Far East Journal of Mathematical Sciences (FJMS)
Volume 76, Number 2, 2013, Pages 235-243
Published Online: June 2013
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

ON THE STEADY STATE THERMOELASTIC PROBLEM FOR A THIN ANNULAR DISC

B. R. Sontakke and A. A. Navlekar

Department of Mathematics
Pratishthan Mahavidyalaya
Paithan, Dist. Aurangabad (M.S)
India
e-mail: brsontakke@rediffmail.com

Abstract

This paper is concerned with the direct steady state thermoelastic problem to determine the temperature, displacement and stress functions of a thin annular disc occupying the space $D : a \leq r \leq b$, $-h \leq z \leq h$ with the stated boundary conditions. The finite Hankel transform technique has been used.

1. Introduction

During the second half of the twentieth century, non-isothermal problems of the theory of elasticity became increasingly important because of their applications in diverse fields. First, the high velocity of modern aircrafts gives rise to an aerodynamic heating which produces intense thermal stresses, reducing the strength of aircraft structure. Second, in the nuclear

2010 Mathematics Subject Classification: 35A25, 74M99, 74K20.

Keywords and phrases: annular disc, steady state problem, direct thermoelastic problem, Hankel transform.

Submitted by K. K. Azad

Received September 9, 2012

field, the extremely high temperature and temperature gradients originating inside nuclear reactor influence their design and operation.

Nowacki [6] has determined steady state thermal stresses in a circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and the circular edge. Further, Roychoudhari [7] has succeeded in determining the quasi static thermal stresses in a thin circular plate subjected to transient temperature along the circumference of a circular plate over the upper face with lower face at zero temperature and fixed circular edge thermally insulated.

Wankhede [9] has determined the quasi-static thermal stresses in a thin circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature.

In Khobragade and Wankhede [3, 4] and Khobragade [5], an inverse thermoelastic problem of a thin annular disc is considered to determine the temperature, displacement and thermal stresses.

Hence, in this paper, an attempt has been made to determine the unknown temperature, displacement and stress functions of a thin annular disc of thickness $2h$ occupying the space $D : a \le r \le b, -h \le z \le h.$

2. Result Required

If $f(x)$ satisfies Dirichlet's condition in the interval $(0, a),$ then its finite Hankel transform in that range is defined to be

$$\bar{f}_{\mu}(\xi_i) = \int_0^a xf(x) J_{\mu}(x\xi_i)\, dx, \tag{2.1}$$

where ξ_i is the root of the transcendental equation

$$J_{\mu}(a\xi_i) = 0. \tag{2.2}$$

Then, at any point of $(0,\, a)$ at which the function $f(x)$ is continuous,

$$f(x) = \frac{2}{a^2} \sum_i \bar{f}_\mu(\xi_i) \frac{J_\mu(x\xi_i)}{[J'_\mu(a\xi_i)]^2}, \tag{2.3}$$

where the sum is taken over all the positive roots of equation (2.2).

2.1. Properties of Hankel transform

If $f(X)$ satisfies Dirichlet's conditions in the interval $[0,\, a]$, then

(1) Finite Hankel transform of $\frac{\partial f}{\partial x}$, is

$$H_\mu\left[\frac{\partial f}{\partial x}\right] = \int_0^a \frac{\partial f}{\partial x} x J_\mu(x\xi_i)\, dx$$

$$= \frac{\xi_i}{2\mu}[(\mu - 1)H_{\mu+1} f(x) - (\mu + 1)H_{\mu-1} f(x)],$$

(2)

$$H_\mu\left[\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x}\right] = \frac{\xi_i}{2}\left[-H_{\mu-1}\frac{\partial f}{\partial x} + H_{\mu+1}\frac{\partial f}{\partial x}\right].$$

If $f(x)$ satisfies Dirichlet's conditions in the range $b \le x \le a$ and if its finite Hankel transform in that range is defined to be

$$H[f(x)] = \bar{f}_\mu(\xi_i)$$

$$= \int_b^a x f(x)[J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)]\, dx \tag{2.4}$$

in which, ξ_i is a root of the transcendental equation

$$[J_\mu(\xi_i b)G_\mu(\xi_i a) - J_\mu(\xi_i a)G_\mu(\xi_i b)] = 0, \tag{2.5}$$

then at which the function is continuous

$$f(x) = \sum_i \frac{2\xi_i^2 J_\mu^2(\xi_i b)\bar{f}_\mu(\xi_i)}{J_\mu^2(a\xi_i) - J_\mu^2(b\xi_i)}[J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)]. \tag{2.6}$$

Now, property of Hankel transform defined in (2.4),

$$\int_a^b \left[\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x}\right][J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)]dx$$

$$= -\xi_i^2 \bar{f}_\mu(\xi_i) + a[J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)]_{x=a}$$

$$+ b[J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)]_{x=b}$$

$$= -\xi_i^2 \bar{f}_\mu(\xi_i).$$

3. Statement of the Problem

Consider a thin circular plate of thickness $2h$ occupying the space D: $a \le r \le b,\ -h \le z \le h.$ The differential equation governing the displacement function $U(r, z)$ as (Nowacki [6]) is

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = (1+\nu)a_t T \tag{3.1}$$

$$\text{with } U_r = 0 \text{ at } r = a \text{ and } r = b, \tag{3.2}$$

ν and a_t are the Poisson's ratio and the linear coefficient of the thermal expansion of the material of the plate and T is the temperature of the plate satisfying the differential equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{3.3}$$

subject to the boundary conditions

$$\left[T(r, z) + \frac{\partial T(r, z)}{\partial z}\right]_{z=h} = f(r), \tag{3.4}$$

$$\left[T(r, z) + \frac{\partial T(r, z)}{\partial z}\right]_{z=-h} = g(r), \tag{3.5}$$

$$T(a, z) = 0, \tag{3.6}$$

$$T(b, z) = 0. \tag{3.7}$$

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu \frac{1}{r} \frac{\partial U}{\partial r}, \tag{3.8}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 U}{\partial r^2}, \tag{3.9}$$

where μ is the Lame's constant, while each of the stress functions σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ is zero with in the plate in the plane state of stress.

Equations (3.1) to (3.9) constitute the mathematical formulation of the problem under consideration.

4. Solution of the Problem

Applying finite Hankel transform stated in [8] to (3.3) to (3.5) and using (3.6) and (3.7), one obtains

$$\frac{d^2\overline{T}}{dz^2} - \lambda_n^2 \overline{T} = 0, \tag{4.1}$$

$$\left[\overline{T}(\lambda_n, z) + \frac{d\overline{T}(\lambda_n, z)}{dz}\right]_{z=h} = \bar{f}(\lambda_n), \tag{4.2}$$

$$\left[\overline{T}(\lambda_n, z) + \frac{d\overline{T}(\lambda_n, z)}{dz}\right]_{z=-h} = \overline{g}(\lambda_n), \tag{4.3}$$

where $\overline{T}$ denotes the finite Hankel transform of T and λ_n is the Hankel transform parameter.

Equation (4.1) is a second order differential equation whose solution is given by

$$\overline{T}(\lambda_n, z) = Ae^{\lambda_n z} + Be^{-\lambda_n z}, \tag{4.4}$$

where A and B are constants.

Using (4.2) and (4.3) in (4.4), we obtain the values of A and B. Substituting these values in (4.4) and then inversion of finite Hankel transform lead to

$$T(r,\ z) = 2\sum_{n=1}^{\infty}\lambda_n^2 \bar{f}(\lambda_n)\left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b) - J_0^2(\lambda_n a)}\right]$$

$$\cdot\left[\frac{\sinh(\lambda_n(z+h)) + \lambda_n \cosh(\lambda_n(z+h))}{(1+\lambda_n^2)\sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h)}\right]$$

$$\cdot\,[J_0(r\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(r\lambda_n)]$$

$$-\,2\sum_{n=1}^{\infty}\lambda_n^2 \bar{g}(\lambda_n)\left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b) - J_0^2(\lambda_n a)}\right]$$

$$\cdot\left[\frac{\sinh(\lambda_n(z-h)) - \lambda_n \cosh(\lambda_n(z-h))}{(1+\lambda_n^2)\sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h)}\right]$$

$$\cdot\,[J_0(r\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(r\lambda_n)], \tag{4.5}$$

where

$$\bar{f}(\lambda_n) = \int_a^b rf(r)[J_0(r\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(r\lambda_n)]dr,$$

$$\bar{g}(\lambda_n) = \int_a^b rg(r)[J_0(r\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(r\lambda_n)]dr$$

and λ_n are the positive roots of the transcendental equation

$$[J_0(a\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(a\lambda_n)] = 0,$$

equation (4.5) is the desired solution of the given problem.

5. Determination of Thermoelastic Displacement

Substituting the value of $T(r,\ z)$ from (4.5) in (3.1) one obtains the thermoelastic displacement function $U(r, z)$ as

$$U(r, z) = -2(1+\nu)a_t \sum_{n=1}^{\infty} \bar{f}(\lambda_n)\left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b) - J_0^2(\lambda_n a)}\right]$$

$$\cdot \left[\frac{\sinh(\lambda_n(z+h)) + \lambda_n \cosh(\lambda_n(z+h))}{(1+\lambda_n^2)\sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h)}\right]$$

$$\cdot [J_0(r\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(r\lambda_n)]$$

$$+ 2(1+\nu)a_t \sum_{n=1}^{\infty} \bar{g}(\lambda_n)\left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b) - J_0^2(\lambda_n a)}\right]$$

$$\cdot \left[\frac{\sinh(\lambda_n(z-h)) - \lambda_n \cosh(\lambda_n(z-h))}{(1+\lambda_n^2)\sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h)}\right]$$

$$\cdot [J_0(r\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(r\lambda_n)]. \tag{5.1}$$

6. Determination of Stress Functions

Using (5.1) in (3.8) and (3.9), the stress functions are obtained as

$$\sigma_{rr} = \frac{4\mu(1+\nu)a_t}{r} \sum_{n=1}^{\infty} \lambda_n \bar{f}(\lambda_n)\left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b) - J_0^2(\lambda_n a)}\right]$$

$$\cdot \left[\frac{\sinh(\lambda_n(z+h)) + \lambda_n \cosh(\lambda_n(z+h))}{(1+\lambda_n^2)\sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h)}\right]$$

$$\cdot [J_0(r\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(r\lambda_n)]$$

$$- \frac{4\mu(1+\nu)a_t}{r} \sum_{n=1}^{\infty} \lambda_n \bar{g}(\lambda_n)\left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b) - J_0^2(\lambda_n a)}\right]$$

$$\cdot \left[\frac{\sinh(\lambda_n(z-h)) - \lambda_n \cosh(\lambda_n(z-h))}{(1+\lambda_n^2)\sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h)}\right]$$

$$\cdot [J_1(r\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_1(r\lambda_n)], \tag{6.1}$$

$$\sigma_{\theta\theta} = 4\mu(1+\nu)a_t \sum_{n=1}^{\infty} \lambda_n^2 \bar{f}(\lambda_n) \left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b) - J_0^2(\lambda_n a)} \right]$$

$$\cdot \left[\frac{\sinh(\lambda_n(z+h)) + \lambda_n \cosh(\lambda_n(z+h))}{(1+\lambda_n^2)\sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h)} \right]$$

$$\cdot \left[J_1'(r\lambda_n) G_0(b\lambda_n) - J_0(b\lambda_n) G_1'(r\lambda_n) \right]$$

$$- 4\mu(1+\nu)a_t \sum_{n=1}^{\infty} \lambda_n^2 \bar{g}(\lambda_n) \left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b) - J_0^2(\lambda_n a)} \right]$$

$$\cdot \left[\frac{\sinh(\lambda_n(z-h)) - \lambda_n \cosh(\lambda_n(z-h))}{(1+\lambda_n^2)\sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h)} \right]$$

$$\cdot \left[J_1'(r\lambda_n) G_0(b\lambda_n) - J_0(b\lambda_n) G_1'(r\lambda_n) \right]. \tag{6.2}$$

7. Conclusions

In this paper, we discussed completely the direct steady state problem of thermoelastic deformation of a thin annular disc of thickness $2h$, with boundary condition of the third kind is maintained on the lower plane surface at $g(r)$ while on upper plane surface, it is maintained at $f(r)$ which is a known function of r and the temperature is maintained at zero on curved surfaces of the annular disc.

The finite Hankel transform technique is used to obtain the numerical results. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in engineering applications.

References

[1] M. J. Ciałkowski and K. Grysa, On a certain inverse problem of temperature and thermal stress fields, Acta Mechanica 36 (1980), 169-185.

[2] K. Grysa and Z. Kozłowski, One-dimensional problems of temperature and heat flux determination at the surfaces of a thermoelastic slab: Part 1: The analytical solutions, Nuclear Engineering and Design 74 (1982), 1-14.

[3] N. W. Khobragade and P. C. Wankhede, An inverse steady-state thermoelastic problem of a thin annular disc-I, Bulletin of Pure and Applied Sci. 19E(2) (2000), 551-555.

[4] N. W. Khobragade and P. C. Wankhede, An inverse steady-state thermoelastic problem of a thin annular disc, Far East J. Appl. Math. 5(3) (2001), 373-379.

[5] N. W. Khobragade, An inverse unsteady-state thermoelastic problem of a thin annular disc in Marchi-Fasulo transform domain, Far East J. Appl. Math. 11(1) (2003), 17-26.

[6] W. Nowacki, The state of stress in a thick circular plate due to temperature field, Bull. Sci. Acad. Polon Ser. Tech. 5 (1957), 227.

[7] S. K. Roychaudhari, A note on the quasi-static stresses in a thin circular plate due to transient temperature applied along the circumference of a circle over the upper face, Bull. Acad. Sci. Ser. Tech. 20 (1972), 21-24.

[8] I. N. Sneddon, The Use of Integral Transform, McGraw-Hill Book Co. Chap. 3, 1974.

[9] P. C. Wankhede, On the quasi-static thermal stresses in circular plate, Indian J. Pure Appl. Math. 13 (1982), 1273-1277.