorff, locally compact and second countable. Then the collection,* $\mathcal{D} \subset \mathbf{KV}(X, Y)$, *endowed with the compact-open topology* $\mathcal{C}_2$ *of Smithson, and satisfying* $\alpha(\alpha^{-1}(F)) \subset F$ *for each* $F \subset Y$, $\alpha \in \mathcal{D}$ *is first countable.*

Proof. Let Λ be a countable collection of compact subsets with the properties provided by the assumption that X is hemicompact. Choose a countable collection Ω of relatively compact open subsets of Y such that,

www.pphmj.com

for each compact subset K of Y, and open subset W of Y satisfying $K \subset W$, some $P, Q \in \Omega$ satisfies $K \subset P \subset \mathrm{cl}(P) \subset Q \subset W$. For $\alpha \in \mathcal{D}$, let Δ be the countable collection of nonempty subsets of the form $M_1(H \cap \alpha^{-1}(\mathrm{cl}(P)), Q)$, where $H \in \Lambda$, $P, Q \in \Omega$, $\mathrm{cl}(P) \subset Q$. It is now shown that the collection of finite intersections of members of Δ is a countable base at α. Suppose J is compact in X, W is open in Y and $\alpha(J) \subset W$. Choose $P, Q \in \Omega$, $H \in \Lambda$ satisfying $\alpha(J) \subset P \subset \mathrm{cl}(P) \subset Q \subset W$, $J \subset H$. Then $\alpha \in M_1(H \cap \alpha^{-1}(\mathrm{cl}(P)), Q) \subset M_1(H \cap \alpha^{-1}(\mathrm{cl}(P)), W) \subset M_1(J, W)$.

Definition 4. The *point-open topology on a nonempty* $\Theta \subset \mathbf{SV}(X, Y)$ is the smallest topology on Θ for which the set-valued function $H_{\{x\}} : \Theta \to Y$ defined by $H_{\{x\}}(\mu) = \mu(x)$ is continuous for each $x \in X$. The notation $\mathcal{T}_{po}$ is used for this topology.

Proposition 3 may be proved by arguments similar to those in the proof of Proposition 1.

Proposition 3. *If* $\Theta \subset \mathbf{SV}(X, Y)$ *is nonempty, then* $\mathcal{T}_{po}$ *is the topology generated by all subsets* $M_1(\{x\}, W)$, $M_2(\{x\}, W)$, *where* $x \in X$, $W \in \mathcal{O}(Y)$, *as subbase.*

The topology $\mathcal{T}_{po}$ on the nonempty $\Theta \subset \mathbf{SV}(X, Y)$ is the same as the topology T_3 on $\Theta \subset \mathbf{SV}(X, Y)$ defined by Smithson in [7].

Corollary 14 is a generalization of Theorem 4 on page 222 in [4] and is similar to Theorem 2 in [7]. It may be proved by arguments like those in the proofs of Theorems 2.2 and 2.4 in [3].

Corollary 14. (1) *The space* $\mathbf{SV}(X, Y)$ *is Hausdorff if and only if Y is Hausdorff.* (2) *For Y Hausdorff, the space* $\mathbf{SV}(X, Y)$ *is regular* (*completely regular*) *if and only if Y is regular* (*completely regular*).

The topology on the nonempty $\Theta \subset \mathbf{SV}(X, Y)$ generated by subsets $M_1(K, W)$, $M_2(K, W)$ as subbase, where $K \in \mathcal{K}X$ and $W \in \mathcal{O}(Y)$, is called $\mathcal{C}_3$ by Smithson [7]. Clearly, $\mathcal{C}_2 \subset \mathcal{T}_{co} \subset \mathcal{C}_3$.

www.pphmj.com

Theorem 17. *If Y is a Hausdorff space and* $\Theta \subset \mathbf{SV}(X, Y)$ *is compact with the topology* $\mathcal{C}_3$, *then* $\mathcal{T}_{po} = \mathcal{C}_2 = \mathcal{T}_{co} = \mathcal{C}_3$.

Corollary 15. *If Y is a Hausdorff space and* $\Theta \subset \mathbf{SV}(X, Y)$ *is compact with the topology* $\mathcal{C}_3$, *then* $\mathcal{C}_3$ *is characterized in each of the following ways*:

(1) *The weak topology generated by the collection of continuous functions* $\{H_K^* : F \to \mathcal{K}Y : K \in \mathcal{K}X\}$.

(2) *The weak topology generated by the collection of continuous functions* $\{H_{\{x\}}^* : \Theta \to \mathcal{K}Y : x \in X\}$.

(3) *The smallest topology generated by the collection of u.s.c. set-valued functions* $\{H_K : \Theta \to Y : K \in \mathcal{K}X\}$.

References

[1] R. Arens, A topology for spaces of transformations, Ann. Math. 47(3) (1946), 480-495.

[2] J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

[3] T. Edwards, On function spaces with the compact-open topology, New Zealand J. Math. 28(2) (1999), 185-192.

[4] J. Kelley, General Topology, Springer-Verlag, New York, 1975.

[5] R. McCoy, Countability properties of function spaces, Rocky Mountain J. Math. 10(4) (1980), 717-730.

[6] E. Michael, Topologies on spaces of sets, Trans. Amer. Math. Soc. 71 (1951), 152-182.

[7] R. Smithson, Topologies on sets of relations, J. Nat. Sci. Math. (Lahore) 11 (1971), 43-50.

[8] R. Smithson, Multifunctions, Nieuw Archief voor Wiskunde XX(3) (1972), 31-53.

[9] R. Smithson, First countable hyperspaces, Proc. Amer. Math. Soc. 56 (1976), 325-328.

[10] A. Wilansky, Topology for Analysis, Robert E. Krieger Publishing Company, Inc., Malabar, Florida, 1983.

[11] D. Wulbert, Subsets of first countable spaces, Proc. Amer. Math. Soc. 19 (1968), 1273-1277.

www.pphmj.com