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PRIMITIVE IDEMPOTENTS OF IRREDUCIBLE CYCLIC CODES OF LENGTH $p^n q^m$

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Abstract

We obtain the explicit expressions for the $2mn + m + n + 1$ primitive idempotents in the ring $\frac{GF(l)[x]}{\langle x^{p^n q^m} - 1\rangle}$, where p, q are distinct odd primes, multiplicative order of l modulo p^n is $\frac{\phi(p^n)}{2}$, $(n \geq 1)$ and q is primitive root modulo q^m with $\gcd\left(\frac{\phi(p^n)}{2}, \phi(q^m)\right) = 1.$ We generalize the results of Pruthi and Arora [6] and, Batra and Arora [3].

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1. Introduction

Let $GF(l)(F)$ be a field of odd prime order l and G be a cyclic group of order η such that $\gcd(l, n) = 1.$ By Maschke Theorem [11, p. 143], FG (group algebra) is semi-simple and due to Wedderburn Theorem [11, p. 53], every ideal in FG can be written as a direct sum of a finite number of minimal ideals as each ideal in FG is generated by unique idempotent. The generating idempotent of the minimal ideal is called *Primitive Idempotent*. Let $R_\eta = \frac{GF(l)[x]}{\langle x^\eta - 1\rangle}.$ Then $R_\eta \cong FG,$ therefore it is semi-simple. Thus, to describe the complete set of ideals (codes over F) in $R_\eta,$ it is sufficient to find its complete set of primitive idempotents. Let $o(l)_\eta$ denote the order of l modulo η. For $\eta = 2, 4, p^n, 2p^n,$ p is an odd prime and $o(l)_\eta = \phi(\eta),$ the complete set of primitive idempotents in R_η is obtained by Arora and Pruthi [1, 6]. For $\eta = 2^n,$ p^n $(n \geq 1),$ p is an odd prime and $o(l)_\eta = \frac{\phi(\eta)}{2},$ the complete set of primitive idempotents in R_η is obtained by Batra and Arora [3, 4]. For $\eta = p^n q$ $(n \geq 1),$ p and q are distinct odd primes, where l is primitive root modulo p^n and q both with $\gcd\left(\frac{\phi(p^n)}{2}, \frac{\phi(q)}{2}\right) = 1,$ the primitive idempotents in R_η are obtained by Bakshi and Raka [2]. For $\eta = p^n (n \geq 1),$ p is an odd prime, $o(l)_\eta = \frac{\phi(\eta)}{e},$ e is a positive integer, the primitive idempotents in R_η are obtained by Raka et al. [7]. Singh and Pruthi [10] obtained the primitive idempotents of the quadratic residue codes of length $p^n q^m,$ p, q are distinct odd primes and $o(l)_{p^n} = \frac{\phi(p^n)}{2},$ $o(l)_{q^m} = \frac{\phi(q^m)}{2},$ $\gcd\left(\frac{\phi(p^n)}{2}, \frac{\phi(q^m)}{2}\right) = 1.$ Sahni and Sehgal [9] described the primitive idempotents of minimal cyclic codes of length $p^n q$, p, q are distinct odd primes and $o(l)_{p^n} = \phi(p^n)$, $o(l)_q = \phi(q)$, $\gcd\left(\frac{\phi(p^n)}{2}, \frac{\phi(q)}{2}\right) = d$, p does not divide $q - 1$.

In this paper, we consider the case when $\eta = p^n q^m$, where p, q are distinct odd primes and

$$o(l)_{p^n} = \frac{\phi(p^n)}{2}, \quad o(l)_{q^m} = \phi(q^m), \quad \gcd\left(\frac{\phi(p^n)}{2}, \phi(q^m)\right) = 1.$$

Then the complete set of $2mn + 2n + m + 1$ cyclotomic cosets modulo $p^n q^m$ is obtained in Theorem 2.4. Then the corresponding $2mn + 2n + m + 1$ primitive idempotents in R_η are given by Theorems 4.2 and 4.3. In Section 5, as an example, we describe an explicit expression for complete set primitive idempotents of irreducible cyclic codes of length 1089.

2. Cyclotomic Cosets

Let $S = \{0, 1, 2, ..., p^n q^m - 1\}$. For $a, b \in S$, say that $a \sim b$ iff $a \equiv bq^i \pmod{p^n q^m}$ for some integer $i \geq 0$. Then '$\sim$' is an equivalence relation on set S. The equivalence classes of this relation are called *l-cyclotomic coset modulo* $p^n q^m$. The l-cyclotomic coset containing $s \in S$ is $C_s = \{s, sl, ..., sl^{t_s - 1}\}$, where t_s is the smallest positive integer such that $sl^{t_s} \equiv s \pmod{p^n q^m}$.

Lemma 2.1. *Let p, q, l be distinct odd primes and $n \geq 1$, $m \geq 1$ are integers*,

$$o(l)_{p^n} = \frac{\phi(p^n)}{2}, \quad o(l)_{q^m} = \phi(q^m) \quad \textit{and} \quad \gcd\left(\frac{\phi(p^n)}{2}, \phi(q^m)\right) = 1.$$

Then

$$o(l)_{p^{n-j}q^{m-k}} = \frac{\phi(p^{n-j}q^{m-k})}{2}, \text{ for all } j, k;\ 0 \le j \le n-1,\ 0 \le k \le m-1.$$

Proof. Trivial.

Lemma 2.2. *For given distinct odd primes p, q and l, there exists always a fixed integer 'g' satisfying* $(g, pq\,l) = 1$, $1 < g < pq$, $g \not\equiv l^t \pmod{pq}$, *for* $0 \le t \le \frac{\phi(pq)}{2} - 1$. *Further, for* $0 \le j \le n-1$ *and* $0 \le k \le m-1$, *the set*

$$\left\{1, l, l^2, \ldots, l^{\frac{\phi(p^{n-j}q^{m-k})}{2}-1}, g, gl, \ldots, gl^{\frac{\phi(p^{n-j}q^{m-k})}{2}-1}\right\}$$

forms a reduced residue system $(\text{mod } p^{n-j}q^{m-k})$.

Proof. By Lemma 2.1, the lemma follows.

Remark 2.3. On the similar lines, we can prove that the set

$$\left\{1, l, l^2, \ldots, l^{\frac{\phi(p^{n-j})}{2}-1}, g, gl, \ldots, gl^{\frac{\phi(p^{n-j})}{2}-1}\right\}$$

forms a reduced residues system modulo p^{n-j}, for $0 \le j \le n-1$.

Theorem 2.4. *If* $\eta = p^n q^m$ $(n, m \ge 1)$, *then the* $2mn + 2n + m + 1$ *cyclotomic cosets modulo* $p^n q^m$ *are given by*

(i) $C_0 = \{0\}$.

For $0 \le j \le m-1$,

(ii) $C_{p^n q^j} = \{p^n q^j, p^n q^j l, \ldots, p^n q^j l^{\phi(q^{m-j})-1}\}$.

For $0 \leq i \leq n-1$,

$$\text{(iii)}\ C_{p^i q^m} = \left\{ p^i q^m,\ p^i q^m l,\ \ldots,\ p^i q^m l^{\frac{\phi(p^{n-i})}{2}-1} \right\},$$

$$\text{(iv)}\ C_{gp^i q^m} = \left\{ gp^i q^m,\ gp^i q^m l,\ \ldots,\ gp^i q^m l^{\frac{\phi(p^{n-i})}{2}-1} \right\}.$$

For $0 \leq i \leq n-1,\ 0 \leq j \leq m-1$,

$$\text{(v)}\ C_{p^i q^j} = \left\{ p^i q^j,\ p^i q^j l,\ \ldots,\ p^i q^j l^{\frac{\phi(p^{n-i} q^{m-j})}{2}-1} \right\},$$

$$\text{(vi)}\ C_{gp^i q^j} = \left\{ gp^i q^j,\ gp^i q^j l,\ \ldots,\ gp^i q^j l^{\frac{\phi(p^{n-i} q^{m-j})}{2}-1} \right\},$$

where g is defined in Lemma 2.2.

Proof. By repeated application of Lemmas 2.1 and 2.2, the proof follows immediately.

3. Some Lemmas

In this section, we first prove Lemmas 3.4, 3.6-3.16 to describe the complete set of primitive idempotents in $R_{p^n q^m} = \dfrac{GF(l)[x]}{\langle x^{p^n q^m} - 1 \rangle}$.

Definition 3.1. Let α be a fixed primitive $p^n q^m$th root of unity in some extension field of $GF(l)$. For $0 \leq i \leq n-1,\ 0 \leq j \leq m-1$, define

1. $A_{(i,\,j)} = \sum_{s \in C_g} \alpha^{p^i q^j s}$

2. $B_{(i,j)} = \sum_{s \in C_1} \alpha^{p^i q^j s}$

3. $\eta_0 = \sum_{s=0}^{\frac{\phi(p)}{2}-1} (\alpha^{p^{n-1}q^m})^{l^s}$

4. $\eta_1 = \sum_{s=0}^{\frac{\phi(p)}{2}-1} (\alpha^{p^{n-1}q^m})^{gl^s}.$

Note 3.2. Throughout in this paper, we take α as primitive $p^n q^m$th root of unity in some extension field of $GF(l)$.

Remark 3.3. Observe that for $0 \leq i \leq n-1$, $0 \leq j \leq m-1$,

$$A_{(i,j)} = \sum_{s \in C_g} \alpha^{p^i q^j s} = \frac{\phi(p^n q^m)}{\phi(p^{n-i} q^{m-j})} \sum_{s=0}^{\frac{\phi(p^{n-i}q^{m-j})}{2}-1} \beta^{sl^s}$$

and

$$B_{(i,j)} = \frac{\phi(p^n q^m)}{\phi(p^{n-i} q^{m-j})} \sum_{s=0}^{\frac{\phi(p^{n-i}q^{m-j})}{2}-1} \beta^{l^s}.$$

Lemma 3.4. *For* $0 \leq j \leq n-1$, $0 \leq k \leq m-1$, *if* β *is primitive* $p^j q^k$*th root of unity in some extension field of* $GF(l)$ *and* $o(l)_{p^j q^k} = \frac{\phi(p^j q^k)}{2}$, *then*

$$\sum_{s=0}^{\frac{\phi(p^j q^k)}{2}-1} (\beta^{l^s} + \beta^{gl^s}) = \begin{cases} 1 & \text{if } j, k = 1 \\ 0 & \text{if } j, k \neq 1. \end{cases}$$

Proof. By Lemma 2.2, the lemma follows.

Notation 3.5. For presenting the results in compact, we reserve the symbol that $(x, y) \leq (s, t)$ which means that x and y are simultaneously varying with the bounds $0 \leq x \leq s$ and $0 \leq y \leq t$, where s and t are integers.

Lemma 3.6. *For* $0 \leq i \leq n-1,\ 0 \leq j \leq m-1,$

$$A_{(i,j)} + B_{(i,j)} = \begin{cases} p^{n-1}q^{m-1} & \textit{if } (i, j) = (n-1, m-1) \\ 0 & \textit{if } (i, j) \leq (n-2, m-2). \end{cases}$$

Proof. By Definition 3.1, Remark 3.3, Lemma 2.2 and Lemma 3.4, the proof follows.

Lemma 3.7. *For* $0 \leq k \leq m-1,$

$$\sum_{s \in C_{p^n q^k}} \alpha^{p^i q^r s} = \begin{cases} -q^{m-k-1} & \textit{if } r + k = m-1 \\ 0 & \textit{if } r + k < m-1 \\ \phi(q^{m-k}) & \textit{if } r + k \geq m. \end{cases}$$

Proof. Let $\beta = \alpha^{q^{r+k} p^{n+i}}$. Then $\beta = 1$ if $r + k \geq m$ and is primitive q^{m-r-k}th if $r + k \leq m-1$.

Case (i). If $r + k \geq m$, then the sum equals $\phi(q^{m-k})$.

Case (ii). If $r + k \leq m-1$, then $\beta^{l^s} = \beta^{l^r}$ iff $l^s \equiv l^r (\operatorname{mod} q^{m-r-k})$ iff $s \equiv r\ (\operatorname{mod} \phi(q^{m-r-k}))$. In view of the above discussion, we get the desired result.

Lemma 3.8. *For* $0 \leq j \leq n-1,$

$$\sum_{s \in C_{p^j q^m}} \alpha^{p^i q^r s} = \sum_{s \in C_{gp^j q^m}} \alpha^{gp^i q^r s} = \begin{cases} p^{n-j-1}\eta_0 & \textit{for } i + j = n-1 \\ 0 & \textit{for } i + j < n-1 \\ \dfrac{\phi(p^{n-j})}{2} & \textit{for } i + j \geq n. \end{cases}$$

Proof. Trivial.

Lemma 3.9. *For* $0 \le j \le n-1$,

$$\sum_{s \in C_{gp^j q^m}} \alpha^{p^i q^r s} = \sum_{s \in C_{p^j q^m}} \alpha^{gp^i q^r s} = \begin{cases} p^{n-j-1}\eta_1 & \text{for } i+j = n-1 \\ 0 & \text{for } i+j < n-1 \\ \dfrac{\phi(p^{n-j})}{2} & \text{for } i+j \ge n. \end{cases}$$

Proof. Similar as Lemma 3.8.

Lemma 3.10. *For* $0 \le j \le n-1$ *and* $0 \le k \le m-1$,

$$\sum_{s \in C_{p^j q^k}} \alpha^{p^i q^r s} = \sum_{s \in C_{gp^j q^k}} \alpha^{gp^i q^r s}$$

$$= \begin{cases} \dfrac{1}{p^j q^k} B_{(i+j,\, r+k)} & \text{if } (i+j) \le n-1 \text{ and } (r+k) \le m-1 \\ -\dfrac{\phi(p^{n-j}) q^{m-k-1}}{2} & \text{if } (i+j) \ge n \text{ and } (r+k) = m-1 \\ 0 & \text{if } (i+j) \ge n \text{ and } (r+k) < m-1 \\ p^{n-j-1}\phi(q^{m-k})\eta_0 & \text{if } (i+j) = n-1 \text{ and } (r+k) \ge m \\ 0 & \text{if } (i+j) < n-1 \text{ and } (r+k) \ge m \\ \dfrac{\phi(p^{n-j} q^{m-k})}{2} & \text{if } (i+j) \ge n \text{ and } (r+k) \ge m. \end{cases}$$

Proof. By Lemma 3.8 and Remark 3.3, the lemma follows.

Lemma 3.11. *For* $0 \le j \le n-1$, $0 \le r \le m-1$,

$$\sum_{s \in C_{p^i q^r}} \alpha^{p^j q^m s} = \sum_{s \in C_{gp^i q^r}} \alpha^{gp^j q^m s} = \begin{cases} p^{n-i-1}\phi(q^{m-r})\eta_0 & \text{if } (i+j) = n-1 \\ 0 & \text{if } (i+j) < n-1 \\ \dfrac{\phi(p^{n-i} q^{m-r})}{2} & \text{if } (i+j) \ge n. \end{cases}$$

Proof. The proof follows on the similar lines of Lemma 3.10.

Lemma 3.12. *For* $0 \le j \le n-1,\ 0 \le k \le m-1,$

$$\sum_{s \in C_{p^j q^k}} \alpha^{g p^i q^r s} = \sum_{s \in C_{g p^j q^k}} \alpha^{p^i q^r s}$$

$$= \begin{cases} \dfrac{1}{p^j q^k} A_{(i+j, r+k)} & \text{if } (i+j) \le n-1 \text{ and } (r+k) \le m-1 \\ -\dfrac{\phi(p^{n-j}) q^{m-k-1}}{2} & \text{if } (i+j) \ge n \text{ and } (r+k) = m-1 \\ 0 & \text{if } (i+j) \ge n \text{ and } (r+k) < m-1 \\ p^{n-j-1} \phi(q^{m-k}) \eta_1 & \text{if } (i+j) = n-1 \text{ and } (r+k) \ge m \\ 0 & \text{if } (i+j) < n-1 \text{ and } (r+k) \ge m \\ \dfrac{\phi(p^{n-j} q^{m-k})}{2} & \text{if } (i+j) \ge n \text{ and } (r+k) \ge m. \end{cases}$$

Proof. Trivial.

Lemma 3.13. *For* $0 \le j \le n-1$ *and* $0 \le k \le m-1,$

$$\sum_{s \in C_{g p^i q^r}} \alpha^{p^j q^m s} = \sum_{s \in C_{p^i q^r}} \alpha^{g p^j q^m s} = \begin{cases} p^{n-i-1} \phi(q^{m-r}) \eta_1 & \text{if } (i+j) = n-1 \\ 0 & \text{if } (i+j) < n-1 \\ \dfrac{\phi(p^{n-i} q^{m-r})}{2} & \text{if } (i+j) \ge n. \end{cases}$$

Proof. The proof follows on the similar lines of Lemma 3.12.

Lemma 3.14. *For* $0 \le r \le m-1,$

$$\sum_{s \in C_{p^n q^r}} \alpha^{p^n q^k s} = \begin{cases} -q^{m-r-1} & \text{if } (i+j) = n-1 \\ 0 & \text{if } r+k < m-1 \\ \phi(q^{m-r}) & \text{if } r+k \ge m. \end{cases}$$

Proof. The proof follows on the similar lines of Lemma 3.7.

Lemma 3.15. *For* $0 \le i \le n-1$ *and* $0 \le r \le m-1$,

$$\sum_{s \in C_{p^i q^m}} \alpha^{p^j q^r s} = \sum_{s \in C_{gp^i q^m}} \alpha^{gp^j q^r s} = \begin{cases} p^{n-j-1}\eta_0 & \text{for } i+j = n-1 \\ 0 & \text{for } i+j < n-1 \\ \dfrac{\phi(p^{n-i}q^{m-r})}{2} & \text{for } i+j \ge n. \end{cases}$$

Proof. Trivial.

Lemma 3.16. *For* $0 \le i \le n-1$ *and* $0 \le r \le m-1$,

$$\sum_{s \in C_{gp^i q^m}} \alpha^{p^j q^r s} = \sum_{s \in C_{p^i q^m}} \alpha^{gp^j q^r s} = \begin{cases} p^{n-j-1}\eta_1 & \text{if } i+j = n-1 \\ 0 & \text{if } i+j < n-1 \\ \dfrac{\phi(p^{n-i})}{2} & \text{if } i+j \ge n. \end{cases}$$

Proof. Trivial.

4. Main Results

Theorem 4.1. *The primitive idempotent* θ_s *corresponding to the cyclotomic coset* C_s *in* $R_{p^n q^m}$, *is* $\theta_s = \sum_{k=0}^{p^n q^m - 1} \varepsilon_k x^k$, *where* $\varepsilon_k = \frac{1}{p^n q^m} \sum_{j \in C_s} \alpha^{-jk}$ *for* $k = 0, 1, 2, \ldots, p^n q^m - 1$.

Proof. See Theorem 1 [2].

Recall that to describe θ_s, it becomes necessary to compute ε_k. In view of the fact that $-C_1 = C_g$, then $\varepsilon_k = \frac{1}{p^n q^m} \sum_{j \in C_s} \alpha^{-jk} = \frac{1}{p^n q^m} \sum_{j \in C_{gs}} \alpha^{jk}$. Apart from this, ε_k has the same value when k runs through a cyclotomic coset. Therefore, to avoid confusion, we use ε_k^s instead of ε_k whenever k belongs to C_s.

Note. For our further discussion, throughout in this paper, the following notations will be used frequently:

For $0 \leq i,\ j \leq n;\ \ 0 \leq r,\ k \leq 1,$

(i) $\sigma_{p^i q^r}(x) = \sigma_{(i,r)}(x) = \sum\limits_{s \in C_{p^i q^r}} x^s$

(ii) $\sigma_{gp^i q^r}(x) = \sigma_{g(i,r)}(x) = \sum\limits_{s \in C_{gp^i q^r}} x^s.$

Theorem 4.2. *The* $2n + m + 1$ *primitive idempotents corresponding to cyclotomic cosets* $C_0,\ C_{p^n q^k},\ C_{p^j q^m}$ *and* $C_{gp^j q^m}$ *in* $R_{p^n q^m}$ *are*

(i) $\theta_0(x) = \dfrac{1}{p^n q^m}(1 + x + x^2 + \cdots + x^{p^n q^m - 1}).$

(ii) *For* $0 \leq k \leq m - 1,$

$$\theta_{p^n q^k}(x) = \frac{1}{p^n q^m}\left\{\phi(q^{m-k})\left[1 + \sum_{(i,r)=(0,m-k)}^{(n-1,m-1)} (\sigma_{(i,r)}(x) + \sigma_{g(i,r)}(x)) + \sum_{(i,r)=(n,m-k)}^{(n,m-1)} \sigma_{(i,r)}(x) + \sum_{(i,r)=(0,m)}^{(n=1,m)} (\sigma_{(i,r)}(x) + \sigma_{g(i,r)}(x))\right] - q^{m-k-1}\left[\sum_{i=0}^{n-1}(\sigma_{(i,m-k-1)}(x) + \sigma_{g(i,m-k-1)}(x) + \sigma_{(n,m-k-1)}(x))\right]\right\}.$$

(iii) *For* $0 \leq j \leq n - 1,$

$$\theta_{p^j q^m}(x)$$

$$= \frac{1}{p^n q^m}\left\{p^{n-j-1} \sum_{r=0}^{m} (\eta_1 \sigma_{(n-j-1,r)}(x) + \eta_0 \sigma_{g(n-j-1,r)}(x))\right.$$

$$+\frac{\phi(p^{n-j})}{2}\sum_{r=0}^{m}\sigma_{(n,r)}(x)+\frac{\phi(p^{n-j})}{2}\sum_{(i,r)=(n-j,0)}^{(n-1,m)}(\sigma_{(i,r)}(x)+\sigma_{g(i,r)}(x))\Bigg\}.$$

(iv)

$$\theta_{gp^jq^m}(x)=\frac{1}{p^nq^m}\Bigg\{p^{n-j-1}\sum_{r=0}^{m}(\eta_0\sigma_{(n-j-1,r)}(x)+\eta_1\sigma_{g(n-j-1,r)}(x))$$

$$+\frac{\phi(p^{n-j})}{2}\sum_{r=0}^{m}\sigma_{(n,r)}(x)$$

$$+\frac{\phi(p^{n-j})}{2}\sum_{(i,r)=(n-j,0)}^{(n-1,m)}(\sigma_{(i,r)}(x)+\sigma_{g(i,r)}(x))\Bigg\},$$

where $\eta_0=\frac{-1+\sqrt{-p}}{2}$ *and* $\eta_1=\frac{-1-\sqrt{-p}}{2}$.

Proof. By repeated application of Lemmas 3.4, 3.6-3.16, the proof follows.

Theorem 4.3. *For* $0\le j\le n-1$ *and* $0\le k\le m-1$, *the* 2*mn primitive idempotents corresponding to cyclotomic cosets* $C_{p^jq^k}$ *and* $C_{gp^jq^k}$ *in* $R_{p^nq^m}$ *are*

(i)

$$\theta_{p^jq^k}(x)=\frac{1}{p^nq^m}\Bigg\{\frac{1}{p^jq^k}[A_{(n-1,m-1)}\sigma_{(n-j-1,m-k-1)}(x)$$

$$+B_{(n-1,m-1)}\sigma_{g(n-j-1,m-k-1)}(x)]-\frac{\phi(p^{n-j})q^{m-k-1}}{2}$$

$$\cdot\left[\sum_{i=n-j}^{n-1}(\sigma_{(i,m-k-1)}(x)+\sigma_{g(i,m-k-1)}(x))+\sigma_{(n,m-k-1)}(x)\right]$$

$$+\frac{p^{n-j-1}\phi(q^{m-k})}{2}\left[\sum_{r=m-k}^{m}(\eta_1\sigma_{(n-j-1,r)}(x)+\eta_0\sigma_{g(n-j-1,r)}(x))\right]$$

$$+\frac{\phi(p^{n-j}q^{m-k})}{2}\left[\sum_{(i,r)=(n-j,m-k)}^{(n-1,m-1)}(\sigma_{(i,r)}(x)+\sigma_{g(i,r)}(x))\right.$$

$$\left.+\sum_{r=m-k}^{m-1}\sigma_{(n,r)}(x)\right]+\sum_{i=n-j}^{n-1}(\sigma_{(i,m)}(x)+\sigma_{g(i,m)}(x))+1\Bigg\}.$$

(ii)

$$\theta_{gp^jq^k}(x)=\frac{1}{p^nq^m}\left\{\frac{1}{p^jq^k}[B_{(n-1,m-1)}\sigma_{(n-j-1,m-k-1)}(x)\right.$$

$$+A_{(n-1,m-1)}\sigma_{g(n-j-1,m-k-1)}(x)]-\frac{\phi(p^{n-j})q^{m-k-1}}{2}$$

$$\cdot\left[\sum_{i=n-j}^{n-1}(\sigma_{(i,m-k-1)}(x)+\sigma_{g(i,m-k-1)}(x))+\sigma_{(n,m-k-1)}(x)\right]$$

$$+p^{n-j-1}\phi(q^{m-k})\sum_{(i,r)=(n-j-1,m-k)}^{(n-j-1,m)}[\eta_0\sigma_{(n-j-1,r)}(x)$$

$$+\eta_1\sigma_{g(n-j-1,r)}(x)]+\frac{\phi(p^{n-j}q^{m-k})}{2}$$

$$\cdot\left[\sum_{(i,r)=(n-j,m-k)}^{(n-1,m-1)}(\sigma_{(i,r)}(x)+\sigma_{g(i,r)}(x))+\sum_{r=m-k}^{m-1}\sigma_{(n,r)}(x)\right]$$

$$\left.+\sum_{i=n-j}^{n-1}(\sigma_{(i,m)}(x)+\sigma_{g(i,m)}(x))+1\right\},$$

where

$$A_{(n-1,\, m-1)} = \frac{p^{(n-1)}q^{(m-1)}(1 + \sqrt{-p})}{2}$$

and

$$B_{(n-1,\, m-1)} = \frac{p^{(n-1)}q^{(m-1)}(1 - \sqrt{-p})}{2}.$$

Proof. The proof follows on similar lines of Theorem 4.2.

5. Example

$p = 11,\ q = 3,\ l = 5,\ g = 2,\ n = 2,\ m = 2.$

Then $-C_1 = C_2,\ \eta_0 = 0,\ \eta_1 = -1,\ A_0 = 1,\ B_0 = 0.$

$$\theta_0(x) = 4(1 + x + x^2 + \cdots + x^{1088}),$$

$$\theta_{11^2}(x) = 4\{6[1 + \sigma_{(0,2)}(x) + \sigma_{2(0,2)}(x) + \sigma_{(1,2)}(x) + \sigma_{2(1,2)}(x)]$$
$$- 3[\sigma_{(0,1)}(x) + \sigma_{2(0,1)}(x) + \sigma_{(1,1)}(x) + \sigma_{2(1,1)}(x) + \sigma_{(2,1)}(x)]\},$$

$$\theta_{11^2 3}(x) = 4\{2[1 + \sigma_{(0,2)}(x) + \sigma_{2(0,2)}(x) + \sigma_{(1,2)}(x) + \sigma_{2(1,2)}(x)$$
$$+ \sigma_{(0,1)}(x) + \sigma_{2(0,1)}(x) + \sigma_{(1,1)}(x) + \sigma_{2(1,1)}(x) + \sigma_{(2,1)}(x)]$$
$$- [\sigma_{(0,0)}(x) + \sigma_{2(0,0)}(x) + \sigma_{(2,0)}(x)]\},$$

$$\theta_{3^2}(x) = 4\{[\eta_0(\sigma_{2(1,0)}(x) + \sigma_{2(1,1)}(x) + \sigma_{2(1,2)}(x))$$
$$+ \eta_1(\sigma_{(1,0)}(x) + \sigma_{(1,1)}(x) + \sigma_{(1,2)}(x))]\},$$

$$\theta_{3^2 11}(x) = 4\{[\eta_0(\sigma_{2(0,0)}(x) + \sigma_{2(0,1)}(x) + \sigma_{2(0,2)}(x)) + \eta_1(\sigma_{(0,0)}(x)$$
$$+ \sigma_{(0,1)}(x) + \sigma_{(0,2)}(x))]$$
$$+ \sigma_{(1,1)}(x) + \sigma_{2(1,1)}(x) + \sigma_{(1,2)}(x) + \sigma_{2(1,2)}(x)\},$$

$$\theta_{23^2 11}(x) = 4\{[\eta_1(\sigma_{2(0,0)}(x) + \sigma_{2(0,1)}(x) + \sigma_{2(0,2)}(x)) + \eta_0(\sigma_{(0,0)}(x)$$

$$+ \sigma_{(0,1)}(x) + \sigma_{(0,2)}(x))]$$

$$+ \sigma_{(1,1)}(x) + \sigma_{2(1,1)}(x) + \sigma_{(1,2)}(x) + \sigma_{2(1,2)}(x)\},$$

$$\theta_{23^2}(x) = 4\{[\eta_1(\sigma_{2(1,0)}(x) + \sigma_{2(1,1)}(x) + \sigma_{2(1,2)}(x))$$

$$+ \eta_0(\sigma_{(1,0)}(x) + \sigma_{(1,1)}(x) + \sigma_{(1,2)}(x))]\},$$

$$\theta_1(x) = 4\{A_{(1,1)}\sigma_{(1,1)}(x) + B_{(1,1)}\sigma_{2(1,1)}(x)$$

$$+ 3[\eta_0\sigma_{2(1,2)}(x) + \eta_1\sigma_{(1,2)}(x) + 1]\},$$

$$\theta_2(x) = 4\{B_{(1,1)}\sigma_{(1,1)}(x) + A_{(1,1)}\sigma_{2(1,1)}(x)$$

$$+ 3[\eta_1\sigma_{2(1,2)}(x) + \eta_0\sigma_{(1,2)}(x) + 1]\},$$

$$\theta_{11}(x) = 4[A_{(1,1)}\sigma_{(0,1)}(x) + B_{(1,1)}\sigma_{2(0,1)}(x) + 3\eta_0\sigma_{2(0,2)}(x)$$

$$+ \eta_1\sigma_{(0,2)}(x) + \sigma_{(1,2)}(x) + \sigma_{2(1,2)}(x) + 1],$$

$$\theta_{22}(x) = 4\{B_{(1,1)}\sigma_{(0,1)}(x) + A_{(1,1)}\sigma_{2(0,1)}(x)$$

$$+ 3[\eta_1\sigma_{2(0,2)}(x) + \eta_0\sigma_{(0,2)}(x) + \sigma_{(1,2)}(x) + \sigma_{2(1,2)}(x) + 1]\},$$

$$\theta_3(x) = 4\{2[A_{(1,1)}\sigma_{(1,0)}(x) + B_{(1,1)}\sigma_{2(1,0)}(x)]$$

$$+ [\eta_1\sigma_{(1,1)}(x) + \eta_0\sigma_{2(1,1)}(x) + \sigma_{2(1,2)}(x)]\},$$

$$\theta_6(x) = 4\{2[B_{(1,1)}\sigma_{(1,0)}(x) + A_{(1,1)}\sigma_{2(1,0)}(x)]$$

$$+ [\eta_0\sigma_{(1,1)}(x) + \eta_1\sigma_{2(1,1)}(x) + \sigma_{2(1,2)}(x)]\},$$

$$\theta_{33}(x) = 4\{2[A_{(1,1)}\sigma_{(0,0)}(x) + B_{(1,1)}\sigma_{2(0,0)}(x)]$$

$$+ 2[\eta_1(\sigma_{(0,1)}(x) + \sigma_{(0,2)}(x)) + \eta_0(\sigma_{2(0,1)}(x) + \sigma_{2(0,2)}(x))]\},$$

$$\theta_{66}(x) = 4\{2[B_{(1,1)}\sigma_{(0,0)}(x) + A_{(1,1)}\sigma_{2(0,0)}(x)$$

$$+ 2[\eta_0(\sigma_{(0,1)}(x) + \sigma_{(0,2)}(x)) + \eta_1(\sigma_{2(0,1)}(x) + \sigma_{2(0,2)}(x))]\}.$$

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